Dualities in the q-Askey scheme and degenerate DAHA

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Based on M. Mazzocco Nonlinearity '16 and
T. Koornwinder-M. Mazzocco arXiv:1803.02775

Outline

- Askey Wilson polynomials and their properties
- Zhedanov algebra
- q-Askey scheme and its geometric interpretation
- duality
- Non-symmetric AW polynomials, DAHA and degenerations.
- Outlook

Askey-Wilson polynomials

Definition

$$R_n\,[z; a, b, c, d \mid q] := \sum_{k=0}^{n} \frac{(q^{-n}, q^{n-1}, abcd, az, az^{-1}; q)_k}{(ab, ac, ad; q)_k} q^k$$

$(a_1, \ldots, a_r; q)_k := (a_1; q)_k (a_2; q)_k \cdots (a_r; q)_k$,
$(a; q)_k := \Pi_{j=0}^{k-1} (1 - a\, q^j)$

Example

$$R_2\,[z; a, \ldots, d; q] = 1 - \frac{(1+q)(-1+abcdq)(a-z)(-1+az)}{(-1+ab)(-1+ac)(-1+ad)qz} +$$
$$+ \frac{(abcdq-1)(abcdq^2-1)(a-z)(aq-z)(az-1)(aqz-1)}{(ab-1)(ac-1)(ad-1)q(abq-1)(acq-1)(adq-1)z^2}$$

Properties of the Askey-Wilson polynomials

- Symmetric: $R_n[z; a, b, c, d \mid q] = R_n[1/z; a, b, c, d \mid q]$.
- Orthogonal
- q-difference equation $L_z\big(R_n[z]\big) = (q^{-n} + abcdq^{n-1})\, R_n[z]$,

$$L_z\big(f[z]\big) := A(z)\,\big(f[qz] - f[z]\big) + A(1/z)\,\big(f[q^{-1}z] - f[z]\big),$$
$$A(z) = \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}$$

- Three term recursion relation $M_n\big(R_n[z]\big) = (z + z^{-1})\, R_n[z]$,

$$M_n\big(g_n\big) := A_n g_{n+1} + (a + a^{-1} - A_n - C_n) g_n + C_n g_{n-1}.$$

Zhedanov algebra

The operators L_z and $\left(z+\frac{1}{z}\right)$ generate the Zhedanov algebra:

$$(K_0 f)[z] := L_z\big(f[z]\big), \qquad (K_1 f)[z] := (z+z^{-1})f[z],$$

$$(q+q^{-1})K_1K_0K_1 - K_1^2K_0 - K_0K_1^2 = B\,K_1 + C_0\,K_0 + D_0,$$
$$(q+q^{-1})K_0K_1K_0 - K_0^2K_1 - K_1K_0^2 = B\,K_0 + C_1\,K_1 + D_1.$$

B, C_0, C_1, D_0, D_1 constants: $C_0 := (q-q^{-1})^2$ and

$$B = (1-q^{-1})^2(abc+abd+acd+bcd+(a+b+c+d)q),$$
$$C_1 = q^{-1}(q-q^{-1})^2abcd, \qquad D_0\ldots, \qquad D_1 = \ldots$$

