Elliptic families of solutions of the KP equation and compact cycles in the moduli spaces of algebraic curves

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Vanishing properties of $\mathcal{M}_{g,k}$

The moduli spaces $\mathcal{M}_{g,k}$ of *smooth* genus g Riemann surfaces with punctures have curious vanishing properties.

- Diaz' theorem (1986):
  There does not exist a complete (complex) cycle in $\mathcal{M}_g$ of dimension greater than $g-2$

Note, that is the upper bound. The know constructions give complete cycles of dimension of order $\log_3 g$, only.

- Looijenga theorem (1995):
  The tautological ring $R^*(\mathcal{M}_{g,k})$ vanishes in dimensions greater then $g-2+k$

The tautological ring $R^*(\mathcal{M}_{g,k})$ is generated by classes

$$\psi_i = c_1(L_i), \ \ \kappa_i = p_*(\psi_1^{i+1}) \in H^*(\mathcal{M}_g).$$

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Faber's conjecture

- Faber conjectured (1999) that:
  $R^*(\mathcal{M}_{g,k})$ *looks "like" the cohomology ring of a compact complex variety of dimension* $g - 2 + k$

Conjectural geometric explanations

Widely accepted by experts "geometric explanation" of vanishing properties of $\mathcal{M}_{g,k}$ is the existence of its stratification by certain number of affine strata or the existence of a cover of $\mathcal{M}_{g,k}$ by certain number of open affine sets.

Historically, Arbarello first realized that a stratification of $\mathcal{M}_g$ could be useful for a study of its geometrical properties. He studied the stratification (known already for Rauch)

$$\mathcal{W}_2 \subset \mathcal{W}_3 \subset \cdots \subset \mathcal{W}_{g-1} \subset \mathcal{W}_g = \mathcal{M}_g,$$

where $\mathcal{W}_n$ if the locus of curves having a Weierstrass point of order at most n, and then conjectured that $\mathcal{W}_n \setminus \mathcal{W}_{n-1}$ is affine.

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Alternative geometric explanation

Relatively recently, the author jointly with S. Grushevsky proposed an alternative approach for geometrical explanation of the vanishing properties of $\mathcal{M}_{g,k}$ motivated by certain constructions of the Whitham perturbation theory of integrable systems. The key elements of the new approach are:

- the moduli space $\mathcal{M}_{g,k}^{(n)}$, $n = (n_1, \ldots, n_k)$ of smooth genus g Riemann surfaces with the fixed n_α-jets of local coordinates in the neighborhoods of labeled points is the total space of a *real-analytic* foliation, whose leaves $\mathcal{L}$ are locally smooth *complex subvarieties* of real codimension $2g$;
- on $\mathcal{M}_{g,k}^{(n)}$ there is an ordered set of $(\dim_{\mathbb{R}} \mathcal{L})$ continuous functions, which restricted onto the leaves of the foliation are piecewise harmonic. Moreover, the first of these function restricted onto $\mathcal{L}$ is a subharmonic function.

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Results

- Proof of Arbarello's conjecture

Theorem

Any compact complex cycle in $\mathcal{M}_g$ of dimension $g - n$ must intersect $\mathcal{W}_n$.

- New upper bound for dimensions of complete (complex) cycles in the moduli space $\mathcal{M}_g^{ct}$ of stable curves of compact type.

Theorem

There do not exist complete complex subvarieties of $\mathcal{M}_g^{ct}$ having non empty intersection with $\mathcal{M}_g$ of dimension greater than $g-1$.
For $g \geq 2$ the maximum dimension of complete complex subvarieties in $\mathcal{M}_g^{ct}$ is $\frac{3}{2}g - 2$.

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Previously known bounds

- Diaz:
  there is no compact cycle in $\mathcal{M}_g^{ct}$ of dimension greater that $2g - 3$.
- Keel and Sadun:
  for $g \geq 3$ there do not exist complete complex subvarieties of $\mathcal{M}_g^{ct}$ of dimension greater than $2g - 4$.

