Complex exceptional orthogonal polynomials

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Bochner's theorem (1929)

Suppose $p_n(x) \in \mathbb{R}[x]$, $n \in \mathbb{Z}_{\geq 0}$, with $\deg p_n = n$ satisfy

$$Tp_n \equiv A_2(x)\frac{d^2 p_n}{dx^2} + A_1(x)\frac{dp_n}{dx} + A_0(x)p_n = E_n p_n(x).$$

- Then $A_j(x) \in \mathbb{R}[x]$ with $\deg A_j \leq j$.
- If, in addition,

$$(p_m, p_n) \equiv \int_a^b p_m(x)p_n(x)w(x)dx = \delta_{mn} g_n,$$

with $w(x) > 0$, then (up to $x \to ax + b$) the $p_n(x)$ are Hermite, Laguerre or Jacobi polynomials.

Exceptional orthogonal polynomials

Let $S \subset \mathbb{Z}_{\geq 0}$ be such that $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

Gómez-Ullate, Kamran and Milson (2010) called $p_n(x) \in \mathbb{R}[x]$, $n \in S$, a system of exceptional orthogonal polynomials if the following conditions are satisfied.

- Eigenvalue equation:

$$Tp_n \equiv A_2(x)\frac{d^2 p_n}{dx^2} + A_1(x)\frac{dp_n}{dx} + A_0(x)p_n = E_n p_n(x).$$

- Orthogonality:

$$(p_m, p_n) \equiv \int_a^b p_m(x)p_n(x)w(x)dx = \delta_{mn} g_n.$$

- Density: $U \equiv \langle p_n : n \in S \rangle$ is dense in $\mathbb{R}[x]$, i.e.

$$(p, p_n) = 0 \; \forall n \in S \;\Rightarrow\; p \equiv 0.$$

Exceptional Hermite polynomials

- Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a double partition:

$$\lambda = \mu^2 = (\mu_1, \mu_1, \mu_2, \mu_2, \ldots, \mu_m, \mu_m),$$

and

$$k_j = \lambda_j + n - j, \quad j = 1, \ldots, n.$$

- Gómez-Ullate, Kamran and Milson (2014) showed that

$$H_{\lambda,l}(x) \equiv \mathrm{Wr}(H_l, H_{k_1}, \ldots, H_{k_n}), \quad l \in \mathbb{Z}_{\geq 0} \setminus \{k_1, \ldots, k_n\},$$

are exceptional orthogonal polynomials.
- According to Crum (1954) and Adler (1994), the relevant weight function

$$w(x) = \frac{e^{-x^2/2}}{W_\lambda^2(x)}, \quad W_\lambda(x) = \mathrm{Wr}(H_{k_1}(x), \ldots, H_{k_n}(x)),$$

is non-singular for $x \in \mathbb{R}$ iff $\lambda = \mu^2$.

Exceptional Hermite polynomials

Our aim: Obtain a natural interpretation of the polynomials

$$H_{\lambda,l}(x) \equiv \mathrm{Wr}(H_l, H_{k_1}, \ldots, H_{k_n})$$

for all partitions λ.

The (complex) harmonic oscillator

- Consider the eigenvalue problem given by the ODE

$$\mathcal{H}\psi \equiv -\frac{d^2\psi}{dz^2} + z^2\psi = E\psi, \quad z \in \mathbb{C},$$

  and boundary conditions

$$\lim_{\mathrm{Re}\, z \to \pm\infty} \psi(z) = 0.$$

- Eigenvalues:

$$E_l = 2l + 1, \quad l \in \mathbb{Z}_{\geq 0} \equiv \{0, 1, 2, \ldots\}.$$

- Eigenfunctions:

$$\psi_l(z) = H_l(z) e^{-z^2/2}.$$

The classical Hermite polynomials

- Rodrigues formula:

$$H_l(z) = (-1)^n e^{z^2} \frac{d^l}{dz^l} e^{-z^2} = e^{z^2/2} \left(-\frac{d}{dz} + z \right)^l e^{-z^2/2}$$