q-Askey scheme

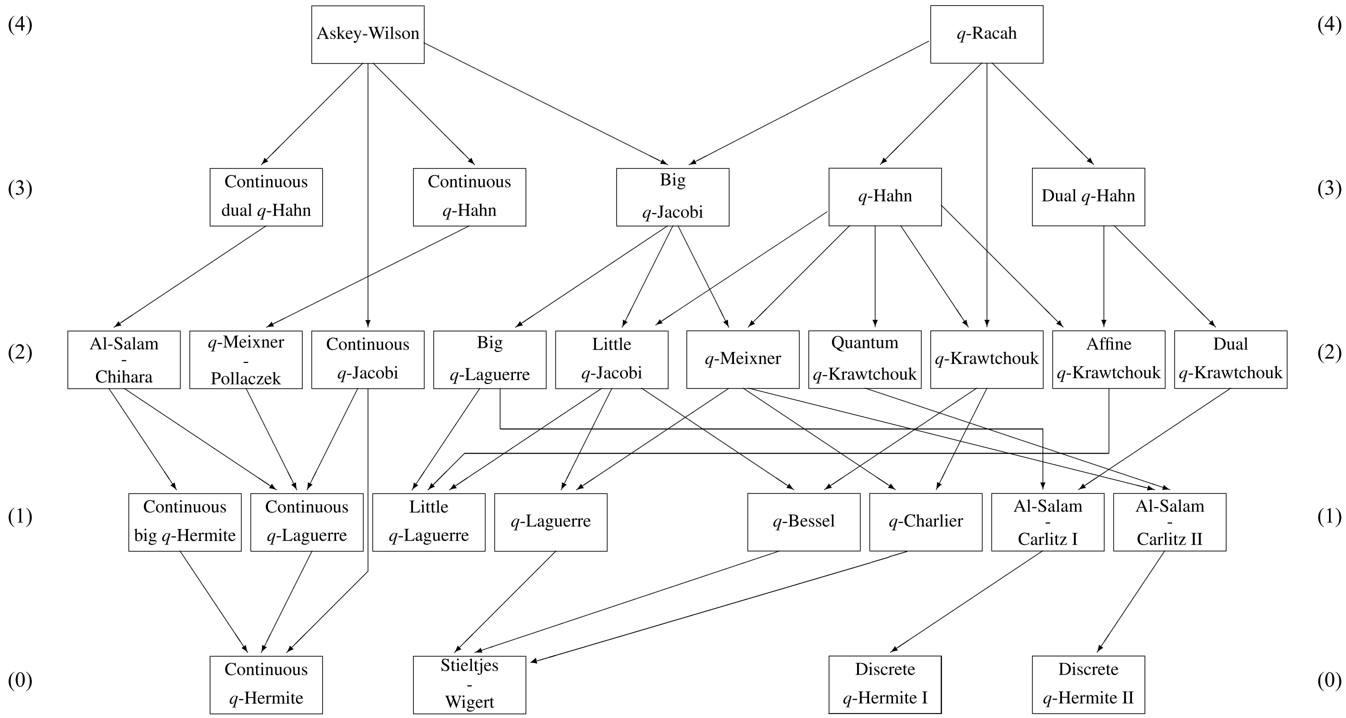
Take degenerations of Askey Wilson polynomials to produce other families of polynomials such that

- Symmetric Laurent or standard polynomials
- Orthogonal
- q-difference equation
- Three-term recursion relation.

Example

$d = 0$ is a good degeneration:
$p_n[z; a, b, c \mid q] := \sum_{k=0}^{n} \frac{(q^{-n}, q^{n-1}, az, az^{-1}; q)_k}{(ab, ac; q)_k} q^k.$
$a = 0$ is not a good degeneration.

q-Askey scheme


(4)
Askey-Wilson
q-Racah
(4)
(3)
Continuous dual q-Hahn
Continuous q-Hahn
Big q-Jacobi
q-Hahn
Dual q-Hahn
(3)
(2)
Al-Salam - Chihara
q-Meixner - Pollaczek
Continuous q-Jacobi
Big q-Laguerre
Little q-Jacobi
q-Meixner
Quantum q-Krawtchouk
q-Krawtchouk
Affine q-Krawtchouk
Dual q-Krawtchouk
(2)
(1)
Continuous big q-Hermite
Continuous q-Laguerre
Little q-Laguerre
q-Laguerre
q-Bessel
q-Charlier
Al-Salam - Carlitz I
Al-Salam - Carlitz II
(1)
(0)
Continuous q-Hermite
Stieltjes - Wigert
Discrete q-Hermite I
Discrete q-Hermite II
(0)

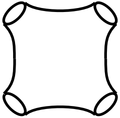
Underlying mechanism

$L_z, \left(z + \frac{1}{z}\right) \to$ Zhedanov algebra

Underlying mechanism

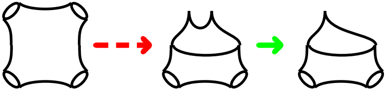
$L_z, (z + \frac{1}{z}) \rightarrow$ Zhedanov algebra

$q \rightarrow 1 \downarrow$



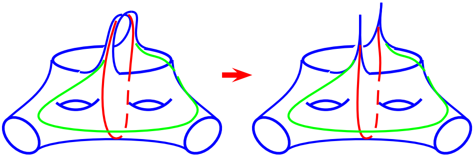
[Oblomkov '04]

Degenerations correspond to *chewing-gum moves* [M. M. Nonlinearity '16]

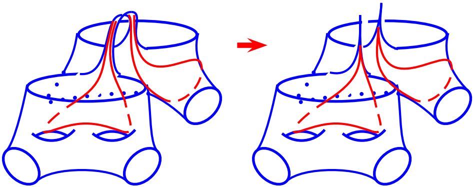


Chewing-gum moves [Chekhov-M. M. Nonlinearity '17]