The proof is by easy induction arguments starting from the base $g = 3$. The proof of the base statement is a corollary of remarkable vanishing result:

- *there do not exist a complete complex subvarieties of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of codimension g.*

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Real normalized differentials

The foliation structure arises through identification of $\mathcal{M}_{g,k}^{(n)}$ with the moduli space of curves with fixed *real-normalized* meromorphic differential. By definition a real normalized meromorphic differential is a differential whose periods over any cycle on the curve are real.

Lemma

For any fixed singular parts of poles with pure imaginary residues, there exists a unique meromorphic differential Ψ*, having prescribed singular part at* p_α *and such that all its periods on* Γ *are real, i.e.*

$$\mathrm{Im}\left(\oint_c \Psi\right) = 0, \ \ \forall\, c \in H^1(\Gamma, \mathbb{Z}).$$

A notion of real normalized differentials is "almost equivalent" to a notion of harmonic functions.

- Indeed,the real normalization implies that the imaginary part $y(p) = \mathrm{Im}\, F(p)$ of the abelian integral $F(p) := \int^p \Psi$ is single-valued, and hence is a *harmonic function* on $\Gamma \setminus \{p_\alpha\}$.
- Conversely, for a given harmonic function $y(p)$ on $\Gamma \setminus \{p_\alpha\}$ there exists a unique up to a constant conjugated harmonic function $x(p)$, i.e. a function $x(p)$ such that $F(p) = x(p) + iy(p)$ is *holomorphic*. Hence $\Psi = dF$ is a real normalized holomorphic differential on $\Gamma \setminus \{p_\alpha\}$.

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Foliation

Definition

A leaf $\mathcal{L}$ of the foliation on $\mathcal{M}_{g,k}^{(n)}$ defined to be the locus along which the periods of the corresponding differentials remain (covariantly) constant.

The leaves $\mathcal{L}$ of the foliation can be regarded as a generalization of the Hurwitz spaces of $\mathbb{P}^1$ covers.

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Coordinates along a leaf

A set of holomorphic coordinates on $\mathcal{M}_{g,k}^{(n)}$ are "critical" values of the corresponding abelian integral $F(p) = c + \int^p \Psi$, $p \in \Gamma$:

At the generic point, where zeros q_s of Ψ are distinct, the coordinates on $\mathcal{L}$ are the evaluation of F at these critical points:

$$\varphi_s = F(q_s), \quad \Psi(q_s) = 0, \quad s = 0, \ldots, d = \dim \mathcal{L}, \tag{1}$$

normalized by the condition $\sum_s \varphi_s = 0$.

A direct corollary of the real normalization is the statement that:

- *imaginary parts $f_s = \mathrm{Im}\varphi_s$ of the critical values depend only on labeling of the critical points*

They can be arranged into decreasing order

$$f_0 \geq f_1 \geq \cdots \geq f_{d-1} \geq f_d.$$

After that f_j can be seen as a well-defined continuous function on $\mathcal{M}_{g,k}^{(n)}$, which restricted onto $\mathcal{L}$ is a piecewise harmonic function. Moreover, f_0 restricted onto $\mathcal{L}$ is a *subharmonic function*, i.e, f_0 *has no local maximum on* $\mathcal{L}$ *unless it is a constant*.

Diaz' theorem revisited

Let X be a complete cycle in $\mathcal{M}_g$ and Z be its preimage under the forgetfull map: $\mathcal{M}_{g,2} \subset \mathcal{C}_g^2 \longmapsto \mathcal{M}_g$.

$\rightarrow$ On Z the function f_0 (defined by critical values of real-normalized differential with two simple poles) must achieve its maximum at some point.

$\rightarrow$ At this point the function f_0 achieves its maximum on $Z \cap \mathcal{L}$.

$\rightarrow$ Hence, it is a constant on $Z \cap \mathcal{L}$.