- The first few polynomials are

$$\begin{aligned} H_0(z) &= 1, & H_1(z) &= 2z, \\ H_2(z) &= 4z^2 - 2, & H_3(z) &= 8z^3 - 12z, \\ H_4(z) &= 16z^4 - 48z^2 + 12, & H_5(z) &= 32z^5 - 160z^3 + 120z. \end{aligned}$$

- Orthogonality:

$$\int_{\mathbb{R}} H_j(z) H_l(z) e^{-z^2} dz = \delta_{jl} 2^l l! \sqrt{\pi}.$$

Darboux transformations

- Letting

$$D_k = \frac{d}{dz} - \frac{\psi_k'}{\psi_k},$$

  we have

$$\begin{aligned} D_k^* D_k &= \left(-\frac{d}{dz} - \frac{\psi_k'}{\psi_k}\right)\left(\frac{d}{dz} - \frac{\psi_k'}{\psi_k}\right) \\ &= -\frac{d^2}{dz^2} + \frac{\psi_k''}{\psi_k} \\ &= -\frac{d^2}{dz^2} + z^2 - E_k. \end{aligned}$$

- In other words

$$\mathcal{H} = D_k^* D_k + E_k.$$

Darboux transformations

- Reversing order,

$$D_k D_k^* = \left(\frac{d}{dz} - \frac{\psi_k'}{\psi_k}\right)\left(-\frac{d}{dz} - \frac{\psi_k'}{\psi_k}\right)$$
$$= -\frac{d^2}{dz^2} + \frac{\psi_k''}{\psi_k} - 2\left(\frac{\psi_k'}{\psi_k}\right)',$$

  gives

$$\mathcal{H}_k \equiv D_k D_k^* + E_k = -\frac{d^2}{dz^2} + z^2 - 2\left(\frac{\psi_k'}{\psi_k}\right)'.$$

- Introducing

$$\psi_{k,l} = D_k \psi_l = \frac{\mathrm{Wr}(\psi_l, \psi_k)}{\psi_k}, \quad l \neq k,$$

  we get

$$\mathcal{H}_k \psi_{k,l} = D_k (D_k^* D_k + E_k)\psi_l = D_k \mathcal{H} \psi_l = E_l \psi_{k,l}.$$

Example: $k = 1$

- Schrödinger operator:

$$\mathcal{H}_1 = -\frac{d^2}{dz^2} + z^2 + \frac{2}{z^2} + 2.$$

- The first few eigenfunctions are

$$\begin{aligned}
&\psi_{1,0}(z) = \tfrac{1}{z} e^{-z^2/2}, \\
&\psi_{1,2}(z) = -\tfrac{2+4z^2}{z} e^{-z^2/2}, \qquad && \psi_{1,3}(z) = -16z^2 e^{-z^2/2}, \\
&\psi_{1,4}(z) = \frac{12(1+4z^2-4z^4)}{z} e^{-z^2/2}, && \psi_{1,5}(z) = 64z^2(5-2z^2)e^{-z^2/2}.
\end{aligned}$$

- Note the pole at $x = 0$ for even l.

n-fold Darboux transformations

- Darboux transformations at levels $k_n < k_{n-1} < \cdots < k_1$ yields

$$\mathcal{H}_\lambda = -\frac{d^2}{dz^2} + z^2 - 2\frac{d^2}{dz^2}\left(\log \mathrm{Wr}(\psi_{k_1}, \ldots, \psi_{k_n})\right),$$

  where

$$\lambda = (\lambda_1, \ldots, \lambda_n), \quad \lambda_j = k_j - (n - j).$$

- $\mathcal{H}_\lambda$ related to $\mathcal{H}$ by

$$\mathcal{D}_\lambda \circ \mathcal{H} = \mathcal{H}_\lambda \circ \mathcal{D}_\lambda,$$

  with

$$\mathcal{D}_\lambda \psi = \frac{\mathrm{Wr}(\psi, \psi_{k_1}, \ldots, \psi_{k_n})}{\mathrm{Wr}(\psi_{k_1}, \ldots, \psi_{k_n})}.$$

n-fold Darboux transformations

- Hence, the functions

$$\psi_{\lambda,l} = \mathcal{D}_\lambda \psi_l = \frac{\mathrm{Wr}(\psi_l, \psi_{k_1}, \dots, \psi_{k_n})}{\mathrm{Wr}(\psi_{k_1}, \dots, \psi_{k_n})}, \quad l \neq k_j,$$

satisfy

$$\mathcal{H}_\lambda \psi_{\lambda,l} = \mathcal{H}_\lambda \mathcal{D}_\lambda \psi_l = \mathcal{D}_\lambda \mathcal{H} \psi_l = (2l+1)\psi_{\lambda,l}$$

