Noncommutative Poisson geometry and integrable systems

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JGPW, 15 September 2020

Plan for the talk

Plenary talk :

1. **Double brackets and associated structures**
2. Relation to integrable systems

Parallel talk :

1. IS from double Poisson brackets
2. IS from double quasi-Poisson brackets

Motivation

Following [Kontsevich, '93] and [Kontsevich-Rosenberg, '99]

$$\begin{array}{ccc} \text{associative } \mathbb{C}\text{-algebra} & \to & \text{commutative } \mathbb{C}\text{-algebra} \\ A & \longrightarrow & \mathbb{C}[\mathrm{Rep}(A,n)] \end{array}$$

$\mathbb{C}[\mathrm{Rep}(A,n)]$ is generated by symbols a_{ij}, $\forall a \in A$, $1 \leq i,j \leq n$.
Rules : $1_{ij} = \delta_{ij}$, $(a+b)_{ij} = a_{ij} + b_{ij}$, $(ab)_{ij} = \sum_k a_{ik} b_{kj}$.

Goal : Find a property P_{nc} on A that gives the usual property P on $\mathbb{C}[\mathrm{Rep}(A,n)]$ for all $n \in \mathbb{N}^\times$

Towards double brackets (1)

Setup : M is a space parametrised by matrices $a^{(1)}, \ldots, a^{(r)} \in \mathfrak{gl}_n(\mathbb{C})$

$\Rightarrow \mathbb{C}[M]$ is generated by $a^{(1)}_{ij}, \ldots, a^{(r)}_{ij}$, for $1 \leq i, j \leq n$

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Assume that M has a Poisson bracket $\{-,-\}$ which has a **nice** form : for any $a, b = a^{(1)}, \dots, a^{(r)}$

$$\{a_{ij}, b_{kl}\} = c_{kj} d_{il} \,, \tag{1}$$

for some $c, d \in W_M := \mathbb{C}\langle a^{(1)}, \dots, a^{(r)} \rangle$

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Can we *symbolically* understand the Poisson bracket with matrices?

Towards double brackets (2)

Trick : write $\{a_{ij}, b_{kl}\} = c_{kj} d_{il}$ as

$$\{\!\{a, b\}\!\}_{kj,il} := \{a_{ij}, b_{kl}\} = (c \otimes d)_{kj,il} \,. \tag{2}$$

As a map $\{\!\{-,-\}\!\} : W_M \times W_M \to W_M \otimes W_M$
(Recall $W_M = \mathbb{C}\langle a^{(1)}, \ldots, a^{(r)} \rangle$. Here $\otimes = \otimes_{\mathbb{C}}$)

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Antisymmetry $\Rightarrow$ $\{\!\{a, b\}\!\} = -\tau_{(12)} \{\!\{b, a\}\!\}$

Leibniz rules :

$$\{\!\{a, bc\}\!\} = (b \otimes \mathrm{Id}_n) \{\!\{a, c\}\!\} + \{\!\{a, b\}\!\} (\mathrm{Id}_n \otimes c) \,,$$
$$\{\!\{ad, b\}\!\} = (\mathrm{Id}_n \otimes a) \{\!\{d, b\}\!\} + \{\!\{a, b\}\!\} (d \otimes \mathrm{Id}_n) \,.$$

Jacobi identity? . . . *a bit of work!*

Double brackets

We follow [Van den Bergh, *double Poisson algebras*,'08]

A denotes an arbitrary f.g. associative $\mathbb{C}$-algebra, $\otimes = \otimes_{\mathbb{C}}$

For $d \in A^{\otimes 2}$, set $d = d' \otimes d''(= \sum_k d'_k \otimes d''_k)$, and $\tau_{(12)} d = d'' \otimes d'$.
Multiplication on $A^{\otimes 2}$: $(a \otimes b)(c \otimes d) = ac \otimes bd$.

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Definition

A *double bracket* on A is a $\mathbb{C}$-bilinear map $\{\!\{-,-\}\!\} : A \times A \to A^{\otimes 2}$ which satisfies

1. $\{\!\{a, b\}\!\} = -\tau_{(12)} \{\!\{b, a\}\!\}$ (cyclic antisymmetry)
2. $\{\!\{a, bc\}\!\} = (b \otimes 1) \{\!\{a, c\}\!\} + \{\!\{a, b\}\!\} (1 \otimes c)$ (outer derivation)
3. $\{\!\{ad, b\}\!\} = (1 \otimes a) \{\!\{d, b\}\!\} + \{\!\{a, b\}\!\} (d \otimes 1)$ (inner derivation)

Double Poisson bracket

Recall $d = d' \otimes d'' \in A^{\otimes 2}$ (notation) $\rightsquigarrow \{\!\{a,b\}\!\} = \{\!\{a,b\}\!\}' \otimes \{\!\{a,b\}\!\}''$

