Geometric optimization of the eigenvalues of Laplace operator and mathematical physics

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INTEGRABILITY IN ALGEBRA, GEOMETRY AND PHYSICS: NEW TRENDS
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Outline

Geometric optimization of eigenvalues of Δ
- Laplace operator in $\mathbb{R}^n$
- Laplace-Beltrami operator
- Geometric optimization of eigenvalues
- Known results about particular surfaces

Minimal submanifolds of a sphere and extremal spectral property of their metrics
- Two important theorems
- New method

Symmetry reduction
- Hsiang-Lawson reduction theorem
- Otsuki tori
- Generalized Lawson τ-surfaces

Spectral problem for the Laplace operator

- Laplace operator in $\mathbb{R}^n$

$$\Delta f = -\frac{\partial^2 f}{\partial x_1^2} - \cdots - \frac{\partial^2 f}{\partial x_n^2}$$

- Let Ω be a domain in $\mathbb{R}^n$
- Dirichlet spectral problem

$$\begin{cases} \Delta f = \lambda f \\ f|_{\partial\Omega} = 0 \end{cases}$$

- The spectrum consists only of eigenvalues

$$0 \leqslant \lambda_1(\Omega, D) \leqslant \lambda_2(\Omega, D) \leqslant \ldots$$

Example: $\dim = 1$ (string)

- $\Omega = [0, l] \subset \mathbb{R}$
- Dirichlet spectral problem

$$\begin{cases} -u'' = \lambda u \\ u(0) = u(l) = 0 \end{cases}$$

- Eigenvalues $\lambda_n = \left(\frac{\pi n}{l}\right)^2$, where $n = 1, 2, 3, \ldots$
- Eigenfunctions $u_n = \sin(\frac{\pi n}{l} x)$

Example: dim $= 2$, rectangular membrane

- Rectangular membrane $[0, a] \times [0, b] \subset \mathbb{R}^2$

$$-\frac{\partial^2}{\partial x^2} u(x, y) - \frac{\partial^2}{\partial y^2} u(x, y) = \lambda u(x, y)$$

- Separation of variables
- Eigenvalues

$$\lambda_{m,n} = \left(\frac{\pi m}{a}\right)^2 + \left(\frac{\pi n}{b}\right)^2, \quad m, n = 1, 2, 3, \ldots$$

- We should order them to obtain $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$
- $\lambda_1 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$

Geometric optimization of eigenvalues for domains in $\mathbb{R}^n$

- Eigenvalues are functionals on the "set of domains"

$$\Omega \longmapsto \lambda_i(\Omega, D)$$

- Naïve question: can we find

$$\min_{\Omega} \lambda_i(\Omega, D), \quad \max_{\Omega} \lambda_i(\Omega, D)?$$

Toy example: rectangular membranes

- Let us consider now only $\Omega = [0, a] \times [0, b]$ and

$$\lambda_1 = \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2$$

- Naïve question

$$\min_{a,b} \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2, \quad \max_{a,b} \left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2?$$

- Trivial answer: no min or max, but $\inf = 0$, $\sup = +\infty$

Rescaling

- What happens if $a \longmapsto ka$, $b \longmapsto kb$?
- Then

$$\lambda_1(ka, kb) = \left(\frac{\pi}{ka}\right)^2 + \left(\frac{\pi}{kb}\right)^2 =$$

$$= \frac{1}{k^2}\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2\right] = \frac{1}{k^2}\lambda_1(a, b)$$

- One should fix the area! Let Area $= 1 \iff b = \frac{1}{a}$, then

$$\lambda_1(a) = \left(\frac{\pi}{a}\right)^2 + (\pi a)^2$$

- If Area $= 1$ then

$$\min_a \lambda_1(a) = \lambda_1(1) = 2\pi^2, \quad \sup_a \lambda_1(a) = +\infty$$

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Toy example: rectangular membranes

- $\min\limits_{a} \lambda_1(a) = \lambda_1(1)$ means that *a drumhead of square shape produces the lowest possible sound among all rectangular drumheads of given area*

Right question

- Find
$$\inf_{\substack{\Omega\subset\mathbb{R}^n \\ \mathrm{Vol}(\Omega)=c}} \lambda_i(\Omega, D)$$
- In the $i = 1$ 2D case this means *"A drumhead of which shape produces the lowest possible sound among all drumheads of given area?"*
- J. W. Strutt, baron Rayleigh, *The Theory of Sound,* Vol. I, II, London : Macmillan, 1877, 1878.