- Using $\mathrm{Wr}(gf_1, \dots, gf_n) = g^n \mathrm{Wr}(f_1, \dots, f_n)$, we get

$$\psi_{\lambda,l} = H_{\lambda,l} \frac{e^{-x^2/2}}{W_\lambda},$$

with

$$H_{\lambda,l} = \mathrm{Wr}(H_l, H_{k_1}, \dots, H_{k_n}), \quad W_\lambda = \mathrm{Wr}(H_{k_1}, \dots, H_{k_n}).$$

Exceptional Hermite polynomials

Theorem (Crum (1954) and Adler (1994))

The Wronskian $W_\lambda(x)$ has no zeros on the real line if and only if

$$\lambda = \mu^2 = (\mu_1, \mu_1, \ldots, \mu_m, \mu_m).$$

- Let
$$H_l^{(\mu)} = H_{\mu^2, l}, \quad l \in \mathbb{N} \setminus \{k_1 + 1, k_1, \ldots, k_n + 1, k_n\}$$
(where $k_j = \mu_j + 2(n - j)$).
- Then
$$\begin{aligned} \deg H_l^{(\mu)} &= \deg \mathrm{Wr}(H_l, H_{k_1+1}, H_{k_1}, \ldots, H_{k_n+1}, H_{k_n}) \\ &= 2|\mu| - 2n + l. \end{aligned}$$
- Missing $2|\mu|$ degrees:
$$0, 1, \ldots, 2|\mu| - 2n - 1,$$
$$2|\mu| - 2n + k_j,\ 2|\mu| - 2n + k_j + 1.$$

Exceptional Hermite polynomials

Theorem (Goméz-Ullate, Grandati & Milson (2014))

The polynomials $H_l^{(\mu)}$ satisfy

$$\int_{\mathbb{R}} H_j^{(\mu)}(x) H_l^{(\mu)}(x) \frac{e^{-x^2/2}}{W_{\mu^2}(x)^2} dx = \delta_{jl} \sqrt{\pi} 2^l l! \prod_{i=1}^{n} (l - k_i)(l - k_i - 1),$$

and

$$\left\langle H_l^{(\mu)} : l \neq k_j, k_j + 1 \right\rangle \text{ is dense in } L^2\left(\mathbb{R}, e^{-x^2/2} / W_{\lambda^2}(x)^2\right).$$

- Hence, they are called exceptional Hermite polynomials. (Finite number of degrees missing, but constitute an orthogonal and complete basis in the corresponding Hilbert space.)
- Our aim: Obtain a natural interpretation of the polynomials $H_{\lambda,l}$ for all partitions λ.

Trivial monodromy

- $\mathcal{H} = -d^2/dz^2 + u(z)$ is said to have trivial monodromy if all solutions of
$$\mathcal{H}\psi = E\psi$$
are meromorphic for all E.
- Local conditions on $u(z)$ at poles $z = z_i$ (Duistermaat & Grünbaum 1986):
$$u(z) = \sum_{j \geq -2} c_j (z - z_i)^j$$
with
$$c_{-2} = m_i(m_i + 1), \quad m_i \in \mathbb{N},$$
$$c_1 = c_3 = \cdots = c_{2m_i - 1} = 0.$$
- In such a case, each $\psi(z)$ is quasi-invariant at $z = z_i$:
  1. $\psi(z)(z - z_i)^{m_i}$ analytic at $z = z_i$,
  2. $(\psi(z)(z - z_i)^{m_i})^{(2j-1)}|_{z=z_i} = 0$, for all $j = 1, \ldots, m_i$.

