S.Novikov

Singular Solitons and Isospectral Deformations

June-July 2015

Collaborators: P.Grinevich

References: Novikov's Homepage
www.mi.ras.ru/~snovikov
click Publications, items 175,176,182, 184. New Results were published recently in the Journal of Brazilian Math Society, December 2013, special volume dedicated to the 60th Anniversary of IMPA and in the Journal Russian Math Surveys, September-October 2014

A lot of singular solutions to the famous KdV equation are known. Problem: Can one extend the idea of Isospectral Deformation to the Singular Solutions of KdV?

$$u_t = 6uu_x - u_{xxx}$$

Is it possible to construct Spectral Theory on the real line for the corresponding 1D Schrodinger Operators $L = -\partial_x^2 + u(x,t)$ such that time dependence of operator is an isospectral deformation?

Some people investigated "weakly singular" solutions to KdV. For example, Tao proved about 10 years ago that KdV dynamics is well-posed in the Sobolev spaces H^{-s} for $s \leq 3/10$ at the real x-line (circle). No ideas of Inverse Scattering (Spectral) Transform were used, no claims about isospectral properties has been made. Later Kappeler and Topalov using finite-gap approximation proved this for $s \leq 1$ including isospectral property. This is probably the final limit for the ordinary spectral theory.

However, the well-known fundamental class of exact solutions contains "singular multisolitons" and "singular finite-gap KdV solutions" (algebro-geometric solutions). We cannot find them in the Sobolev spaces above. They have stronger singularities. What can one say about corresponding Schrodinger Operators?

Our basic observation is following: All solutions (for all λ) $L\psi = \lambda\psi$ are x-meromorphic on the real line for all t. This property we take as a definition of s-meromorphic operators.

Our construction: Consider space of functions $f(x) \in F, x \in R$, which are C^∞ plus isolated poles x_j of finite order. We assume that negative parts belong to some fixed finite-dimensional spaces Q_j of polynomials from the variables $y^{-1} = (x - x_j)^{-1}$. We assume also that positive parts are such that all products fg for $f, g \in F$ don't have terms of the order

y^{-1} in all singular points $x_j \in R$.
Define indefinite inner product

$$< f, g >= \int_R f(x)\bar{g}(\bar{x})dx$$

integrating along R outside singularities and avoiding singular point by any small contours. This definition is correct but inner product is indefinite.

Space F defined by all solutions $Lf = \lambda f$ to the algebrogeometric (AG) or "singular finite-gap" operators belong to this class. The singularities of potentials are following

$$u = \frac{n_j(n_j+1)}{(x-x_j)^2} + \Sigma_k b_{jk}(x-x_j)^{2k} + o((x-x_j)^{2n_j}) \quad \text{for } k \geq 0$$

We consider general real potentials with discrete set of such singularities x_j, finite at every period for periodic case or finite at the whole line for the rapidly decreasing case.

The spectral theory should be developed in the space $f \in F$ generated by functions which are C^∞ plus isolated singularities at the real line. Assume for simplicity that every function is meromorphic in some small domain near singularity
$f(x) = \Sigma_{k \le n_j} q_k (x - x_j)^{-n_j + 2k} + o((x - x_j)^{n_j})$ for $k \ge 0$, nearby of every real singularity x_j of potential u for given moment t. We

call it $F = F_{x_1,\ldots,x_M;n_1,\ldots,n_M} = F_{X;N}$

The inner product in the space $F = F_{X;N}$ was defined above
$< f, g >= \int f(x)\bar{g}(\bar{x})dx$

It is well-defined here using complex contours avoiding singularities because all residues of the product are equal to zero.

The operator L is symmetric in respect to this inner product, which is indefinite.

We consider either functions rapidly decreasing at infinity ($T=\infty$) or quasiperiodic with Bloch-Floquet condition $f(x+T)=\varkappa f(x), \psi \in F_{X,N}(\varkappa)$ for $|\varkappa|=1$. The number of negative squares of inner product in the space $F_{X,N}(\varkappa)$ is equal to

$m_{X;N}=\Sigma_j[(n_j+1)/2]$; (It is the

Integral of KdV dynamics. Even more, the time deformation is isospectral).