From a double bracket $\{\!\{-,-\}\!\}$, define $\{\!\{-,-,-\}\!\} : A^{\times 3} \to A^{\otimes 3}$

$$\begin{aligned} \{\!\{a,b,c\}\!\} = & \quad \left\{\!\!\left\{a, \{\!\{b,c\}\!\}'\right\}\!\!\right\} \otimes \{\!\{b,c\}\!\}'' \\ & + \tau_{(123)} \left\{\!\!\left\{b, \{\!\{c,a\}\!\}'\right\}\!\!\right\} \otimes \{\!\{c,a\}\!\}'' \\ & + \tau_{(132)} \left\{\!\!\left\{c, \{\!\{a,b\}\!\}'\right\}\!\!\right\} \otimes \{\!\{a,b\}\!\}'' \,, \quad \forall a,b,c \in A \end{aligned}$$

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Definition

A double bracket $\{\!\{-,-\}\!\}$ is *Poisson* if $\{\!\{-,-,-\}\!\} : A^{\times 3} \to A^{\otimes 3}$ vanishes.

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Example

1. $A = \mathbb{C}[x]$, $\{\!\{x,x\}\!\} = x \otimes 1 - 1 \otimes x$.
2. $A = \mathbb{C}\langle x,y \rangle$, $\{\!\{x,x\}\!\} = 0 = \{\!\{y,y\}\!\}$, $\{\!\{x,y\}\!\} = 1 \otimes 1$.

A first result

(notation) $\rightsquigarrow \{\!\{a,b\}\!\} = \{\!\{a,b\}\!\}' \otimes \{\!\{a,b\}\!\}''$

Proposition (Van den Bergh,'08)

If A has a double bracket $\{\!\{-,-\}\!\}$, then $\mathbb{C}[\mathrm{Rep}(A,n)]$ has a unique antisymmetric biderivation $\{-,-\}_{\mathrm{P}}$ satisfying

$$\{a_{ij}, b_{kl}\}_{\mathrm{P}} = \{\!\{a,b\}\!\}'_{kj} \, \{\!\{a,b\}\!\}''_{il} \,. \tag{3}$$

If $\{\!\{-,-\}\!\}$ is Poisson, then $\{-,-\}_{\mathrm{P}}$ is a Poisson bracket.

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If $\{\!\{-,-\}\!\}$ is Poisson, then $\{-,-\}_{\mathrm{P}}$ is a Poisson bracket.

Example

$A = \mathbb{C}[x]$, $\{\!\{x,x\}\!\} = x \otimes 1 - 1 \otimes x$ endows $\mathfrak{gl}_n(\mathbb{C}) = \mathbb{C}[\mathrm{Rep}(A,n)]$ with

$$\{x_{ij}, x_{kl}\}_{\mathrm{P}} = x_{kj}\delta_{il} - \delta_{kj}x_{il}\ .$$

This is (up to sign) KKS Poisson bracket on $\mathfrak{gl}_n \simeq \mathfrak{gl}_n^*$

A first dictionary

Algebra A	Geometry $\mathbb{C}[\mathrm{Rep}(A,n)]$
double bracket $\{\!\{-,-\}\!\}$	anti-symmetric biderivation $\{-,-\}_{\mathrm{P}}$
double Poisson bracket $\{\!\{-,-\}\!\}$	Poisson bracket $\{-,-\}_{\mathrm{P}}$

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Problem : "nice" integrable systems usually live on reduced phase spaces

(e.g. Calogero-Moser, Ruijsenaars-Schneider systems)

Poisson reduction

Lemma (Van den Bergh,'08)

If A has a double Poisson bracket $\{\!\{-,-\}\!\}$, the following defines a Lie bracket on $H_0(A) = A/[A,A]$

$$\{a,b\} = \{\!\{a,b\}\!\}' \, \{\!\{a,b\}\!\}'' \tag{4}$$

(for (4) we take lifts in A then $A \xrightarrow{\{\!\{-,-\}\!\}} A \otimes A \xrightarrow{\mathrm{m}} A \to H_0(A)$)

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Let $\mathcal{X}(a)$ be such that $\mathcal{X}(a)_{ij} = a_{ij} \in \mathbb{C}[\mathrm{Rep}(A,n)]$

Then $\mathrm{tr}\, \mathcal{X}(a) \in \mathbb{C}[\mathrm{Rep}(A,n)]^{\mathrm{GL}_n} = \mathbb{C}[\mathrm{Rep}(A,n)/\!/ \, \mathrm{GL}_n]$

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Proposition (Van den Bergh,'08)

The Poisson structure $\{-,-\}_{\mathrm{P}}$ on $\mathrm{Rep}(A,n)$ descends to $\mathrm{Rep}(A,n)/\!/\,\mathrm{GL}_n$ in such a way that

$$\{\mathrm{tr}\,\mathcal{X}(a), \mathrm{tr}\,\mathcal{X}(b)\}_{\mathrm{P}} = \mathrm{tr}\,\mathcal{X}(\{a,b\})\,. \qquad (5)$$