Rayleigh-Faber-Krahn theorem

- If $i = 1$ then the minimum is reached on a ball of given volume, i.e.

$$\min_{\substack{\Omega \subset \mathbb{R}^n \\ \mathrm{Vol}(\Omega) = c}} \lambda_1(\Omega, D) = \lambda_1(B, D),$$

  where B is the ball of volume c in $\mathbb{R}^n$.
- This means that the optimal drumhead form is the disc.

Krahn-Szegö theorem

- If $i = 2$ then the minimum is reached on the union of two identical balls, i.e.

$$\min_{\substack{\Omega\subset\mathbb{R}^n \\ \mathrm{Vol}(\Omega)=c}} \lambda_2(\Omega, D) = \lambda_2(B \sqcup B, D),$$

where $B \sqcup B$ is the union of two identical balls in $\mathbb{R}^n$ such that $\mathrm{Vol}(B \sqcup B) = c$.

What about $i \geqslant 3$?

- We do not know the answer even in the case of planar domains

No	Optimal union of discs		Computed shapes	
3		46.125		46.125
4		64.293		64.293
5		82.462		78.47
6		92.250		88.96
7		110.42		107.47
8		127.88		119.9
9		138.37		133.52
10		154.62		143.45

Laplace-Beltrami operator on manifolds

- Laplace-Beltrami operator on a Riemannian manifold

$$\Delta f = -\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}g^{ij}\frac{\partial f}{\partial x^j}\right),$$

where g_{ij} is the metric tensor, g^{ij} are the component of the matrix inverse to g_{ij} and $g = \det g$.

Spectral problem for the Laplace-Beltrami operator

- Spectral problem for the Laplace-Beltrami operator on a Riemannian manifold M without boundary

$$\Delta f = \lambda f$$

- The spectrum consists only of eigenvalues

$$0 = \lambda_0(M, g) < \lambda_1(M, g) \leqslant \lambda_2(M, g) \leqslant \dots$$

Geometric optimization of eigenvalues

- Let us fix M. Then $\lambda_i(M, g)$ is a functional on the space of Riemannian metrics on M

$$g \longmapsto \lambda_i(M, g)$$

- Natural geometric optimization problem: find

$$\sup_g \lambda_i(M, g),$$

where g belongs to the the space of Riemannian metrics on M such that $\mathrm{Vol}(M) = 1$

- This is a good question only for surfaces

Upper bounds

- Yang and Yau (1980): for an orientable surface M of genus γ we have
$$\lambda_1(M,g) \leqslant 8\pi(\gamma+1).$$
- Korevaar (1993): there exists a constant C such that for any $i > 0$ and any compact surface M of genus γ the functional $\lambda_i(M,g)$ is bounded,
$$\lambda_i(M,g) \leqslant C(\gamma+1)i.$$

Eigenvalues as functions of a metric

- The functional $\lambda_i(M, g)$ depends continuously on the metric g, but this functional is not differentiable.
- However, it was shown by Berger, Bando & Urakawa, El Soufi & Ilias that for analytic deformations g_t the left and right derivatives of the functional $\lambda_i(M, g_t)$ with respect to t exist.

Eigenvalues as functions of a metric

- **Definition** (Nadirashvili, 1986, El Soufi and Ilias, 2000). A Riemannian metric g on a closed surface M is called *extremal metric* for the functional $\lambda_i(M, g)$ if for any analytic deformation g_t such that $g_0 = g$ the following inequality holds,

$$\frac{d}{dt}\lambda_i(M, g_t)\Big|_{t=0+} \cdot \frac{d}{dt}\lambda_i(M, g_t)\Big|_{t=0-} \leqslant 0.$$

What can we say about particular surfaces?