$\rightarrow$ If f_0 is a constant then (inductively) all the other functions f_j are constants.

$\rightarrow$ Then, $Z \cap \mathcal{L}$ is at most zero-dimensional, i.e. Z intersects $\mathcal{L}$ transversally.

$\rightarrow$ $\dim X \leq g - 2$

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Calogero-Moser curves revisited

The elliptic CM system is a system of N particles with the Hamiltonian

$$H_2 = \frac{1}{2}\sum_{i=1}^{N} p_i^2 - 2\sum_{i \neq j} \wp(q_i - q_j),$$

where $\wp(q)$ is the Weierstrass $\wp$-function. It is completely integrable and admits the Lax representation $\dot{L} = [L, M]$, where $L = L(t, z)$ and $M = M(t, z)$ are $(N \times N)$ matrices depending on a spectral parameter z.

The spectral curves of the CM system are defined by the characteristic equation for the Lax matrix

$$\Gamma_{N,\tau}^{spec} \subset C \times E_\tau : R(k, z) = \det(k \cdot I - L(t, z)) = 0$$

They form a N-parametric family. The parameters are the commuting Hamiltonians H_i. For generic values of the parameters the spectral curves are smooth curves of genus N.

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For particular values of the parameters the spectral curve are singular.

Let $\mathcal{K}_{g,N,\tau} \subset \mathcal{M}_g$ be a family of *smooth genus g* algebraic curves Γ that are the normalization $\Gamma \longmapsto \Gamma^{spec}_{N,\tau}$ of N-particle CM system. It can be shown that:

- $\mathcal{K}_{g,N,\tau}$ is $g-1$-dimensional *affine* subvariety of $\mathcal{M}_g$.
- The closer of $\mathcal{K}_{g,N,\tau}$ as $N \to \infty$ is the whole moduli space $\mathcal{M}_g$.

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In terms of the real normalized differentials the locus $\mathcal{K}_g = \cup_{N,\tau} \mathcal{K}_{g,N,\tau}$ is characterized as:

- the locus of curves on which there exists a pair of meromorphic differentials Ψ_1, Ψ_2 with the only pole of the second order at a puncture p_0 and with integer periods

$$\oint_\gamma \Psi_i \in \mathbb{Z}$$

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For $\Gamma \in \mathcal{K}_g$ the parameters N and τ are recovered by the formulae

$$N = < \oint \Psi_1, \oint \Psi_2 >, \quad \tau = \frac{\Psi_1}{\Psi_2}(p_0)$$

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The theory of the elliptic CM system is isomorphic to the theory of elliptic solutions of the KP equation. Namely, a function $u(x, y, t)$ which is an elliptic function with respect to the variable x satisfies the KP equation if and only if it has the form

$$u(x, y, t) = -2 \sum_{i=1}^{N} \wp(x - q_i(y, t)) + c, \tag{2}$$

and its poles q_i as functions of y satisfy the equations of motion of the elliptic CM system.

It can be shown directly that if $u(x, y, t, \lambda)$ is an elliptic family of solutions of the KP equation, i.e. for fixed (x, y, t) the function u is an elliptic function of the variable $\lambda \in E_\tau$, then it has the form

$$u = -2 \sum_{i=1}^{N} \left[\lambda_{i\,x}^2 \wp(\lambda - \lambda_i) - \lambda_{i\,xx} \zeta(\lambda - \lambda_i)\right] + c(x, y, t), \quad \lambda_i = \lambda_i(x, y, t). \tag{3}$$

Elliptic families of the KP equation solutions

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The sum of the residues vanishes for an elliptic function u. Therefore, $\sum_i \lambda_{ixx} = 0$. We say that the poles λ_i, $i = 1, \ldots, N$ are *balanced* if they can be presented in the form

$$\lambda_i(x, y, t) = q_i(x, y, t) - hx, \qquad \sum_{i=1}^{N} q_i(x, y, t) = const, \qquad (4)$$

where h is an arbitrary non-zero constant.