- **Hooking holes:**

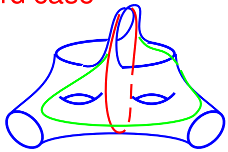


- **Pinching two sides of the same hole:**



Bordered cusped character variety

Standard case



Fundamental group: $\pi_1(\Sigma_{g,s})$

Representations:

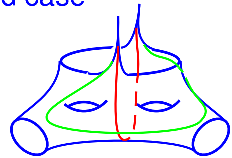
$\mathrm{Hom}\left(\pi_1(\Sigma_{g,s}) \to \mathbb{S}L_2(\mathbb{C})\right)$

Character variety:

$\mathrm{Hom}\left(\pi_1(\Sigma_{g,s}) \to \mathbb{S}L_2(\mathbb{C})\right) /_{\mathbb{S}L_2(\mathbb{C})}$

$\dim(\mathcal{C}_{g,s}) = 6g - 6 + 3s$

Cusped case



Fundamental groupoid of arcs: $\pi_{\mathfrak{a}}(\Sigma_{g,s,n})$

Representations:

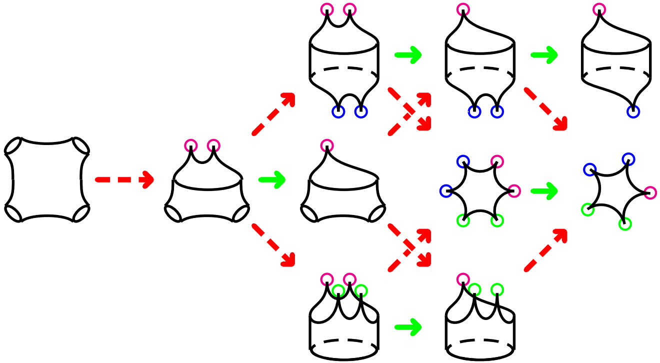
$\mathrm{Hom}\left(\pi_{\mathfrak{a}}(\Sigma_{g,s,n}), \mathbb{S}L_2(\mathbb{C})\right)$

Cusped Character Variety:

$\mathrm{Hom}\left(\pi_{\mathfrak{a}}(\Sigma_{g,s,n}), \mathbb{S}L_2(\mathbb{C})\right) /_{\prod_{j=1}^n U_j}$

$\dim(\mathcal{C}_{g,s,n}) = 6g - 6 + 3s + 2n$

Geometric q-Askey scheme



Duality

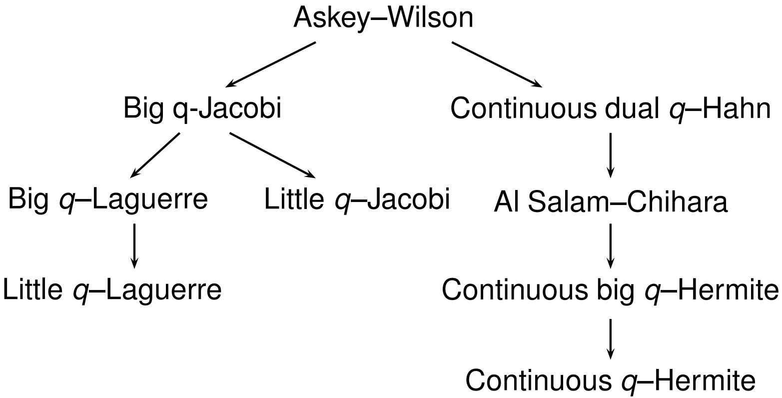
- The operators $L_z, z+z^{-1}$ act on $Sym[z]$.
- They generate the Zhedanov algebra $\langle L_z, (z+\frac{1}{z})\rangle$
- on Askey-Wilson polynomials
$L_z\big(R_n[z]\big) = (q^{-n} + abcdq^{n-1})\, R_n[z]$ and
$(z+z^{-1})\, R_n[z] = M_n\big(R_n[z]\big)$

Duality: $L_z, (z+z^{-1}) \leftrightarrow M_n, \Lambda_n$ [Noumi-Stokman '04].

