Kähler metrics on the moduli space of punctured Riemann surfaces

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Plan

1. Metrics on the moduli space $\mathscr{M}_{g,n}$
2. Potential of the TZ metric on $\mathscr{W}_{0,n}$
3. Potentials and Chern forms on $\mathfrak{S}_{g,n}$

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- If $X \simeq \Gamma\backslash\mathbb{H}$, where $\mathbb{H} = \{z = x + \sqrt{-1}y \in \mathbb{C} : y > 0\}$ is Lobachevsky plane and Γ is Fuchsian group of type (g,n), then $q(z)$ is a cusp form of weight 4 for Γ and $\mu(z) = y^2\overline{q(z)}$.

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- The WP metric has global potentials on the configuration space $\mathscr{W}_{0,n} \to \mathscr{M}_{0,n}$ and on the Schottky space $\mathfrak{S}_g \to \mathscr{M}_g$, given by the classical Liouvlle action (P. Zograf & L.T.).

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- Let Γ_i be the cyclic subgroup $\langle S_i \rangle$ and let $\sigma_i \in \mathrm{PSL}(2, \mathbb{R})$ be such that $\sigma_i \infty = z_i$ and $\sigma_i^{-1} S_i \sigma_i = \left(\begin{smallmatrix} 1 & \pm 1 \\ 0 & 1 \end{smallmatrix}\right)$.

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- Let $E_i(z, s)$ be the Eisenstein-Maass series associated with the cusp z_i, which for $\mathrm{Re}\, s > 1$ is defined by the following absolutely convergent series

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- The Eisenstein-Maass series are Γ-automorphic functions on $\mathbb{H}$ satisfying

$$\Delta E_i(z, s) = s(1 - s) E_i(z, s),$$

where

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

is the Laplace-Beltrami operator on the Lobachevsky plane.

Cusp (TZ) metric cont.

- The inner products

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- Curvature properties of the TZ metric?

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- **Proposition (J. Park, L.P. Teo & L.T., 2015)**

Put $h_i = |a_i(1)|^2$, $i = 1, \dots, n-1$, and $h_n = |a_n(-1)|^2$. Then $-\log h_i$ and $\log h_n$ are Kähler potentials for the metrics $\langle\,,\,\rangle_i$ and $\langle\,,\,\rangle_n$, and $\log h = \log h_n - \log h_1 - \cdots - \log h_{n-1}$ is a Kähler potential for the metric $\langle\,,\,\rangle_{\mathrm{TZ}}$.

Potential on $\mathscr{W}_{0,n}$, remarks

- Explicit description of potentials h_i in terms of the hyperbolic metric $e^{\varphi(w)}|dw|^2$ on $X = \mathbb{P}^1 \setminus \{w_1, \ldots, w_{n-3}, 0, 1, \infty\}$:

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- Let $S_L : \mathscr{W}_{0,n} \to \mathbb{R}$ be the classical Liouville action (A. Polyakov 1983, P. Zograf & L.T., 1985). Then the function $\mathscr{S} = S_L + \pi h$ on $\mathscr{W}_{0,n}$ satisfies

$$\partial \mathscr{S} = 2\mathscr{R}$$

$$\bar{\partial}\partial \mathscr{S} = -2\sqrt{-1}\left(\omega_{\mathrm{WP}} - \frac{4\pi^2}{3}\omega_{\mathrm{TZ}} \right),$$

where $d = \partial + \bar{\partial}$ — de Rham differential on $\mathscr{W}_{0,n}$.

The space $\mathfrak{S}_{g,n}$

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- Let Σ be marked, normalized Schottky group of rank $g > 1$ with domain of discontinuity $\Omega \subset \mathbb{C}$. The fiber over a point $[\Sigma] \in \mathfrak{S}_g$ is a configuration space of n points in $\Sigma \backslash \Omega$.

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- The Schottky uniformization of the compact surface X with marked points $x_1, \ldots, x_n$ and the Fuchsian uniformization of a punctured surface X_0 are related by the covering map

$$J : \mathbb{H}^* \to \Omega$$

such that the image of the cusps is the orbit $\Sigma \cdot \{w_1, \ldots, w_n\} \subset \Omega$.

- Let $z_1, \ldots, z_n$ be a complete set of cusps for Γ. Then

$$J(\sigma_i z) = w_i + \sum_{k=1}^{\infty} a_i(k) e^{2\pi\sqrt{-1}kz}, \quad i = 1, \ldots, n,$$

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- $h_i = |a_i(1)|^2$ determines Hermitian metric on the line bundles $\mathscr{L}_i$ over $\mathfrak{S}_{g,n}$, $i = 1, \dots, n$.

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- Let $\mathscr{C}_i$ be a tautological line bundle over $\mathscr{M}_{g,n}$ whose fiber at $(X; x_1, \ldots, x_n) \in \mathscr{M}_{g,n}$ is the cotangent line $T^*_{x_i}X$, $i = 1, \ldots, n$.
- Let $\mathscr{L}_i = p^*(\mathscr{C}_i)$ under the projection $p\colon \mathfrak{S}_{g,n} \to \mathscr{M}_{g,n}$.
- $h_i = |a_i(1)|^2$ determines Hermitian metric on the line bundles $\mathscr{L}_i$ over $\mathfrak{S}_{g,n}$, $i = 1, \ldots, n$.
- $H = \exp\left\{\dfrac{S_L}{\pi}\right\}$ determines Hermitian metric on the line bundle $\mathscr{L} = \mathscr{L}_1 \otimes \cdots \otimes \mathscr{L}_n$ over $\mathfrak{S}_{g,n}$.

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  - *Up to the factor* $1/12$ *this combination of metrics appears in the local index theorem for families on punctured Riemann surfaces for* $k = 0, 1$ *(P. Zograf & L.T., 1991).*