- $\lambda_1(\mathbb{S}^2, g)$. Hersch proved in 1970 that $\sup \lambda_1(\mathbb{S}^2, g) = 8\pi$ and the maximum is reached on the canonical metric on $\mathbb{S}^2$. This metric is the unique extremal metric.
- $\lambda_1(\mathbb{R}P^2, g)$. Li and Yau proved in 1982 that $\sup \lambda_1(\mathbb{R}P^2, g) = 12\pi$ and the maximum is reached on the canonical metric on $\mathbb{R}P^2$. This metric is the unique extremal metric.
- $\lambda_1(\mathbb{T}^2, g)$. Nadirashvili proved in 1996 that $\sup \lambda_1(\mathbb{T}^2, g) = \frac{8\pi^2}{\sqrt{3}}$ and the maximum is reached on the flat equilateral torus. El Soufi and Ilias proved in 2000 that the only extremal metric for $\lambda_1(\mathbb{T}^2, g)$ different from the maximal one is the metric on the Clifford torus.

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What can we say about particular surfaces?

- $\lambda_1(\mathbb{K}, g)$. Jakobson, Nadirashvili and I. Polterovich proved in 2006 that the metric on a Klein bottle realized as the Lawson bipolar surface $\tilde{\tau}_{3,1}$ is extremal. El Soufi, Giacomini and Jazar proved in the same year that this metric is the unique extremal metric and the maximal one. Here $\sup \lambda_1(\mathbb{K}, g) = 12\pi E\left(\frac{2\sqrt{2}}{3}\right)$, where E is a complete elliptic integral of the second kind,

$$E(k) = \int_0^1 \frac{\sqrt{1-k^2\alpha^2}}{\sqrt{1-\alpha^2}} d\alpha.$$

What can we say about particular surfaces?

- $\lambda_2(\mathbb{S}^2, g)$. Nadirashvili proved in 2002 that $\sup \lambda_2(\mathbb{S}^2, g) = 16\pi$ and maximum is reached on a singular metric which can be obtained as the metric on the union of two spheres of equal radius with canonical metric glued together. The proof contained some gaps filled later by Petrides (2012).
- $\lambda_3(\mathbb{S}^2, g)$. Nadirashvili and Sire proved in 2015 that $\sup \lambda_3(\mathbb{S}^2, g) = 24\pi$ and maximum is reached on a singular metric which can be obtained as the metric on the union of three spheres of equal radius with canonical metric glued together.

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What can we say about particular surfaces?

- $\lambda_i(\mathbb{T}^2, g), \lambda_i(\mathbb{K}, g)$. Several series of extremal metrics on tori and Klein bottles:
- Bipolar Lawson tau-surfaces $\tilde{\tau}_{r,k}$ (Lapointe, 2008),
- Lawson tau-surfaces $\tau_{r,k}$ (A.P., 2012),
- Otsuki tori $O_{\frac{p}{q}}$ (A.P., 2013),
- Bipolar Otsuki tori $\tilde{O}_{\frac{p}{q}}$ (Karpuhin, 2014)
- ...

A classical theorem

- Let N be a submanifold of $\mathbb{R}^n$. Let Δ be the Laplace-Beltrami operator on N equipped with the induced metric.
- **Theorem.** The restrictions $x^1|_N, \ldots, x^n|_N$ on N of the standard coordinate functions of $\mathbb{R}^{n+}$ are harmonic iff N is a minimal submanifold of $\mathbb{R}^n$.

Takahashi theorem

- Let N be a d-dimensional submanifold of $\mathbb{R}^{n+1}$. Let Δ be the Laplace-Beltrami operator on N equipped with the induced metric.
- **Theorem.** The functions $x^1|_N, \ldots, x^{n+1}|_N$ are eigenfunctions of Δ with eigenvalue $\frac{d}{R^2}$ iff N is a minimal submanifold of the sphere S^n_R of radius R.

Recent theorem by Nadirashvili, El Soufi & Ilias

- Let us introduce the Weyl eigenvalues counting function

$$N(\lambda) = \#\{\lambda_i | \lambda_i < \lambda\}.$$

- **Theorem.** The metric g_0 induced on N by minimal immersion $N \subset \mathbb{S}^n$ is an extremal metric for the functional $\lambda_{N\left(\frac{d}{R^2}\right)}(N, g)$.

How to find extremal metrics?