Classical Theory: Spectral Theory of Rapidly Decreasing and Periodic Schrodinger Operators L requires NONSINGULARITY of Potential $u(x)$ as well as physical derivation of KdV in the Theory of Solitons.

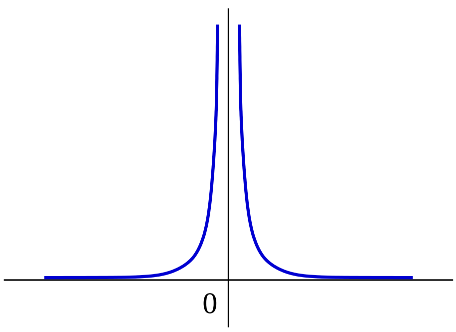
However, a number of other applications of KdV theory was discovered later which do not require nonsingularity. In particular, since late 1970s a number of works were written studying the motion of poles for the singular KdV Solutions. Especially Rational and Elliptic Solutions were popular. We consider here only real solutions.

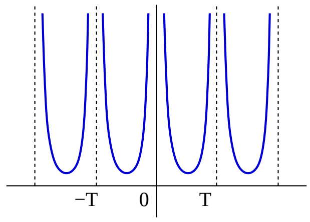
Example: For $j = 1, ..., \frac{n(n+1)}{2}$ there are Real Rational and Elliptic Solutions (here $x_j(t)$ may be nonreal)

$$u(x,t) = \sum_j 2/(x - x_j(t))^2$$

$$u(x,t) = \sum_j 2\wp(x - x_j(t))$$

let $u(x,0) = n(n+1)/x^2$



and $u(x,0) = n(n+1)\wp(x)$; 

(the famous Lame' Potentials.)

Hermit found Spectrum with Dirichlet boundary conditions for $x = 0, T$. Here T is a real period. No spectral theory on the real line was discussed in the classical literature prior to our works. For $n = 1$ this solution is a SINGULAR TRAVELING WAVE $u =$

$2\wp(x-at)$ with 2nd order pole in the point $x = at$ for real a. Don't Confuse it with NONSINGULAR TRAVELING WAVE $u = 2\wp(x + i\omega' - at)$ where $2i\omega'$ is an imaginary period. It was a first nontrivial example of smooth periodic finite-gap potentials as it was found first time in 1950s. In the classical theory of Lame potentials since Hermit only cases with poles at the real axis appeared.

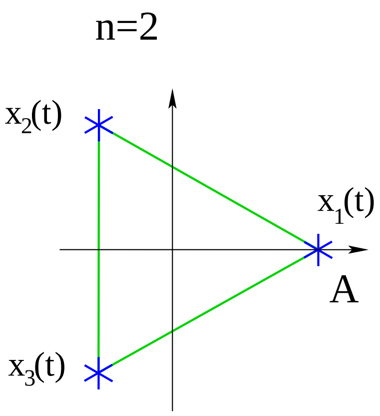
The evolution of Lame' Potentials

$$u(x,0) = n(n+1)\wp(x)$$

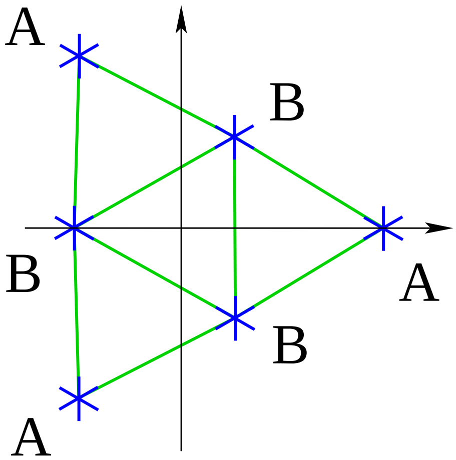
or $u(x,0) = n(n+1)/x^2$ leads to singular solutions

Important Question:

How many real poles these solutions have for $t > 0$?