Quasi-invariants

Since $\mathcal{H}_\lambda$ has trivial monodromy, each

$$H_{\lambda,l} = \mathrm{Wr}(H_l, H_{k_1}, \ldots, H_{k_n}), \quad l \neq k_j, \;\; j = 1, \ldots, n,$$

is contained in

$$\mathcal{Q}_\lambda = \left\{ p \in \mathbb{C}[z] : \psi(z) = p(z) \frac{e^{-z^2/2}}{W_\lambda(z)} \text{ q.} - \text{inv. at } z = z_i, \forall \text{ poles } z_i \right\}.$$

Proposition

We have

$$\mathbb{C}\langle H_{\lambda,l} : l \neq k_j \rangle = \mathcal{Q}_\lambda.$$

Proof.

The subspaces have the same codimension in $\mathbb{C}[z]$, since

$$\# \text{ degrees missing } = \# \text{ q.} - \text{inv. conditions.}$$

□

A Hermitian product

Let

$$\mathcal{Q}_{\lambda,\mathbb{R}} = \left\{ p \in \mathbb{C}[z] : \psi(z) = p(z)\frac{e^{-z^2/2}}{W_\lambda(z)} \text{ q.} - \text{inv. at } z = z_i, \forall \text{ real poles } z_i \right\}.$$

Definition

Let $\xi \in \mathbb{R}$ be s.t.

$$0 < |\xi| < \min_{z_i \notin \mathbb{R}} |\mathrm{Im}\, z_i|.$$

Then, we set

$$\langle p, q \rangle = \int_{i\xi + \mathbb{R}} p(z)\bar{q}(z)\frac{e^{-z^2}}{W_\lambda(z)^2} dz, \quad p, q \in \mathcal{Q}_{\lambda,\mathbb{R}},$$

where

$$\bar{q}(z) = \overline{q(\bar{z})}.$$

A Hermitian product

Lemma

$\langle \cdot, \cdot \rangle$ *does not depend on the value of* ξ.

Proof.

By the residue thm and quasi-invariance, we have

$$\begin{aligned}
\langle p, q \rangle_{\xi} &= \langle p, q \rangle_{-\xi} + \sum_{z_i \in \mathbb{R}} \left((z - z_i)^{2m_i} p(z) \bar{q}(z) \frac{e^{-z^2}}{W_\lambda(z)^2} \right)^{(2m_i - 1)} \Big|_{z = z_i} \\
&= \langle p, q \rangle_{-\xi} + \sum_{z_i \in \mathbb{R}} \sum_{j=0}^{2m_i - 1} \binom{2m_i - 1}{j} \left((z - z_i)^{m_i} p(z) \frac{e^{-z^2/2}}{W_\lambda(z)} \right)^{(j)} \Big|_{z = z_i} \\
&\qquad \times \left((z - z_i)^{m_i} \bar{q}(z) \frac{e^{-z^2/2}}{W_\lambda(z)} \right)^{(2m_i - 1 - j)} \Big|_{z = z_i} \\
&= \langle p, q \rangle_{-\xi}.
\end{aligned}$$

□

A Hermitian product

Proposition

$\langle \cdot, \cdot \rangle$ *is Hermitian.*

Proof.

Introducing

$$w(z) = \frac{e^{-z^2}}{W_\lambda(z)^2}$$

and observing $\bar{w}(z) = w(z)$, we deduce

$$\begin{aligned}\langle p, q \rangle_\xi &= \overline{\int_{\mathbb{R}} p(i\xi + x)\bar{q}(i\xi + x)w(i\xi + x)dx} \\ &= \overline{\int_{\mathbb{R}} \bar{p}(-i\xi + x)q(-i\xi + x)w(-i\xi + x)dx} \\ &= \overline{\langle q, p \rangle_{-\xi}}.\end{aligned}$$

Hence, the assertion follows from the lemma. □

Orthogonality

Theorem

We have

$$\langle H_{\lambda,j}, H_{\lambda,l}\rangle = \delta_{jl} 2^l l! \sqrt{\pi} \prod_{m=1}^{n} 2(l - k_m).$$

Proof.

