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Bi-flat F-manifolds, complex reflection groups and integrable systems of conservation laws.

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Based on joint works with Alessandro Arsie

Plan of the talk

1. Flat and bi-flat F-manifolds.
2. Bi-flat F-manifolds and Painlevé transcendents.
3. Bi-flat F-manifolds and complex reflection groups.
4. Integrable systems of conservation laws.

Part I. Flat and bi-flat F manifolds

Flat F-manifolds (Manin)

Definition

A **flat F-manifold** *(or* **F-manifold with compatible flat structure***)* $(M, \circ, \nabla, e)$ *is a manifold equipped with a product* $\circ : TM \times TM \to TM$ *on the tangent spaces (with structure constants* c^i_{jk}*), a connection* ∇ *(with Christoffel symbols* Γ^i_{jk}*) and a distinguished vector field* e *s.t.*

1. *the one parameter family of connections* $\nabla_{(\lambda)}$ *with Christoffel symbols*
$$\Gamma^i_{jk} + \lambda c^i_{jk}$$
is flat and torsionless for any λ*.*
2. e *is the unit of the product.*
3. e *is flat:* $\nabla e = 0$*.*

Manifolds equipped with a product $\circ$, a connection ∇ and a vector field e satifying conditions 1 and 2 will be called **almost flat F-manifolds**.

For a given λ the torsion is

$$T_{ij}^{(\lambda)k} = \Gamma_{ij}^k - \Gamma_{ji}^k + \lambda(c_{ij}^k - c_{ji}^k)$$

and the curvature is

$$R_{ijl}^{(\lambda)k} = R_{ijl}^k + \lambda(\nabla_i c_{jl}^k - \nabla_j c_{il}^k) + \lambda^2(c_{im}^k c_{jl}^m - c_{jm}^k c_{il}^m),$$

where R_{ijl}^k is the Riemann tensor of ∇.

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We obtain

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and the curvature is

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where R^k_{ijl} is the Riemann tensor of ∇.

We obtain

1. the connection ∇ is torsionless,
2. the product $\circ$ is commutative,
3. the connection ∇ is flat

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We obtain

1. the connection ∇ is torsionless,
2. the product $\circ$ is commutative,
3. the connection ∇ is flat,
4. the tensor field $\nabla_l c_{ij}^k$ is symmetric in the lower indices,

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3. the connection ∇ is flat,
4. the tensor field $\nabla_l c_{ij}^k$ is symmetric in the lower indices,
5. the product $\circ$ is associative.

Generalized WDVV associativity equations

From conditions $1, 2, 3, 4$ it follows that, in **flat coordinates** for ∇, we have

$$c^i_{jk} = \partial_j \partial_k A^i.$$

Condition 5 tell us that the vector potential A^i satisfies the associativity equations:

$$\partial_j \partial_l A^i \partial_k \partial_m A^l \quad = \quad \partial_k \partial_l A^i \partial_k \partial_m A^l$$

The invariant metric

Invariant metric η:

- $\nabla\eta = 0$.
- $\eta_{il} c^l_{jk} = \eta_{jl} c^l_{ik}$.

A consequence: $A^i = \eta^{il}\partial_l F$ and generalized WDVV associativity equations become WDVV associativity equations:

$$\partial_j\partial_h\partial_i F \eta^{il} \partial_l\partial_k\partial_m F \quad = \quad \partial_j\partial_k\partial_i F \eta^{il} \partial_l\partial_h\partial_m F$$

Euler vector field and Frobenius manifolds

A vector field satisfying the conditions

$$[e, E] = e, \quad \mathrm{Lie}_E c^i_{jk} = c^i_{jk}$$

is called an **Euler vector field**.

Frobenius manifolds are flat F-manifolds endowed with an invariant metric and a **linear** Euler vector field ($\nabla\nabla E = 0$).

Almost duality

Let us consider the contravariant metric $g = (E\circ)\,\eta^{-1}$ (the **intersection form**). It turns out that

- the Levi-Civita connection $\tilde{\nabla}$ of g,

- the **dual product** defined as $X * Y = (E\circ)^{-1}\,X \circ Y, \quad \forall X, Y$,

- and the Euler vector field E.

define an almost flat structure with invariant metric g^{-1} at the points where $E\circ$ is invertible.