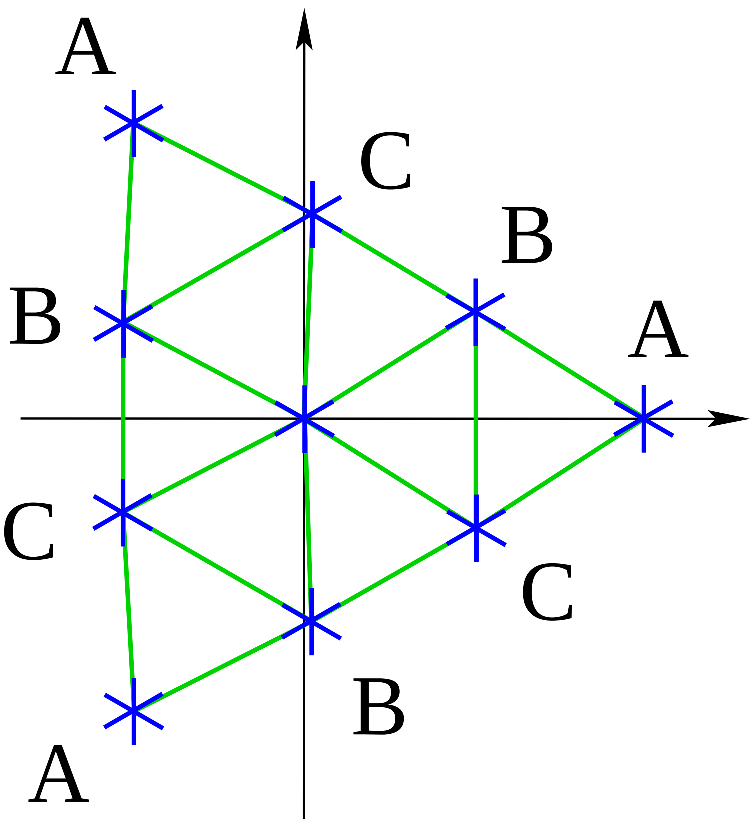


n=3



We have $1, 1, 2, 2, 3, \ldots$ real poles for $n = 1, 2, 3, 4, 5 \ldots .$

n=4 $\frac{n(n+1)}{2}$ =10



$x_j \sim r_j t^{1/3}$

The symmetry group $Z/3Z$ acts here

$r_j \to \zeta r_j, \zeta^3 = 1$

Our Result: The number of real

poles is equal to $[(n+1)/2]$. For $t > 0$ the number of real poles is obviously equal to the number of negative squares for the Inner Product because every "simple" singularity of the type $2/x^2$ gives exactly one negative square. For $t = 0$ this number is also equal to the number of negative squares of inner product because the multiple singularity of the type $n(n+1)/x^2$ gives exactly $[(n+1)/2]$ negative squares as one can see from the formulas above. We proved Completeness Theorem for the spectral decomposition in the space of functions $F = F_{X;N}$ for the algebrogeometric operators L. So we conclude that multiple singularity for $t = 0$ really splits into the number of simple ones equal to the number of negative squares of inner product. It is therefore an Integral of Time Dynamics. Many years ago

Arkad'ev, Polivanov and Pogrebkov constructed some sort of scattering coefficients for the potentials with "simple" singularities like $2/(x-x_k)^2$, but inner product and spectral theory were not discussed in this pioneering work. We found no other works somehow crossing these ideas. It looks like many authors who studied motion of poles avoided obvious question—how many of them remain in the real axis after bifurkation.

Second Part:

Important Examples. Indefinite Spectral Theory and R-Fourier Transform.