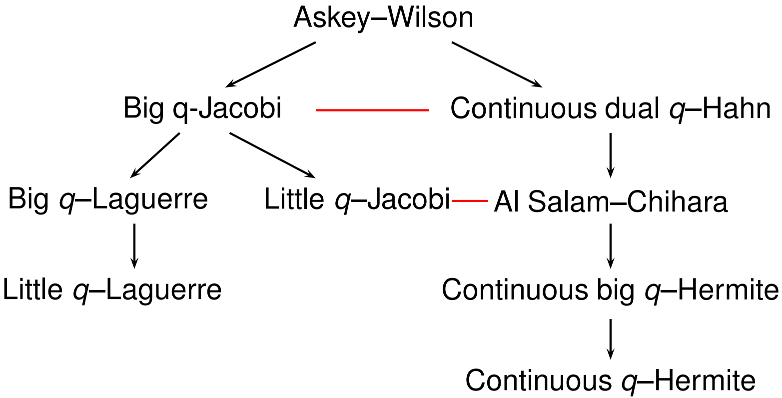
$$R_n\left[a^{-1}q^{-m}; a, b, c, d \mid q\right] = R_m\left[\tilde{a}^{-1}q^{-n}; \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \mid q\right]$$

$\tilde{a} = (q^{-1}abcd)^{\frac{1}{2}} \quad \tilde{b} = ab/\tilde{a}, \quad \tilde{c} = ac/\tilde{a}, \quad \tilde{d} = ad/\tilde{a}.$

q-Askey scheme



q-Askey scheme



Non symmetric Askey-Wilson polynomials

$$E_n[z] := R_n[z; a, b, c, d \mid q] - \frac{q^{1-n}(1-q^n)(1-q^{n-1}cd)}{(1-qab)(1-ab)(1-ac)(1-ad)} \times az^{-1}(1-az)(1-bz)R_{n-1}[z; qa, qb, c, d \mid q] \quad n \geq 0,$$

$$E_{-n}[z] := R_n[z; a, b, c, d \mid q] - \frac{q^{1-n}(1-q^n ab)(1-q^{n-1}abcd)}{(1-qab)(1-ab)(1-ac)(1-ad)} \times b^{-1}z^{-1}(1-az)(1-bz)R_{n-1}[z; qa, qb, c, d \mid q] \quad n \geq 1,$$

[Koornwinder '07]

Properties of the non-symmetric Askey-Wilson polynomials

- They form a basis in the space of Laurent polynomials.
- Orthogonal
- q-difference equation

$$\begin{aligned} YE_n &= q^{n-1}abcd\, E_n \qquad (n = 0, 1, 2, \ldots), \\ YE_{-n} &= q^{-n}\, E_{-n} \qquad (n = 1, 2, \ldots). \end{aligned}$$

- Three term recursion relation

$$M_n\big(E_n[z]\big) = z^{-1}\, E_n[z].$$

[Koornwinder '07]

DAHA

Z, Y and T generate the DAHA of type $\check{C}_1 C_1$:

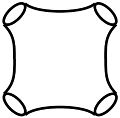
$$\begin{aligned}
(T+ab)(T+1) &= 0,\\
(T^{-1}Y+q^{-1}cd)(T^{-1}Y+1) &= 0,\\
(aZ^{-1}T^{-1}+1)(bZ^{-1}T^{-1}+1) &= 0,\\
(c+qZ^{-1}T^{-1}Y)(d+qZ^{-1}T^{-1}Y) &= 0,\\
ZZ^{-1} = 1 &= Z^{-1}Z.
\end{aligned}$$

[Sahi '99] Can we degenerate DAHA as well?

Underlying mechanism

$L_z, R_n \rightarrow$ Zhedanov algebra

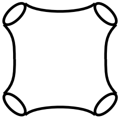
$q \rightarrow 1$



Underlying mechanism

$L_z, R_n \to$ Zhedanov algebra $\hookrightarrow$ DAHA $\check{C}_1 C_1$

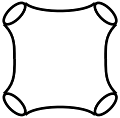
$\downarrow q \to 1$



Underlying mechanism

$L_z, R_n \to$ Zhedanov algebra $\hookrightarrow$ DAHA $\check{C}_1 C_1$

$\downarrow q \to 1$ $\qquad\qquad$ $\downarrow q \to 1$



$Hol(\nabla, \Sigma_{0,4})$

Degenerations of DAHA were constructed in M.M. '16.
Chewing-gum moves correspond to confluences of poles in ∇

[Chekhov,M.M, Rubtsov'18]

Outlook

- R.h.s. of $q - Askey$ scheme.
- Macdonald polynomials.
- The cusped character variety carries a cluster algebra structure - what role does this play in the theory?