Saito metric and determinant on Coxeter discriminant strata

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Overview

Coxeter groups/Root systems

Let $V = \mathbb{R}^n$, $u, \alpha \in V$ and $(\,,\,)$ the standard bilinear form in V.

Definition

A reflection is a linear operator s_α on V defined by

$$u \mapsto s_\alpha u = u - 2\frac{(u,\alpha)}{(\alpha,\alpha)}\alpha.$$

It fixes a subspace of V of codimension 1, called a mirror (reflecting hyperplane). A **finite** group generated by reflections will be called *finite reflection group* and will be denoted by $W \subset O(V)$.

Definition

Let R be a set of non-zero vectors in V s.t

1. $R \cap \mathbb{R}\alpha = \{-\alpha, \alpha\}$,
2. $s_\alpha R = R$,

$\forall \alpha \in R$. The set R is called a root system with associated reflection group $W = \langle s_\alpha | \alpha \in R \rangle$.

Note that W is necessarily finite in this case.

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Definition

Let $\Delta \subset R$. We call Δ a simple system if

- it is a basis for the $\mathbb{R}$-span of R in V, and
- each $\alpha \in R$ is a linear combination of elements of Δ with coefficients **all** of the same sign.

Theorem (Coxeter '34)

Let $\Delta \subset R_+$ be a simple system. Then W is generated by the set $S = \{s_\alpha | \alpha \in \Delta\}$ subject to the relations

$$(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1, \quad \alpha, \beta \in \Delta,$$

where $m(\alpha, \alpha) = 1$, $\forall \alpha$ and $m(\alpha, \beta) \in \mathbb{Z}_{\geq 0}$.

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Any (finite) group W having such a presenation relative to a generating set S is called a (finite) Coxeter group. The pair (W, S) is called a Coxeter system.

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Remark

W is determined up to an isomorphism by $m(\alpha, \beta)$. One can encode this information in a graph with vertex set in one-to-one correspondence with Δ. It gives rise to the notion of a **Coxeter graph**.

The symmetric group

$W = S_n$, the symmetric group:

- Let ϵ_i, $i = 1, \ldots, n$ be the standard orthonormal basis in V,
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$$m(\alpha, \beta) = \begin{cases} 2, & \text{disjoint vertices}, \\ 3, & \textit{otherwise}. \end{cases}$$

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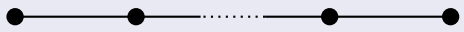
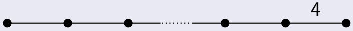
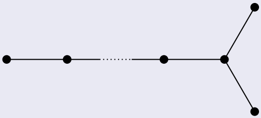
It fixes pointwise the line $L = \{\mathbb{R}\beta\}$, $\beta = \epsilon_1 + \cdots + \epsilon_n$. Hence, we usually denote W by A_{n-1}.

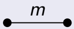
Classification-Coxeter '35

Coxeter diagrams of (irred.) finite Coxeter groups/ Classical series

A_n, $(n \geq 1)$: 

B_n, $(n \geq 2)$: 

D_n, $(n \geq 4)$: 

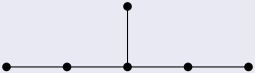
$I_2(m)$, $(m \geq 5)$: 

Code for graphs- B. McKay

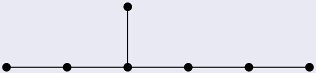
Classification-Coxeter '35

Coxeter diagrams of (irred.) finite Coxeter groups / Exceptional series

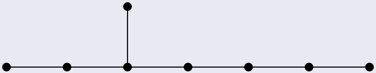
E_6:



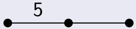
E_7:



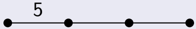
E_8:



H_3:



H_4:



F_4:

Orbit spaces

- Let $V = \mathbb{C}^n$, g the W-invariant standard constant metric given by
$$g(e_i, e_j) = \delta_{ij},$$
where $e_i, i = 1, \ldots, n$ is the standard basis in V and let $\{x^i\}_{i=1}^n$ be the corresponding orthogonal coordinates.