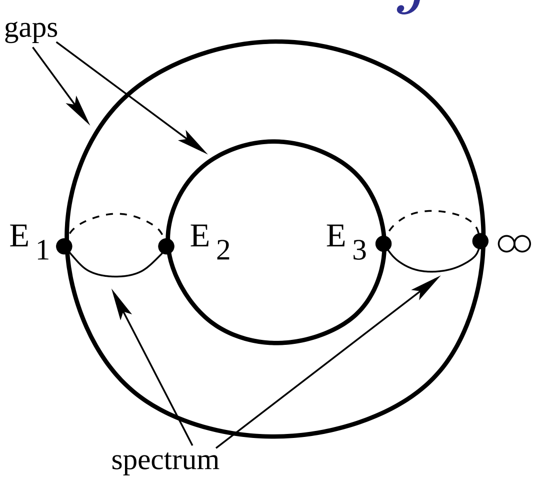
How Singular Solitons can be used? We used them to define right analog of Fourier Transform on Riemann Surfaces. There are many orthonormal bases in Mathematics and Applications (”Wavelets for example) but Fourier base has remarkable multiplicative properties.

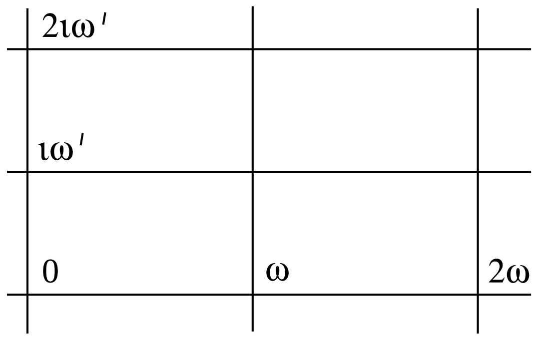
They are important for Nonlinear Problems. The notion of Resonances is based on multiplication.

It was critical to have bases with good MULTIPLICATIVE properties on Riemann Surfaces for the operator quantization of strings.

The spectrum of operator (see below) is equal to the projection on the complex λ-plane of the so-called Canonical Contour κ_0.

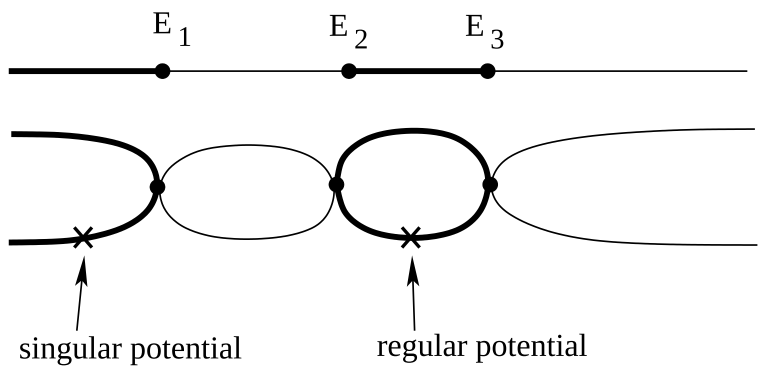
Example 1: Let $g = 1$ (Γ is a torus)
and all E_j are real, $j = 1, 2, 3$:





The lattice of periods of the Weierstrass $\wp$-function in this case is **rectangular with periods** $2\omega, 2i\omega'$.

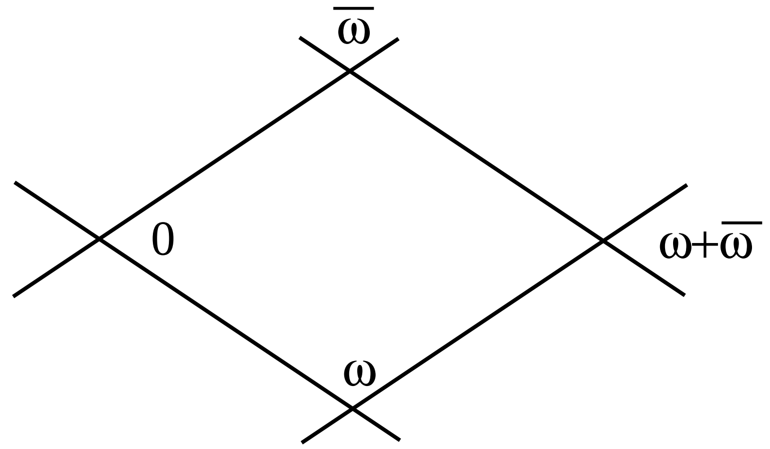
The spectrum is real , and spectral gaps are $[-\infty, E_1]$ and $[E_2, E_3]$, $\tau = id$ at κ_0