This is called the **almost dual structure** (Dubrovin). In general $\tilde{\nabla} E \neq 0$. Replacing $\tilde{\nabla}$ with $\nabla^* = \tilde{\nabla} + \bar{\lambda} *$ (with a suitable $\bar{\lambda}$) one obtains a flat connection satisfying $\nabla^* E = 0$.

From Frobenius manifolds to bi-flat F-manifolds

Any Frobenius manifold $(M, \eta, \circ, e, E)$ is equipped with:

- the flat structure $(\nabla, \circ, e)$,
- the flat structure $(\nabla^*, *, E)$.

It turns out that

$$(d_\nabla - d_{\nabla^*})(X\circ) = 0, \quad \forall X,$$

where d_∇ is the exterior covariant derivative:

$$(d_\nabla \omega)^l_{i_0\,\dots\,i_k} = \sum_{j=0}^{k} (-1)^j \nabla_{i_j} \omega^l_{i_0\,i_1\,\dots\,\hat{i}_j\,\dots\,i_k}.$$

Bi-flat F-manifolds

Definition

A **bi-flat** F**-manifold** *is a manifold equipped with two different flat structures* $(\nabla, \circ, e)$ *and* $(\nabla^*, *, E)$ *related by the following conditions*

1. E *is an Euler vector field.*
2. $*$ *is the dual product.*
3. $(d_{\nabla} - d_{\nabla^*})(X \circ) = 0, \quad \forall X.$

Part II. Bi-flat F manifolds and Painlevé transcendents.

A system of PDEs for semisimple bi-flat F-manifolds

In the semisimple case bi-flat F manifolds can be constructed from the solution of the following system of PDEs

$$\partial_k \Gamma^i_{ij} = -\Gamma^i_{ij}\Gamma^i_{ik} + \Gamma^i_{ij}\Gamma^j_{jk} + \Gamma^i_{ik}\Gamma^k_{kj}, \quad i \neq k \neq j \neq i, \tag{1}$$

$$\sum_{i=1}^{n} \partial_i(\Gamma^i_{ij}) = 0, \qquad i \neq j \tag{2}$$

$$\sum_{i=1}^{n} u^i \partial_i(\Gamma^i_{ij}) = -\Gamma^i_{ij}, \qquad i \neq j \tag{3}$$

for the $n(n-1)$ unknown functions $\Gamma^i_{ij}(\mathbf{u})$. The above system is compatible and thus its general solution depends on $n(n-1)$ arbitrary constants.

The system (1) plays a crucial role in the theory of integrable system of hydrodynamic type (Tsarev)

The case $n = 3$

From (2) and (3) it follows that

$$\Gamma^i_{ij} = \frac{F_{ij}\left(\frac{u^2-u^3}{u^1-u^2}\right)}{u^i - u^j}. \tag{4}$$

Imposing (1) and introducing $z = \frac{u^2-u^3}{u^1-u^2}$, we obtain

$$\begin{aligned}
\frac{dF_{12}}{dz} &= \frac{(F_{12}F_{13} - F_{12}F_{23})z + F_{12}F_{23} - F_{13}F_{32}}{z(z-1)},\\
\frac{dF_{21}}{dz} &= \frac{(F_{21}F_{23} - F_{21}F_{13})z + F_{23}F_{31} - F_{21}F_{23}}{z(z-1)},\\
\frac{dF_{13}}{dz} &= \frac{(F_{12}F_{23} - F_{12}F_{13})z - F_{12}F_{23} + F_{13}F_{32}}{z(z-1)},\\
\frac{dF_{31}}{dz} &= \frac{(F_{31}F_{12} - F_{32}F_{21})z - F_{31}F_{32} + F_{32}F_{21}}{z(z-1)},\\
\frac{dF_{23}}{dz} &= \frac{(F_{21}F_{13} - F_{21}F_{23})z - F_{23}F_{31} + F_{21}F_{23}}{z(z-1)},\\
\frac{dF_{32}}{dz} &= \frac{(F_{32}F_{21} - F_{31}F_{12})z + F_{31}F_{32} - F_{32}F_{21}}{z(z-1)}.
\end{aligned} \tag{5}$$

System (5) admits three linear first integrals

$$\begin{aligned} I_1 &= F_{12} + F_{13}, \\ I_2 &= F_{23} + F_{21}, \\ I_3 &= F_{31} + F_{32}, \end{aligned}$$

and one quadratic first integral

$$I_4 = F_{31}F_{13} + F_{12}F_{21} + F_{23}F_{32}.$$

Using these first integrals we can reduce (5) to the sigma form of the generic Painlevé VI equation. A similar analysis can be repeated in regular non-semisimple case. The role of canonical coordinates is played by a distinguished set of coordinates found by David and Hertling (2015).

