Cluster varieties, toric degenerations, and mirror symmetry

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Ongoing joint work with Lara Bossinger, Juan Bosco Frías Medina, and Alfredo Nájera Chávez

Outline

1. Cluster varieties and their compactifications
2. Toric degenerations of cluster varieties
3. Connections to Batyrev-Borisov mirror symmetry

Moral definitions and examples

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Calabi-Yau variety

A complex projective variety with a notion of complex volume

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Example

Compact torus $\mathbb{C}/\mathbb{Z}^2$, $\Omega = dz$

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Log Calabi-Yau variety

A smooth complex variety U with a unique volume form Ω having at worst a simple pole along any divisor in *any* compactification of U

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Carefully glued tori

$$U = \bigcup_i T_i / \sim$$

$$\mu_{ij} : T_i \dashrightarrow T_j, \qquad \mu_{ij}^* (\Omega_j) = \Omega_i$$

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- Lattice $N \cong \mathbb{Z}^n$
- Skew-form $\{\cdot, \cdot\} : N \times N \to \mathbb{Z}$
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Gives a pair of dual tori and bases for their character lattices:

$$T_{M;s} := \operatorname{Spec}\left(\mathbb{C}\left[z^{\pm e_1}, \ldots, z^{\pm e_n}\right]\right)$$

$$T_{N;s} := \operatorname{Spec}\left(\mathbb{C}\left[z^{\pm e_1^*}, \ldots, z^{\pm e_n^*}\right]\right)$$

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Recursive rule for changing basis $s \mapsto s'$, together with birational maps $\mu_{s,s'} : T_{M;s} \dashrightarrow T_{M;s'}$ and $\mu_{s,s'} : T_{N;s} \dashrightarrow T_{N;s'}$

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$\mathcal{X}$ has a Poisson structure:

$$\{z^{n_1}, z^{n_2}\} = \{n_1, n_2\} \, z^{n_1+n_2}$$

$\mathcal{A}$ has a degenerate symplectic structure, with 2-form:

$$\sum_{i,j} \frac{dz^{e_i^*}}{z^{e_i^*}} \wedge \frac{dz^{e_j^*}}{z^{e_j^*}}$$

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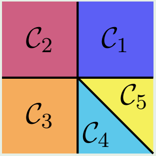
Example (A_2 cluster varieties)



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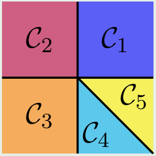

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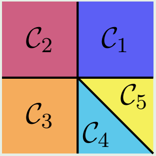

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Theorem ([GHKK16])

The cluster complex Δ^+ has a simplicial fan structure.

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$$\mathcal{X} = \bigcup_i \operatorname{Spec}\left(\mathbb{C}\left[\mathcal{C}_i^\vee \cap N\right]^{\mathrm{gp}}\right)$$

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Much more natural object from toric geometry perspective:

Special completion of $\mathcal{X}$ [FG15]

$$\widehat{\mathcal{X}} = \bigcup_i \operatorname{Spec}\left(\mathbb{C}\left[\mathcal{C}_i^\vee \cap N\right]\right)$$

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- $C(\mathrm{Gr}_k(\mathbb{C}^n)) \setminus \bigcup_i D_i$ is a cluster $\mathcal{A}$-variety.
- Each $p_{[i+1,i+k]}$ is a frozen variable, so $C(\mathrm{Gr}_k(\mathbb{C}^n))$ is such a compactification.

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Note:

- If we restrict to the tori $T_{N;s} \subset \mathcal{A}$ and $T_{M;s} \subset \mathcal{X}$, then $A_{i;s} = z^{e^*_{i;s}}$ and $X_{i;s} = z^{e_{i;s}}$.

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Note:

- If we restrict to the tori $T_{N;s} \subset \mathcal{A}$ and $T_{M;s} \subset \mathcal{X}$, then $A_{i;s} = z^{e_{i;s}^*}$ and $X_{i;s} = z^{e_{i;s}}$.
- On other tori $T_{N;s'}$ and $T_{M;s'}$, $\mu_{s',s}^*(A_{i;s})$ and $\mu_{s',s}^*(X_{i;s})$ will be rational functions in the variables $A_{j;s'}$ and $X_{j;s'}$.

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- $\mathcal{X}$-variables extend canonically to family.
- Extension of $X_{i;s}$ homogeneous under some natural torus action, with weight $\mathbf{c}_{i;s}$.

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Basic idea:

Add coefficients carefully to get

$$\begin{array}{ccc} \widehat{\mathcal{X}}_{\mathbf{t}} & \hookrightarrow & \widehat{\mathscr{X}}_{s_0} = \bigcup_s \operatorname{Spec}\left(R\left[X_{1;s}, \ldots, X_{n;s}\right]\right) \\ & & \downarrow \\ & & \mathbb{A}^n_{s_0} = \operatorname{Spec}\left(R\right) \end{array}$$

where $R := \mathbb{C}\left[t_{1;s_0}, \ldots, t_{n;s_0}\right]$.

