Spin Ruijsenaars-Schneider systems from cyclic quivers

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Overview

A fruitful research direction in non-commutative algebraic geometry consists in following the *Kontsevich-Rosenberg principle*: given a classical structure P defined over commutative algebras, a structure P_{nc} on an associative algebra A has algebro-geometric meaning if it induces P on the representation spaces of A. The work of Van den Bergh [5] deals with the introduction of non-commutative Poisson geometry in this context, and it encodes the non-commutative version of (quasi-)Hamiltonian reduction. We explain how to obtain integrable systems in this framework by extending cyclic quivers.

1. Background

Given a unital associative algebra A over $\mathbb{C}$ and $N \in \mathbb{N}^\times$, the representation space $\operatorname{Rep}(A,N)$ is the affine scheme defined by the coordinate ring generated by symbols a_{ij} for $a \in A$, $1 \leq i,j \leq N$, $\mathbb{C}$-linear in a and satisfying

$$\Sigma_j \, a_{ij} b_{jk} = (ab)_{ik}\,, \qquad 1_{ij} = \delta_{ij}\,.$$

If we write $\mathcal{X}(a)$ for the $N \times N$ matrix (a_{ij}) representing a, we get the rules $\mathcal{X}(a)\mathcal{X}(b) = \mathcal{X}(ab)$ and $\mathcal{X}(1) = \operatorname{Id}_N$.

There is a natural $\operatorname{GL}_N(\mathbb{C})$ action on $\operatorname{Rep}(A,N)$ by simultaneous conjugation.

We want a Poisson structure on $\operatorname{Rep}(A,N)$ completely determined on A. Following [5], we put

$$\{a_{ij}, b_{kl}\} := \{\!\{a,b\}\!\}'_{kj}\, \{\!\{a,b\}\!\}''_{il}\,, \qquad (1)$$

where $\{\!\{a,b\}\!\} = \{\!\{a,b\}\!\}' \otimes \{\!\{a,b\}\!\}'' \in A \otimes A$ is obtained from a **double Poisson bracket**

$$\{\!\{-,-\}\!\} : A^{\otimes 2} \to A^{\otimes 2}\,.$$

This bilinear map satisfies non-commutative skewsymmetry/derivation rules, and a Jacobi identity in $A^{\otimes 3}$, making (1) a Poisson bracket.

An element $\mu_A \in A$ is a **moment map** if

$$\{\!\{\mu_A, a\}\!\} = a \otimes 1 - 1 \otimes a\,.$$

Theorem 1 ([5]) *Fix $(A, \{\!\{-,-\}\!\}, \mu_A)$ as above. Using $\mathcal{X}(\mu_A) : \operatorname{Rep}(A,N) \to \mathfrak{gl}_N$, $\lambda \in \mathbb{C}$, the space*

$$\mathcal{X}(\mu_A)^{-1}(\lambda \operatorname{Id}_N) /\!/ \operatorname{GL}_N(\mathbb{C})$$

inherits the Poisson bracket of $\operatorname{Rep}(A,N)$ which is determined by $\{\!\{-,-\}\!\}$ through (1).

Remark 2 *We will use an analogue of Theorem 1 in the quasi-Poisson setting. We end up with a genuine Poisson bracket on a reduced space [5].*

Remark 3 *We can construct double brackets from quivers [5]. We then use a reduction by some diagonal subgroup $\prod_s \operatorname{GL}_{n_s}(\mathbb{C}) \subset \operatorname{GL}_N(\mathbb{C})$.*

References

[1] Chalykh, O., Fairon, M.: *Multiplicative quiver varieties and generalised Ruijsenaars-Schneider models*. J. Geom. Phys. 121, 413–437 (2017). arXiv:1704.05814

[2] Chalykh, O., Fairon, M.: *On the Hamiltonian formulation of the trigonometric spin Ruijsenaars-Schneider system*. Lett. Math. Phys. 110, 2893–2940 (2020). arXiv:1811.08727