Three dimensional regular case and Painlevé trascendents

Theorem (A.Arsie, P.L. 2015): Three dimensional regular bi-flat F-manifolds are locally parameterized by solutions of the full Painlevé IV, V, and VI equations according to the Jordan canonical form J of $L = E\circ$. More precisely,

- *PVI* in the case
$$J = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$
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$$J = \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}.$$

Part III. Complex reflection groups and bi-flat F manifolds

Complex reflection groups

A complex (pseudo)-reflection is a unitary transformation of $\mathbb{C}^n$ of finite order that leaves invariant a hyperplane. A finite complex reflection group is a finite group generated by complex reflections.

Finite complex reflection groups were classified by Shephard and Todd, and consist in an infinite family depending on 3 positive integers and 34 exceptional cases.

The ring of invariant polynomials of a complex reflection group is generated by n algebraically independent invariant polynomials $(u_1, ..., u_n)$, where n is the dimension of the complex vector space on which the group acts.

Frobenius manifolds from Coxeter groups

Theorem (Dubrovin)
The orbit space of a Coxeter group is equipped with a Frobenius manifold structure $(\eta, \circ, e, E)$ where

1. The flat coordinates for η are basic invariants $(u_1, ..., u_n)$ of the group called *Saito flat coordinates*.
2. In the Saito flat coordinates

$$e = \frac{\partial}{\partial u_n}, \; E = \sum_{i=1}^{n} \left(\frac{d_i}{d_n} \right) u_i \frac{\partial}{\partial u_i}.$$

where d_i are the degrees of the invariant polynomials u_i and $2 = d_1 < d_2 \leq d_3 \leq ... \leq d_{n-1} < d_n$ (d_n is the Coxeter number).

An almost flat structures associated with Coxeter groups

Let G be a Coxeter group acting on a euclidean space $\mathbb{E}^n$ with euclidean coordinates $(p_1, ..., p_n)$. Let g be the euclidean metric and ∇^* the associated Levi-Civita connection. Then the data

$$\left(\nabla^*, \quad * = \sum_{H \in \mathcal{H}} \frac{d\alpha_H}{\alpha_H} \otimes \sigma_H \pi_H, \quad E = \sum p_k \frac{\partial}{\partial p_k}\right)$$

where

- $\mathcal{H}$ is the collection of the reflecting hyperplanes H,
- α_H is a linear form defining a reflecting hyperplane H,
- π_H is the orthogonal projection onto the orthogonal complement of H,
- the collection of weights σ_H is G-invariant and satisfy

$$\sum_{H \in \mathcal{H}} \sigma_H \pi_H = Id.$$

define a flat structure with invariant metric g. This an equivalent reformulation of a result of Veselov (1999). The almost dual structure of the Frobenius manifold structure on the orbit space of Coxeter group has the Veselov's form with all the weights σ_H equal to each other (Dubrovin, 2003).

Two flat structure associated with complex reflection group

Starting from a complex reflection group it is possible two construct two different flat structure:

- The first structure has been obtained by Kato-Mano-Sekiguchy generalizing Dubrovin-Saito construction to well generated finite complex reflection groups (a complex reflection group of rank n is said to be well-generated if its minimal generating set consists of n reflections).
- The second one is obtained starting from a Dunkl-Kohno-type connection associated with complex reflection groups considered by Looijenga.