As it was shown by Akmetshin, Volvovsky and Kr, the balance poles of u satisfy equations that are the equations of motion of a hamiltonian system with the phase space that is the space of functions $(q_1(x), \ldots, q_N(x), p_1(x), \ldots, p_N(x))$ with the Poisson brackets

$$\{q_i(x), p_j(\tilde{x})\} = \delta_{ij}\delta(x - \tilde{x})$$

and with the Hamiltonian $\widehat{H} = \int H(x)\, dx$,

$$H = \sum_{i=1}^{N} p_i^2(h - q_{i\,x}) - \frac{1}{Nh}\left(\sum_{i=1}^{N} p_i(h - q_{i\,x})\right)^2 - U(q), \qquad (5)$$

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where

$$U(q) = \sum_{i=1}^{N} \frac{q_{i\,xx}^2}{4(h - q_{i\,x})} - \frac{1}{2} \sum_{j \neq i} \left[(h - q_{j\,x}) q_{i\,xx} - (h - q_{i\,x}) q_{j\,xx} \right] \zeta(q_i - q_j)$$

$$+ \frac{1}{2} \sum_{j \neq i} \left[(h - q_{j\,x})^2 (h - q_{i\,x}) + (h - q_{j\,x})(h - q_{i\,x})^2 \right] \wp(q_i - q_j)$$

$$+ \frac{\partial}{\partial x} \left(\frac{h}{2} \sum_{i \neq j} (q_{i\,x} - q_{j\,x}) \zeta(q_i - q_j) \right).$$

Example.

For $N = 2$ this system is a hamiltonian system on the space of two functions $q(x)$, $p(x)$, where we set

$$q = q_1 = -q_2, \qquad \frac{1}{h}p(h^2 - q_x^2) = p_1(h - q_x) = -p_2(h - q_x),$$

The Poisson brackets are canonical, i.e. $\{q(x), p(\tilde{x})\} = \delta(x - \tilde{x})$. The Hamiltonian density H in the coordinates $\{p, q\}$ is equal to

$$H = \frac{2}{h}p^2(h^2 - q_x^2) - h\frac{q_{xx}^2}{2(h^2 - q_x^2)} - 2h(h^2 - 3q_x^2)\wp(2q).$$

It was noticed by A. Shabat that the equations of motion given by this Hamiltonian are equivalent to Landau – Lifshitz equation.

The spectral curves Γ giving elliptic families of the KP solutions can be characterized in two equivalent ways:

- they are curves whose Jacobian contains an elliptic curve, i.e. $E_\tau \subset J(G)$

  This is a nontrivial constraint and the space of corresponding algebraic curves has codimension $g-1$ in the moduli space of all the curves.

- the locus $\widehat{\mathcal{K}}_g$ of curves on which there exists a pair of meromorphic differentials Ψ_1, Ψ_2 with the only pole of order at most g at a puncture p_0, with integer periods $\oint_\gamma \Psi_i \in \mathbb{Z}$, and such that $dz = \tau\Psi_1 - \Psi_2$ is a holomorphic differential.

Note, that the holomorphic differential dz defines a map $\Gamma \to E_\tau$.

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Idea of the proof of the main theorem

Let X be a compact cycle in $\mathcal{M}_g^{ct}$ and let Y be the preimage Y of X under the forgetful map $\mathcal{M}_{g,1}^{ct,(g)} \to \mathcal{M}_g^{ct}$. If the dimension of X is at least g then:

$\to$ the intersection of Y with $\widehat{\mathcal{K}}_g$ is at least one-dimensional

$\to$ the compactness of X implies that the value of τ along this intersection is constant.

$\to$ the subvariety $\widehat{\mathcal{K}}_{g,N,\tau}$ is affine. Hence, $Y \cap \widehat{\mathcal{K}}_{g,N,\tau}$ intersects the boundary.

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HAPPY BIRTHDAY SASHA !