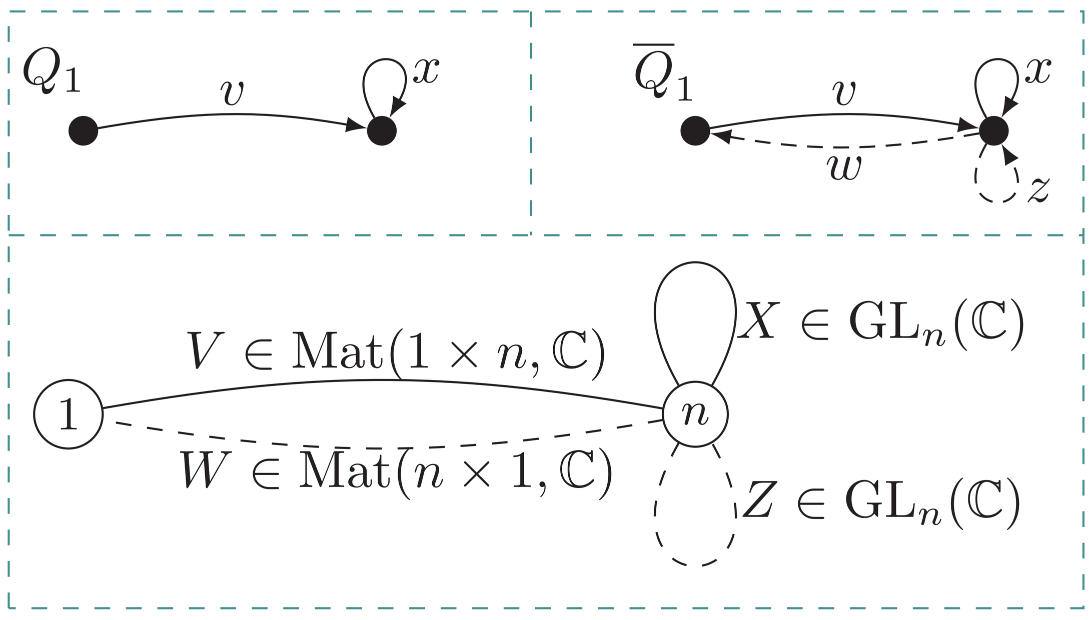
[3] Fairon, M.: *Spin versions of the complex trigonometric Ruijsenaars-Schneider model from cyclic quivers*. J. of Int. Syst. 4, no. 1, xyz008 (2019). arXiv:1811.08717

[4] Fairon, M.: *Multiplicative quiver varieties and integrable particle systems*. PhD thesis, University of Leeds (2019). Available at http://etheses.whiterose.ac.uk/24498/

[5] Van den Bergh, M.: *Double Poisson algebras*. Trans. Amer. Math. Soc., 360 no. 11, 5711–5769 (2008). arXiv:math/0410528

2. Ruijsenaars-Schneider system from a quiver

Idea: We derive a space whose Poisson bracket is determined by a double quasi-Poisson bracket associated with a quiver. We follow the general scheme outlined in Part 1.



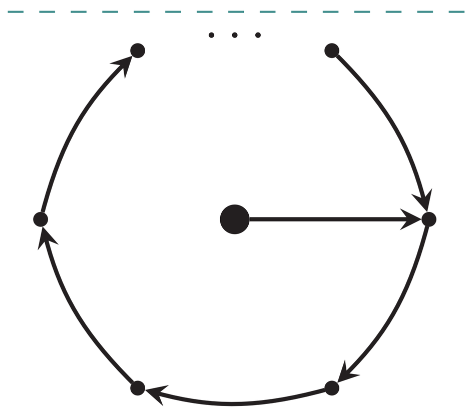
Step 1: Form the double $\overline{Q}_1$ of Q_1. We can define a double quasi-Poisson bracket $\{\!\{-,-\}\!\}$ on a localisation A_1 of the path algebra $\mathbb{C}\overline{Q}_1$.

Step 2: $\operatorname{Rep}(A_1,(1,n))$ is formed of (X,Z,V,W) (see left) with $1 + VW \neq 0$, and inherits a quasi-Poisson bracket by Equation (1).

Step 3: Fixing $q \in \mathbb{C}^\times$, we get a Poisson variety $\mathcal{C}_{n,q} := \{XZX^{-1}Z^{-1} = q(\operatorname{Id}_n + VW)\} /\!/ \operatorname{GL}_n(\mathbb{C})$ and the functions $\left(\operatorname{tr}(Z^k)\right)_{k\in\mathbb{Z}}$ Poisson commute.

Result: We can understand the Poisson structure on $\mathcal{C}_{n,q}$ using the double bracket $\{\!\{-,-\}\!\}$. In local coordinates, Z is the Lax matrix of the complex trigonometric Ruijsenaars-Schneider (RS) system [1].

3.1. First cyclic case

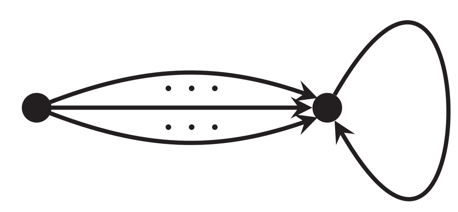


cyclic quiver $Q_1^{(m)}$ on $m \geq 2$ vertices with an extra arrow

Starting with $Q_1^{(m)}$, we follow Steps 1-3 of Part 2 to get a Poisson variety $\mathcal{C}^{(m)}_{n,\mathbf{q}}$ which is locally isomorphic to some $\mathcal{C}_{n,q}$ as a Poisson variety.

We can realise the RS system on $\mathcal{C}^{(m)}_{n,\mathbf{q}}$, as well as cyclic generalisations of this system [1]. Quantum analogues of these different systems have appeared in supersymmetric gauge theory, or in relation to Double Affine Hecke Algebras and MacDonald theory [1].