The first flat structure

Theorem (Kato, Mano and Sekiguchi, 2015)
The orbit space of a well generated complex reflection group is equipped with a flat structure $(\nabla, \circ, e, E)$ with **linear** Euler vector field where

1. The flat coordinates for ∇ are basic invariants $(u_1, ..., u_n)$ of the group.
2. In the Saito flat coordinates

$$e = \frac{\partial}{\partial u_n}, \;\; E = \sum_{i=1}^{n} \left(\frac{d_i}{d_n} \right) u_i \frac{\partial}{\partial u_i}.$$

The second flat structure

Let G be an irreducible complex reflection group acting on $\mathbb{C}^n$. Then

$$\left(\nabla^* = \nabla - \sum_{H\in\mathcal{H}} \frac{d\alpha_H}{\alpha_H} \otimes \tau_H \pi_H,\ * = \sum_{H\in\mathcal{H}} \frac{d\alpha_H}{\alpha_H} \otimes \sigma_H \pi_H,\ E = \sum p_k \frac{\partial}{\partial p_k}\right)$$

where

- $\mathcal{H}$ is the collection of the reflecting hyperplanes H,
- α_H is a linear form defining a reflecting hyperplane H,
- π_H is the unitary projection onto the unitary complement of H,
- the collections of weights σ_H and τ_H are G-invariant and satisfy

$$\sum_{H\in\mathcal{H}} \sigma_H \pi_H = \sum_{H\in\mathcal{H}} \tau_H \pi_H = Id.$$

- ∇ is the standard flat connection on $\mathbb{C}^n$,

define a flat structure.

Complex reflection groups and bi-flat F-manifolds

The standard choice is to choose σ_H and τ_H proportional to the order of the corresponding reflection.

There are other choices that lead to bi-flat F-manifolds. For instance, in the case of Weyl groups of rank 2, 3, and 4 and for the groups $I_2(m)$ we checked that there is a $(m-1)$-parameter family of bi-flat structures where m is the number of orbits for the action of G on the collection of reflecting hyperplane.

Conjecture: The bi-flat structures associated with Coxeter groups admitting a dual structure of the above form depend on $m-1$ parameters.

the flat coordinates for the connection ∇ are basic polynomial invariants of the group.

The case of B_2

We have

$$\nabla^* = \nabla - \sum_{H \in \mathcal{H}} \frac{d\alpha_H}{\alpha_H} \otimes \tau_H \pi_H, \quad * = \sum_{H \in \mathcal{H}} \frac{d\alpha_H}{\alpha_H} \otimes \sigma_H \pi_H$$

with

$$\alpha_1 = [1, 0], \qquad \alpha_2 = [0, 1], \qquad \alpha_3 = [1, -1] \qquad \alpha_4 = [1, 1]$$

and

$$\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = \frac{1}{2}, \quad \tau_1 = \tau_2 = 2c + 1, \; \tau_3 = \tau_4 = -2 - 2c$$

For any value of c we get different flat basic invariants:

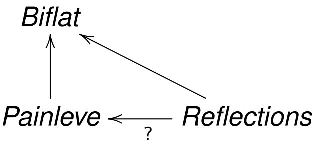
$$u_1 = p_1^2 + p_2^2, \qquad u_2 = p_1^4 + p_4^2 + c u_1^2.$$

and a different vector potential:

$$\begin{aligned} A^1_{B_2} &= -\frac{2}{3}\left(c + \frac{3}{4}\right) u_1^3 + u_1 u_2, \\ A^2_{B_2} &= -\frac{1}{6}(c+1)(2c+1)u_1^4 + \frac{1}{2}u_2^2. \end{aligned}$$

For $c = -\frac{3}{4}$ the vector potential comes from a Frobenius potential and the flat basic invariants coincide with the Saito flat coordinates.

Remark



There are algebraic solutions of Painlevé VI equation coming from complex reflection groups (Boalch, 2003).

Part IV. Flat F manifolds and integrable systems of conservation laws

Principal hierarchy for flat F-manifolds

Integrable hierarchy:

$$v^i_{t_{(p,l+1)}} = c^i_{jk} X^k_{(p,l)} v^k_x, \qquad p = 1, ..., n \qquad l = -1, 0, 1, 2, 3, ...$$

Primary flows:

$$\nabla_j X^i_{(p,-1)} = 0$$

Higher flows:

$$\nabla_j X^i_{(p,l+1)} = c^i_{jk} X^k_{(p,l)}.$$

This is a generalization of Dubrovin's principal hierarchy (P.L., M. Pedroni, A. Raimondo, 2010).