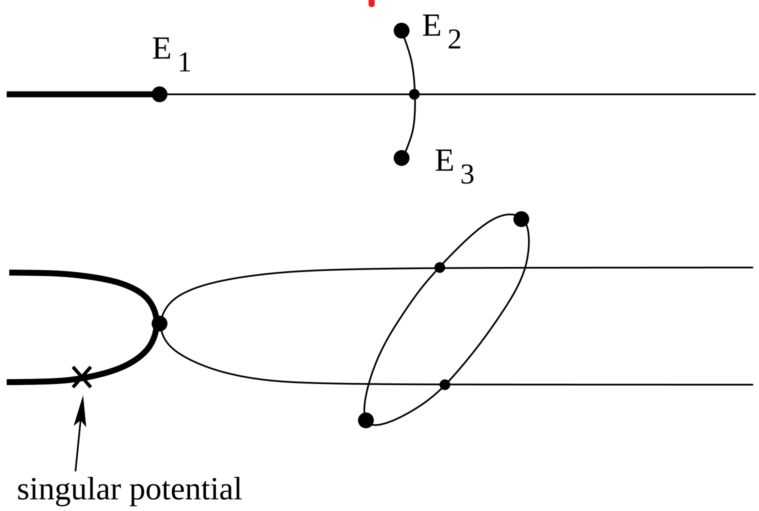
κ_0 is represented by fine lines.

The contour κ_0 has 2 components here: infinite and finite. There is only one pole γ: For Regular Case it belongs to the finite gap, for the Singular Case it belongs to the infinite gap
(They both are the shifted Hermit-Lame Operators but in regular case the shift is imaginary, in singular case the shift is real).

Example 2. Let $g = 1, E_1 \in R, E_3 = \overline{E_2}$:



The lattice of periods is



rombic. κ_0

given by fine lines.

The spectrum on the whole line coincides with the projection of the contour κ_0 on the $E-plane$. It contains complex arc joining $E_2, \bar{E}$ and $\tau \neq id$ at κ_0

Define the "spectral measure" $d\mu$. Let λ_j= projection of poles:

$$d\mu = \frac{(E-\lambda_1)\ldots(E-\lambda_g)dE}{2\sqrt{(E-E_1)\ldots(E-E_{2g+1})}}$$

For every smooth function on the contour κ_0 with decay fast at infinity, we define

Direct and Inverse Spectral Transform:

$$\tilde{\phi}(x) = \frac{1}{\sqrt{2\pi}} \int_{\kappa_0} \phi(\lambda)\Psi(\sigma\lambda, x) d\mu(\lambda(E)) \quad (1)$$

$$\phi(\lambda) = \frac{1}{\sqrt{2\pi}} \int_R \tilde{\phi}(x)\Psi(\lambda, x) dx \quad (2)$$

We call it R-Fourier Transform if all $\lambda_j = \infty$; $d\mu^F = dE/2\sqrt{(E-E_1)\ldots(E-E_{2g+1})}$, Our base has good multiplicative properties:

$$\Psi(x,\lambda)\Psi(y,\lambda) = l\Psi(x+y,\lambda)$$
$$l = (\partial_z^g + \zeta(z)\partial_z^{g-1} + \ldots)$$
$$\lambda = (E, \pm), z = x + y$$

In the Regular Case $\tau = id$ at κ_0 and measure is positive. This Spectral Transform is an Isometry between the Hilbert spaces with positive inner products

$$< \psi_1, \psi_2 >_{\kappa_0} =$$
$$\int_{\kappa_0} \psi_1(\lambda)\overline{\psi}_2(\tau\lambda)d\mu(\lambda)$$

$$< f_1, f_2 >_R = \int_R f_1(x)\overline{f}_2(x)dx$$

Consider Singular Potentials

1) Formula for the Spectral Transform remains valid; For the Inverse Transform it remains valid after a natural regularization.