By induction on $n = \ell(\lambda)$. Letting $\hat{\lambda} = (\lambda_2, \ldots, \lambda_n)$, we deduce

$$\begin{aligned}\langle H_{\lambda,j}, H_{\lambda,l}\rangle &= \int_{i\xi+\mathbb{R}} (D_1 \psi_{\hat{\lambda},j})(z)(\overline{D_1 \psi_{\hat{\lambda},j}})(z)dz \\ &= \int_{i\xi+\mathbb{R}} \psi_{\hat{\lambda},j}(z)(\overline{D_1^* D_1 \psi_{\hat{\lambda},j}})(z)dz.\end{aligned}$$

Using $D_1^* D_1 = \mathcal{H}_{\hat{\lambda}} - 2k_1 - 1$, we obtain

$$\langle H_{\lambda,j}, H_{\lambda,l}\rangle = 2(l - k_1)\langle H_{\hat{\lambda},j}, H_{\hat{\lambda},l}\rangle.$$

□

Density

Theorem

The subspace $\mathcal{Q}_\lambda$ is dense in $\mathcal{Q}_{\lambda,\mathbb{R}}$ in the sense that

$$\langle p, q\rangle = 0,\ \forall q \in \mathcal{Q}_\lambda \implies p \equiv 0.$$

Proof.

Observe that

$$q_{\lambda,l} := W_\lambda^2 H_l \in \mathcal{Q}_\lambda, \quad \forall l \in \mathbb{Z}_{\geq 0}.$$

Hence

$$0 = \langle p, q_{\lambda,l}\rangle = \int_{i\xi+\mathbb{R}} p(z)\bar{H}_l(z)e^{-z^2}dz.$$

Taking $\xi \to 0$ and expanding p in the H_j, it follows from

$$\int_{\mathbb{R}} H_j(z)H_l(z)e^{-z^2}dz = \delta_{jl}2^l l!\sqrt{\pi}$$

that $p \equiv 0$. □

Free particle on a cylinder

- Consider instead

$$\mathcal{H}\psi \equiv -\frac{d^2\psi}{dx^2} = E\psi, \quad x \in \mathbb{C}/2\pi\mathbb{Z}.$$

- Eigenvalues:

$$E_l = l^2, \quad l \in \mathbb{Z}.$$

- Eigenfunctions:

$$e_l(x) = \exp(ilx).$$

- Note that the eigenvalues have multiplicity 2.

n-fold Darboux transformations

- Darboux transformations at levels $k_n < k_{n-1} < \cdots < k_1$ parameterised by

$$\theta = (\theta_1, \dots, \theta_n), \quad \theta_k \in \mathbb{C}/2\pi\mathbb{Z}.$$

- Letting

$$\phi_{k_j}(\theta_j; x) = 2\cos(k_j x + \theta_j),$$

  the resulting Schrödinger operators can be written

$$\mathcal{H}_\lambda = -\frac{d^2}{dx^2} - 2\frac{d^2}{dx^2}\left(\log \mathrm{Wr}(\phi_{k_1}, \dots, \phi_{k_n})\right).$$

- Eigenfunctions are of the form

$$\phi_{\lambda,l}(x) = \frac{P_{\lambda,l}(\exp(ix))}{\mathcal{W}_\lambda(\exp(ix))},$$

  with $P_{\lambda,l}(z)$, $\mathcal{W}_\lambda(z)$ Laurent polynomials.

Main results

- We obtain a Laurent orthogonality relation of the form

$$(P_{\lambda,j}, P_{\lambda,l}) := \frac{1}{2\pi i} \int_{|z|=\mu} P_{\lambda,j}(z) P_{\lambda,l}(z) \mathcal{W}_\lambda(z)^{-2} \frac{dz}{z}$$
$$= \delta_{j+l,0} \prod_{m=1}^{n} (l^2 - k_m).$$

- The $P_{\lambda,l}$ do not span the space of quasi-invariants and $(\cdot,\cdot)$ has a non-trivial kernel. Need to consider the minimal complex Euclidean extension.
- In the case $|a_k| = 1$, we obtain a natural density result.

Reference

A detailed account of our results with complete proofs is available in the preprint

W. A. Haese-Hill, M. H. & A. P. Veselov (2015). Complex exceptional orthogonal polynomials and quasi-invariance, arXiv:1509.07008.