Flat and canonical coordinates

In flat coordinates $(u^1, ..., u^n)$ the flows of the hierarchy are systems of conservation laws:

$$u^i_{t_{(p,l)}} = c^i_{jk} X^j_{(p,l-1)} u^k_x = \partial_x X^i_{(p,l)}.$$

In the product is semisimple, the in canonical coordinates $(r^1, ..., r^n)$ we have (by definition)

$$c^i_{jk} = \delta^i_j \delta^i_k$$

and morover

$$r^i_{t_{(p,l+1)}} = X^i_{(p,l)}(\mathbf{r}) r^i_x, \qquad i = 1, ...n$$

Principal hierarchy with additional structures

In the case of F-manifolds with invariant metric the principal hierarchy becomes Hamiltonian w.r.t. the Dubrovin-Novikov bracket associated with η^{-1}.

In the case of flat Frobenius manifolds the principal hierarchy becomes bi-Hamiltonian: the second Hamiltonian structure the Dubrovin-Novikov bracket associated with the intersection form.

The principal hierarchy for B_2

In flat coordinates we have

$$u^i_{t_{(p,l+1)}} = \left(\partial_j \partial_k A^i_{B_2}\right) X^j_{(p,l)} u^k_x, \qquad i = 1, 2,$$

where $A^1_{B_2} = -\frac{2}{3}\left(c + \frac{3}{4}\right) u_1^3 + u_1 u_2$, $A^2_{B_2} = -\frac{1}{6}(c+1)(2c+1) u_1^4 + \frac{1}{2} u_2^2$

The recursion relations read

$$\partial_j X^i_{(p,-1)} = 0, \qquad \partial_j X^i_{(p,l+1)} = \left(\partial_j \partial_k A^i_{B_2}\right) X^k_{(p,l)}.$$

By straightforward computations we get:

$$X_{(1,-1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; X_{(1,0)} = \begin{pmatrix} u_2 \\ \frac{1}{12} u_1^3 \end{pmatrix}, \; X_{(1,1)} = \begin{pmatrix} \frac{1}{2} u_2^2 + \frac{1}{48} u_1^4 \\ \frac{1}{12} u_1^3 u_2 \end{pmatrix}, \ldots$$

$$X_{(2,-1)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; X_{(2,0)} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \; X_{(2,1)} = \begin{pmatrix} u_1 u_2 \\ \frac{1}{2} u_2^2 + \frac{1}{16} u_1^4 \end{pmatrix}, \ldots$$

The parameter c

- $c \neq -1, -\frac{1}{2}, -\frac{3}{4}$ (generic case): the principal hierarchy is not hamiltonian.
- $c = -\frac{3}{4}$: the principal hierarchy is (bi)-Hamiltonian. For this value of the parameter we have a Frobenius manifold.
- $c = -1, c = -\frac{1}{2}$: one of the chains of the principal hierarchy is degenerate.

Integrable deformations of the principal hierarchy

Deformations of the form

$$u_t^i = \partial_x \left[X^i(\mathbf{u}) + \epsilon (Y_j^i(\mathbf{u}) u_x^j \right) + \epsilon^2 \left(W_j^i(\mathbf{u}) u_{xx}^j + W_{jk}^i(\mathbf{u}) u_x^j u_x^k \right) + \dots \Big]$$

Values of c	**Integrable first order deformations**	**Integrable second order deformations**
$c \neq -\frac{3}{4}, -1, -\frac{1}{2}$	No functional parameters	Two functional parameters of a single variable
$c = -\frac{3}{4}$ (Frobenius)	No functional parameters	Two functional parameters of a single variable
$c = -\frac{1}{2}, -1$	One functional parameter of a single variable	Two additional functional parameters of a single variable

Some open questions

- We obtained integrability up to order 2 in ϵ. Is it possible to continue this procedure in principle to all orders, or are there obstructions?
- To construct examples where the right hand side contains a finite number of terms.
- Special values of the functional parameters seem related to a generalization of DR hierarchy.
- Are these hierarchies (bi)-Hamiltonian w.r.t. suitable non local Hamiltonian structures?

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