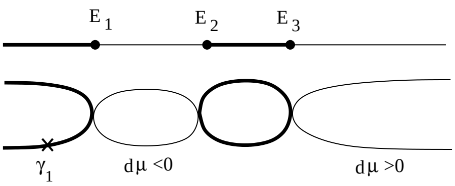
2) Spectral Transform is an isometry between the spaces with indefinite metric described above.

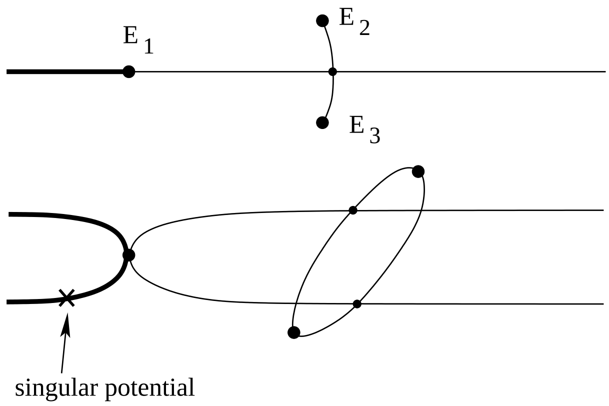
All singularities have a form described above

Example 1. All branching points are real: τ acts identically on κ_0, the form $d\mu$ is negative somewhere. For R-Fourier Transform we have: $d\mu^F/dp > 0$ exactly in every second component starting from the infinite one; So we have $[(g+1)/2]$ "negative" finite components in κ_0.

Example 2. Some pair of branching points is complex adjoint: τ is not identity in the nonreal components of κ_0; So the inner product is nonlocal and therefore indefinite.

We proved Completeness Theorem in the spaces $F_{X,N}(\varkappa)$ which are similar to the Pontryagin-Sobolev spaces





Every function on the line $f(x) \in \mathcal{L}_2(R)$ can be written as a direct integral of the Bloch-Floquet spaces such that $f(\varkappa, x+T) = \varkappa f(\varkappa, x)$. The space $F_{X,N}$ also is a direct integral of Bloch-Floquet spaces $f \in F_{X,N}(\varkappa), |\varkappa| = 1$: Our inner product has r negative squares in the space $F_{X,N}(\varkappa), r = [(g+1)/2]$ for the R-Fourier case.

Multidimensional Problem:

We already extended our results to the higher order OD Algebrogeometric Operators and to the case of KP. Is it possible to extend our construction of Inner Product to Singular Algebro-geometric 2D Schrodinger Operators? No nontrivial smooth self-adjoint Periodic Algebrogeometric Schrodinger Operators are known. There exist a theorem for 2D case that they do not exist. However, it is not so for the singular case. Bloch-Floquet eigenfunctions are known for the

$k+1$-particle Moser-Calogero operator with Weierstrass pairwise potential if coupling constant is equal to $n(n+1), n \in Z$. They form a k-dimensional complex algebraic variety. No one eigenfunction is known for $k > 1$ serving the discrete spectrum in the bounded domain inside of poles. Our case in this talk corresponds to $k = 1$. The Dirichlet Problem was solved by Hermit but no extension of his result to higher dimensions is known until now. Our Problem is different:We believe that for all $k > 1$ this family of eigenfunctions also serves spectral problem in some indefinite inner product in the proper space of functions defined in the whole space R^k.