- Find a minimally immersed surface Σ in a unit sphere
- Find $N(2)$
- Then the induced metric on Σ is extremal for $\lambda_{N(2)}$

Hsiang-Lawson reduction theorem

- Let M be a Riemannian manifold with a metric g' and $I(M)$ its full isometry group. Let $G \subset I(M)$ be an isometry group. Let us denote by π the natural projection onto the space of orbits $\pi : M \longrightarrow M/G$.
- The union M^* of all orbits of principal type is an open dense submanifold of M. The subset M^*/G of M/G is a manifold carrying a natural Riemannian structure g induced from the metric g' on M.

Hsiang-Lawson reduction theorem

- Let us define a volume function $V : M/G \longrightarrow \mathbb{R}$: if $x \in M^*/G$ then $V(x) = Vol(\pi^{-1}(x))$
- Let $f : N \longrightarrow M$ be a G-invariant submanifold, i.e. G acts on N and f commutes with the actions of G on N and M.
- A cohomogeneity of a G-invariant submanifold $f : N \longrightarrow M$ in M is the integer $\dim N - \nu$, where $\dim N$ is the dimension of N and ν is the common dimension of the principal orbits.
- Let us define for each integer $k \geqslant 1$ a metric $g_k = V^{\frac{2}{k}} g$.

Hsiang-Lawson reduction theorem

- Theorem (Hsiang-Lawson). Let $f : N \longrightarrow M$ be a G-invariant submanifold of cohomogeneity k, and let M/G be given the metric g_k. Then $f : N \longrightarrow M$ is minimal is and only if $\bar{f} : N^*/G \longrightarrow M^*/G$ is minimal.
- Corollary. If $M = \mathbb{S}^n$, $G = \mathbb{S}^1$ and $\tilde{N} \subset M^*/G$ is a closed geodesic w.r.t. the metric g_1 then $\pi^{-1}(\tilde{N})$ is a minimal torus in $\mathbb{S}^n$.

Otsuki tori

- Let us consider $M = \mathbb{S}^3$ and $G = \mathbb{S}^1$ acting as

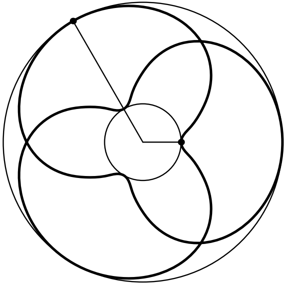
$$\alpha \cdot (x, y, z, t) = (\cos \alpha x + \sin \alpha y, -\sin \alpha x + \cos \alpha y, z, t).$$

- Minimal tori obtained in this case by the described construction are called Otsuki tori.
- Except one particular case (this is a Clifford torus), Otsuki tori are in one-to-one correspondence with rational numbers $\frac{p}{q}$ such that

$$\frac{1}{2} < \frac{p}{q} < \frac{\sqrt{2}}{2}, \quad p, q > 0, \quad (p, q) = 1.$$

Otsuki tori

- We denote these tori by $O_{\frac{p}{q}}$.
- Example: the geodesic corresponding to $O_{\frac{2}{3}}$.



Otsuki tori

A.P., Mathematische Nachrichten, 2013

- **Theorem.** The metric on an Otsuki torus $O_{\frac{p}{q}} \subset \mathbb{S}^3$ is extremal for the functional $\lambda_{2p-1}(\mathbb{T}^2, g)$.

Lawson τ-surfaces

- **Definition** A Lawson tau-surface $\tau_{m,k} \subset \mathbb{S}^3$ is defined by the doubly-periodic immersion $\Psi_{m,k} : \mathbb{R}^2 \longrightarrow \mathbb{S}^3 \subset \mathbb{R}^4$ given by the following explicit formula,

$$\Psi_{m,k}(x, y) =$$
$$= (\cos mx \cos y, \sin mx \cos y, \cos kx \sin y, \sin kx \sin y).$$

- In complex form

$$(e^{imx} \cos y, e^{ikx} \sin y) \subset \mathbb{C}^2 = \mathbb{R}^4.$$

Lawson τ-surfaces

- This family of surfaces is introduced in 1970 by Lawson. He proved that for each unordered pair of positive integers (m, k) with $(m, k) = 1$ the surface $\tau_{m,k}$ is a distinct compact minimal surface in $\mathbb{S}^3$. Let us impose the condition $(m, k) = 1$. If both integers m and k are odd then $\tau_{m,k}$ is a torus. We call it a Lawson torus. If one of integers m and k is even then $\tau_{m,k}$ is a Klein bottle. We call it a Lawson Klein bottle. The torus $\tau_{1,1}$ is the Clifford torus.