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- Let $y^1(x), \dots, y^n(x)$ be a hom. basis in the ring of invariant polynomials $S(V^*)^W = \mathbb{C}[x^1, \dots, x^n]^W = \mathbb{C}[x]^W = \mathbb{C}[y^1, \dots, y^n]$.

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- Let $d_i = \deg y^i$, $i = 1, \ldots, n$ and fix the ordering
$$h = d_1 > \cdots \geq d_n = 2.$$

We call h the **Coxeter number** of W.

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- Let $g^{\alpha\beta}$ be the corresponding contravariant metric. g is defined on $M_W \setminus \Sigma$, $\det(g^{\alpha\beta}(y)) = 0$ on Σ.

Definition (K.Saito et. al '80, Dubrovin '94)

The metric $\eta^{\alpha\beta} = \mathcal{L}_e g^{\alpha\beta}$ is called the *Saito* metric, it is defined up to proportionality and it is flat, where $e = \partial_{y^1}$.

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There exists a distinguished basis $t^i \in \mathbb{C}[x]^W$, $(1 \leq i \leq n)$ s.t η is constant and antidiagonal,

$$\eta^{\alpha\beta} = \delta^{n+1,\alpha+\beta}.$$

Such coordinates are called *Saito* polynomials.

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Proposition

The determinant of the covariant Saito metric η in the x coordinates is given as

$$\det \eta(x) = c \prod_{\alpha \in R_+} g(\alpha, x)^2, \quad c \in \mathbb{C}^\times.$$

Frobenius manifolds

Definition

An algebra $(\mathcal{A}, \circ, <,>)$ over $\mathbb{C}$ is called Frobenius if

- it is commutative, associative, with unity e,
- $<,>: \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ is a non-degenerate bilinear form s.t

$$< a \circ b, c > = < a, b \circ c > \quad \forall a, b, c \in \mathcal{A}.$$

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Definition (Dubrovin '94)

$(M, \circ, e, <,>, g, E)$ is a Frobenius manifold if each tangent space T_pM is a Frobenius algebra, varying smoothly over M with some additional axioms.

Theorem (Dubrovin'94)

There exists, up to an equivalence, a Frobenius structure on M_W with the metric $\eta =<,>$ and

- *the Euler vector field $E = \sum_{i=1}^{n} \frac{1}{h} d_i y^i \partial_{y^i}$,*
- *the identity vector field $e := \partial_{y^1}$.*

The metrics η, g form a **flat pencil** of metrics on M_W.

Coxeter discriminant

Definition (Strachan '04)

Let M be a Frobenius manifold. A natural submanifold N of M is a submanifold $N \subset M$ s.t

- $TN \circ TN \subset TN$,
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Definition

Σ is called a Coxeter discriminant. It is the image of the union of the mirrors under the natural projection map

$$\pi : V \to M_W.$$

A stratum $\pi(D) \subset \Sigma$ is the image of the intersection subspace $D = \cap_{\beta \in B} \Pi_\beta$, where $B \subset R$, $\Pi_\beta = \{x \in V | g(x, \beta) = 0\}$.

- **Discriminant strata** are shown to be natural submanifolds of the Frobenius manifold M_W (Strachan '04; Feigin, Veselov '07, AFS'17)

- $\pi : D \to \pi(D)$ is a diffeomorphism near generic point $x_0 \in D$.

- The Saito metric on M_W induces a metric on $\pi(D)$ which is naturally given as the restriction of η to the stratum. Let us denote it by η_D.

- The linear coordinates x^i give rise to coordinates on the stratum D and on $\pi(D)$. These are flat coordinates for the restricted metric g on the stratum D. We denote this metric by g_D.

We are interested in answering the following:

Question

How does $\det \eta_D$ look in the flat coordinates of g_D on discriminant strata?