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where $R := \mathbb{C}\left[t_{1;s_0}, \ldots, t_{n;s_0}\right]$.

- The fiber over 1 will be $\widehat{\mathcal{X}}$. The fiber over 0 will be $\mathrm{TV}\left(\Delta^+\right)$.

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$\mathcal{X}$ gluing defined by

$$\mu_k^*\left(X_i'\right) = \begin{cases} X_k^{-1} & \text{if } i = k \\ X_i\left(1 + X_k^{-\operatorname{sgn}\varepsilon_{ik}}\right)^{-\varepsilon_{ik}} & \text{if } i \neq k \end{cases}$$

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We define $\widehat{\mathscr{X}_{s_0}}$ gluing by

$$\mu_k^*(X_i') = \begin{cases} X_k^{-1} & \text{if } i = k \\ X_i\left(\mathbf{t}^{[\operatorname{sgn}\varepsilon_{ij}\mathbf{c}_k]_+} + \mathbf{t}^{[-\operatorname{sgn}\varepsilon_{ij}\mathbf{c}_k]_+} X_k^{-\operatorname{sgn}\varepsilon_{ik}}\right)^{-\varepsilon_{ik}} & \text{if } i \neq k \end{cases}$$

Toric degenerations of $\widehat{\mathcal{X}}$

Theorem (Bossinger, Frías Medina, Nájera Chávez, M.)

With this gluing, $\widehat{\mathscr{X}}_{s_0}$ satisfies all of the properties we asked our toric degenerations to satisfy.

Toric degenerations of compactified $\mathcal{A}$-varieties

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In [GHKK16] an analogous family is defined for $\mathcal{A}$-varieties compactified by letting frozen variables vanish.

$$\begin{array}{ccc} \overline{\mathcal{A}_{\mathbf{t}}} & \hookrightarrow & \overline{\mathcal{A}_{\mathrm{prin},s_0}} \\ & & \downarrow \\ & & \mathbb{A}^n_{s_0} = \mathrm{Spec}\,(R) \end{array}$$

where $R := \mathbb{C}\left[t_{1;s_0}, \ldots, t_{n;s_0}\right]$.

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Goal of Batyrev-Borisov:

Construct mirror families of Calabi-Yau varieties living in "dual" toric varieties.

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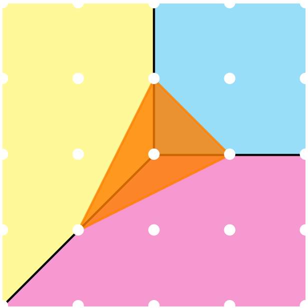

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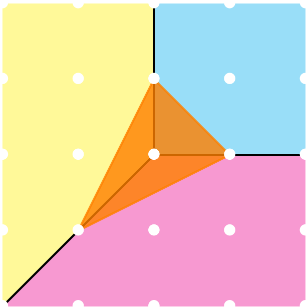

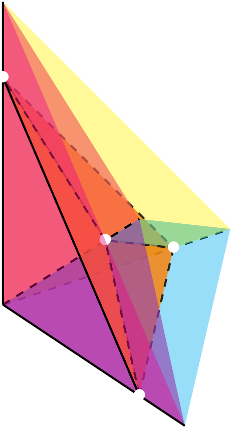

Batyrev-Borisov picture for Grassmannians

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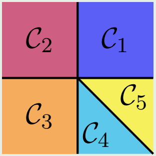
Example ($\mathrm{Gr}_2(\mathbb{C}^5)$)

Start with A_2 scattering diagram:



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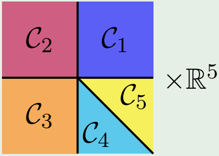
Example ($\mathrm{Gr}_2(\mathbb{C}^5)$)

Add on 5 dimensional linear space, one dimension for each consecutive Plücker:



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Example ($\mathrm{Gr}_2(\mathbb{C}^5)$)

Cut out the cone Ξ generated by the columns of the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

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- Have a fan structure Σ on Ξ.
- Relevant slice of Ξ: $P = \{x \in \Xi \,|\, \langle (1\ 1\ 1\ 1\ 1\ 1\ 1), x \rangle = 5\}$. Unique interior point: $v = (0\ 0\ 1\ 1\ 1\ 1\ 1)^{\mathrm{T}}$.

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- Translate generators of Σ by $-v$ to get a new fan Δ. Σ is "cone over" Δ, and the rays of Δ are generated by vertices of P.

Batyrev-Borisov picture for Grassmannians

Expectations:

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- The construction works the same way for all $\mathrm{Gr}_k(\mathbb{C}^n)$.
- A generalization (higher dimensional version of "cone over"– higher codimension CY subvarieties) works for complete flag varieties.

References

[FG15] V. Fock and A. Goncharov, *Cluster X-varieties at infinity*, (2015), arXiv:1104.0407v2 [math.AG].

[GHKK16] M. Gross, P. Hacking, S. Keel and M. Kontsevich, *Canonical bases for cluster algebras*, preprint (2016), arXiv:1411.1394v2 [math.AG].