3.2. Spin RS system



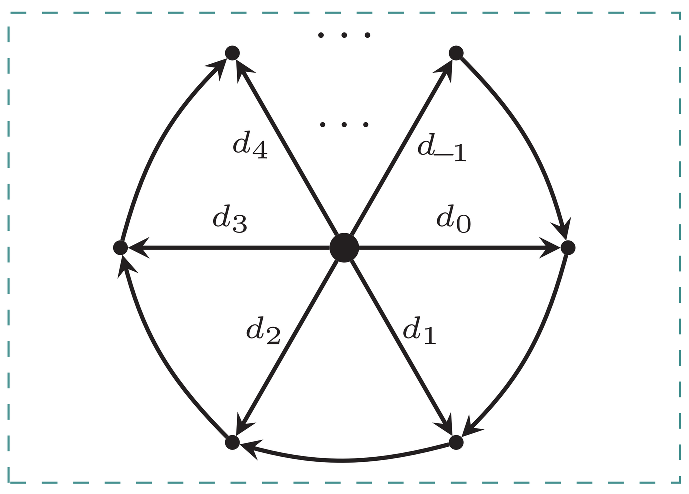
For $d \geq 2$, quiver Q_d obtained from a loop by extension with d arrows

Starting with Q_d, we follow Steps 1-3 of Part 2 to get a Poisson variety $\mathcal{C}_{n,q,d}$ of dimension $2nd$.

We can prove that the functions $\left(\operatorname{tr}(Z^k)\right)_{k\in\mathbb{Z}}$ representing the "double" of the loop-arrow form a degenerate integrable system.

In local coordinates, Z is the Lax matrix of the trigonometric spin RS system [2]. We can also write down the Poisson bracket in terms of those coordinates and solve a conjecture formulated by Arutyunov and Frolov in 1998.

4. Generalised RS systems from cyclic quivers



Fix $m \geq 2$, $\mathbf{d} = (d_s) \in \mathbb{N}^m$, and $\mathbf{q} = (q_s) \in (\mathbb{C}^\times)^m$

Consider $Q^{(m)}_{\mathbf{d}}$ as the cyclic quiver on m vertices with d_s extra arrows to the vertex s in the cycle

We can follow Steps 1-3 of Part 2 to get $\mathcal{C}^{(m)}_{n,\mathbf{q},\mathbf{d}}$, which is a variety with a Poisson bracket induced by a double quasi-Poisson bracket $\{\!\{-,-\}\!\}$

We can explicitly parametrise the space $\mathcal{C}^{(m)}_{n,\mathbf{q},\mathbf{d}}$ in terms of the matrices

$$X_s, Z_s \in \operatorname{GL}_n(\mathbb{C}), \quad V_{s,\alpha} \in \operatorname{Mat}(1 \times n, \mathbb{C}), \; W_{s,\alpha} \in \operatorname{Mat}(n \times 1, \mathbb{C}), \quad 1 \leq \alpha \leq d_s\,,\; 0 \leq s \leq m-1,$$

satisfying the m relations $X_s Z_s X_{s-1}^{-1} Z_{s-1}^{-1} = q_s \prod_{\alpha=1}^{d_s} (\operatorname{Id}_n + W_{s,\alpha} V_{s,\alpha})$, where we take orbits of

$$g \cdot (X_s, Z_s, W_{s,\alpha}, V_{s,\alpha}) = (g_s X_s g_{s+1}^{-1}, g_{s+1} Z_s g_s^{-1}, g_s W_{s,\alpha}, V_{s,\alpha} g_s^{-1})\,, \quad g = (g_s) \in \operatorname{GL}_n(\mathbb{C})^m\,.$$

Result: We can understand the Poisson structure on $\mathcal{C}^{(m)}_{n,\mathbf{q},\mathbf{d}}$ using the double bracket $\{\!\{-,-\}\!\}$. In local coordinates, $Z_\bullet := Z_{m-1} \ldots Z_0$ and $(X_s Z_s)_{s=0}^{m-1}$ can be interpreted as Lax matrices for generalisations of the trigonometric spin RS system, whose symmetric functions are degenerately integrable [4].

The case $\mathbf{d} = (d_0, 0, \ldots, 0)$, $d_0 \geq 2$, is treated in [3]; the subcase $d_0 = 1$ appears in [1] (see Part 3.1).

5. Comments and open problems

- Fix one of the quivers Q described above. The functions forming the integrable system can be lifted to the representation space of $\mathbb{C}\overline{Q}$, where the flows can be constructed explicitly.
- We can understand the action-angle duality of the basic cases as a map "reversing arrows".
- ◦ What is the real version of all these systems?
- ◦ Can we derive other systems (elliptic RS, Van Diejen, ...) from a non-commutative algebra?

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Flash talk: https://tinyurl.com/PosterFairon2021

Availability (Zoom): 13.00-14.00 **BST** on Tuesday 6 and Wednesday 7 April 2021, follow the link https://uofglasgow.zoom.us/j/96090400942 (Meeting ID: 960 9040 0942)