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Let us fix some notation:

- Let $L = \{\gamma_1, \dots, \gamma_k\} \subset R$, $1 \leq k \leq n$ and consider $D = \cap_{\gamma \in L} \Pi_\gamma$ s.t $\dim D = n - k$.
- For any $\beta \in R \setminus \langle L \rangle$, $\widehat{L} = L \cup \{\beta\}$, define $U_\beta = \langle \widehat{L} \rangle \cap R$.

Proposition

The set U_β is a root system and admits the decomposition

$$U_\beta = \bigsqcup_{i=1}^{p} R_i, \tag{1}$$

where $\{R_i\}_{i=1}^{p}$ are irreducible root systems.

Theorem

The determinant of η_D on D is proportional to the product of linear factors

$$\prod_{l \in A} g_D(l, x)^{m_l}, \quad m_l \in \mathbb{N},$$

where A is a collection of non-proportional vectors on D. Furthermore, each $l \in A$ has the form β_D for some $\beta \in R \setminus \langle L \rangle$, where β_D is the orthogonal projection of β on D. The multiplicity m_l equals the Coxeter number of the root system R_q from the decomposition (1), such that $\beta \in R_q$.

Coxeter group, $W = A_4$

Example

Consider a stratum of type A_2 in A_4, let $D = \{x_1 = x_2 = x_3\}$. Coordinates on D are chosen as: $\xi_0 = x_1 = x_2 = x_3$, $\xi_1 = x_4$, $\xi_2 = x_5$. Then,

$$\det \eta_D = c(\xi_0 - \xi_1)^4(\xi_0 - \xi_2)^4(\xi_1 - \xi_2)^2.$$

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Q: How does this match the statement of the Theorem?

1. $\beta = e_3 - e_4$, $U_\beta = \langle e_1 - e_2, e_2 - e_3, \beta\rangle \cap R \cong A_3$, $h(A_3) = 4$,
2. $\beta = e_4 - e_5$, $U_\beta = \langle e_1 - e_2, \beta, \beta\rangle \cap R \cong A_2 \sqcup A_1$, $\beta \in A_1$, $h(A_1) = 2$,
3. $\beta = e_3 - e_5$, $U_\beta = \langle e_1 - e_2, e_2 - e_3, \beta\rangle \cap R \cong A_3$, $h(A_3) = 4$,
4. No other factors in $\det \eta_D$, e.g $(e_2 - e_4)_D = (e_3 - e_4)_D$ etc.

Strata in type A_N

An arbitrary l-dimensional stratum $D \subset V$ has the form ($k \leq l$):

$$
\begin{gathered}
x_1 = \ldots = x_{m_0} = \xi_0, \\
x_{m_0+1} = \ldots = x_{m_0+m_1} = \xi_1 \\
\vdots \\
x_{\sum_{i=0}^{k-1} m_i+1} = \ldots = x_{\sum_{i=0}^{k} m_i} = \xi_k .
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Coordinates on D: $\xi_0, \ldots, \xi_l$, where $\xi_i = x_i$, $i = k+1, \ldots, l$. Then,

$$\boxed{\det \eta_D = c \prod_{0 \leqslant i < j \leqslant l} (\xi_i - \xi_j)^{m_i + m_j}}$$

where $c = (-1)^{\sum_{i=1}^{l} i m_i} (N+1)^{-N} \prod_{a=1}^{l} m_a^2 \prod_{a=0}^{l} m_a^{m_a - 1}$.

Proof of the Theorem ?

For **classical** series we use Landau-Ginzburg superpotential description of the Frobenius structures on the discriminant strata. In type A this superpotential on the stratum D is

$$\lambda(p) = \prod_{i=0}^{n} (p - \xi_i)^{m_i}, \quad m_i \in \mathbb{N}.$$

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The expression for the Saito metric is given as

$$\eta(\partial_i, \partial_j) = \sum_{p_s : \lambda'(p_s)=0} res|_{p=p_s} \frac{\partial_i(\lambda)\partial_j(\lambda)}{\lambda'(p)} dp,$$

where ∂_i denote some vector fields and where first derivatives, $\lambda'(p)$, are taken w.r.t p.

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For **exceptional** series, proof relies heavily on the geometry of root systems.

Thank you for your attention!