Lawson τ-surfaces

- The separation of variables in the equation

$$\Delta\psi = \lambda\psi$$

and the change of variables $\sin y = \operatorname{sn} z$ with modulus $\hat{k} = \frac{\sqrt{m^2-k^2}}{m}$ gives the classical Lamé equation

$$\frac{d^2\varphi}{dz^2} + (h - n(n+1)[\hat{k}\,\operatorname{sn}(z)]^2)\varphi = 0$$

with $n = 1$.

Lawson τ-surfaces

- Three wonderful classical solutions of the Lamé equation with $n = 1$ are given by the Jacobi elliptic functions

$$\mathrm{dn}\, z, \quad \mathrm{cn}\, z, \quad \mathrm{sn}\, z.$$

- The change of variables $\sin y = \mathrm{sn}\, z$ transforms them into

$$\sqrt{1 - \hat{k}^2 \sin y}, \quad \cos y, \quad \sin y.$$

Lawson τ-surfaces

- Using the theory of Magnus-Winkler-Ince equation and the Lamé equation one can prove the following theorem.

Lawson τ-surfaces

A.P., Moscow Math. J. 12 (2012), 173-192.

- 1. Let $\tau_{m,k}$ be a Lawson torus. We can assume that $m, k \equiv 1 \mod 2$, $(m,k) = 1$. Then the induced metric on $\tau_{m,k}$ is an extremal metric for the functional $\Lambda_j(\mathbb{T}^2, g)$, where where

$$j = 2\left[\frac{\sqrt{m^2+k^2}}{2}\right] + m + k - 1.$$

The corresponding value of the functional is

$$\Lambda_j(\tau_{m,k}) = 8\pi m E\left(\frac{\sqrt{m^2-k^2}}{m}\right).$$

Lawson τ-surfaces

A.P., Moscow Math. J. 12 (2012), 173-192.

- 2. Let $\tau_{m,k}$ be a Lawson Klein bottle. We can assume that $m \equiv 0 \mod 2$, $k \equiv 1 \mod 2$, $(m,k) = 1$. Then the induced metric on $\tau_{m,k}$ is an extremal metric for the functional $\Lambda_j(\mathbb{K}, g)$, where

$$j = 2\left[\frac{\sqrt{m^2+k^2}}{2}\right] + m + k - 1.$$

The corresponding value of the functional is

$$\Lambda_j(\tau_{m,k}) = 8\pi m E\left(\frac{\sqrt{m^2-k^2}}{m}\right).$$

Lawson τ-surfaces

- Lawson τ-surfaces are parametrized using $\sin y = \operatorname{sn} z$ and $\cos y = \operatorname{cn} z$. What about the missing $\operatorname{dn} z$?

Generalized Lawson τ-surfaces

A.P., to appear in J. Geom. Analysis

- $(x,y) \mapsto \left(\sqrt{\frac{b^2+c^2-a^2}{2(c^2-a^2)}}\sin y\, e^{iax}, \sqrt{\frac{a^2+c^2-b^2}{2(c^2-b^2)}}\cos y\, e^{ibx},\right.$
$\left.\sqrt{\frac{a^2+b^2-c^2}{2(b^2-c^2)}}\sqrt{1-\frac{b^2-a^2}{c^2-a^2}\sin^2 y}\, e^{icx}\right) \subset \mathbb{C}^3 \cong \mathbb{R}^6.$
- Let
  a) either a, b, c be integers and $|c| > \sqrt{a^2+b^2}$,
  b) or a, b be nonzero integers and $|c| = \sqrt{a^2+b^2}$.
- **Theorem.** The image $T_{a,b,c}$ of this map is a torus/Klein bottle minimal in $\mathbb{S}^5$. The metric on $T_{a,b,c}$ is extremal for $\lambda_{n(a,b,c)}$, where $n(a,b,c)$ is found explicitly.
- This family includes all metrics extremal for λ_1 on the torus and the Klein bottle: Clifford torus $T_{1,1,\sqrt{2}}$, equilateral torus $T_{1,1,2}$ and $\tilde{\tau}_{3,1} \cong T_{1,0,2}$