A second dictionary

Algebra A	Geometry $\mathbb{C}[\mathrm{Rep}(A,n)]$
double bracket $\{\!\{-,-\}\!\}$	anti-symmetric biderivation $\{-,-\}_{\mathrm{P}}$
double Poisson bracket $\{\!\{-,-\}\!\}$	Poisson bracket $\{-,-\}_{\mathrm{P}}$
$(H_0(A),\{-,-\})$ is Lie algebra	$\{-,-\}_{\mathrm{P}}$ is Poisson on $\mathbb{C}[\mathrm{Rep}(A,n)]^{\mathrm{GL}_n}$

Recall $\{-,-\} = \mathrm{m} \circ \{\!\{-,-\}\!\}$ on $H_0(A) = A/[A,A]$

Hamiltonian reduction

double Poisson bracket $\{\!\{-,-\}\!\}$ on $A \rightsquigarrow \{-,-\} = \mathrm{m} \circ \{\!\{-,-\}\!\}$ descends to $H_0(A) = A/[A,A]$

Definition

$\mu_A \in A$ is a *moment map* if $\{\!\{\mu_A, a\}\!\} = a \otimes 1 - 1 \otimes a$, $\forall a \in A$

Hamiltonian reduction

double Poisson bracket $\{\!\{-,-\}\!\}$ on A $\rightsquigarrow$ $\{-,-\} = \mathrm{m} \circ \{\!\{-,-\}\!\}$ descends to $H_0(A) = A/[A,A]$

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$\mu_A \in A$ is a *moment map* if $\{\!\{\mu_A, a\}\!\} = a \otimes 1 - 1 \otimes a$, $\forall a \in A$

For any $\lambda \in \mathbb{C}$, $\{\mu_A - \lambda, a\} = 0$.

$\Rightarrow$ Lie bracket $\{-,-\}$ descends to $H_0(A_\lambda)$ for $A_\lambda := A/\langle \mu_A - \lambda \rangle$

Hamiltonian reduction

double Poisson bracket $\{\!\{-,-\}\!\}$ on $A \rightsquigarrow \{-,-\} = \mathsf{m} \circ \{\!\{-,-\}\!\}$ descends to $H_0(A) = A/[A,A]$

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Proposition (Van den Bergh,'08)

The Poisson structure $\{-,-\}_{\mathrm{P}}$ *on* $\mathrm{Rep}(A,n)$ *descends to* $\mathrm{Rep}(A_\lambda, n)/\!/\,\mathrm{GL}_n$ *in such a way that*

$$\{\mathrm{tr}\,\mathcal{X}(a), \mathrm{tr}\,\mathcal{X}(b)\}_{\mathrm{P}} = \mathrm{tr}\,\mathcal{X}(\{a,b\})\,. \qquad (6)$$

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$(A/[A,A],\{-,-\})$ is Lie algebra	$\{-,-\}_{\mathrm{P}}$ is Poisson on $\mathbb{C}[\mathrm{Rep}(A,n)]^{\mathrm{GL}_n}$
moment map μ_A $A_\lambda = A/(\mu_A - \lambda)$, $\lambda \in \mathbb{C}$ $(A_\lambda/[A_\lambda,A_\lambda],\{-,-\})$ is Lie algebra	moment map $\mathcal{X}(\mu_A)$ slice $S_\lambda := \mathcal{X}(\mu_A)^{-1}(\lambda\,\mathrm{Id}_n)$ $\{-,-\}_{\mathrm{P}}$ is Poisson on $\mathbb{C}[S_\lambda//\,\mathrm{GL}_n]$

Recall $\{-,-\} = \mathrm{m} \circ \{\!\{-,-\}\!\}$ on $H_0(A) = A/[A,A]$

Plan for the talk

Plenary talk :

1. Double brackets and associated structures
2. **Relation to integrable systems**

Parallel talk :

1. IS from double Poisson brackets
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What can we do with double Poisson brackets?

Relation $A \longrightarrow \mathrm{Rep}(A,n)//\,\mathrm{GL}_n(\mathbb{C})$ (or Hamiltonian reduction)

$$\{\mathrm{tr}\,\mathcal{X}(a), \mathrm{tr}\,\mathcal{X}(b)\}_{\mathrm{P}} = \mathrm{tr}\,\mathcal{X}(\{\!\{a,b\}\!\}'\,\{\!\{a,b\}\!\}'') \,.$$

Lemma

If the product $\{\!\{a,b\}\!\}'\,\{\!\{a,b\}\!\}''$ is a commutator, then the functions $\mathrm{tr}\,\mathcal{X}(a), \mathrm{tr}\,\mathcal{X}(b)$ Poisson commute

$\Rightarrow$ We should try to find a "big" family of elements $(a_i)_{i\in I} \subset A$ such that $m \circ \{\!\{a_i, a_j\}\!\} \in [A,A]$

Side remark : The functional independence of the corresponding functions $(\mathrm{tr}\,\mathcal{X}(a_i))_{i\in I}$ seems to be a purely *geometric* feature. I do not see how to understand it at the level of A (yet?)

A first criterion

Lemma (The "Lax Lemma")

Assume that $a \in A$ *satisfies* $\{\!\{a, a\}\!\} = \sum_{s \in \mathbb{N}} (a^s \otimes b_s - b_s \otimes a^s)$ *for finitely many nonzero* $b_s \in A$. *Then the matrix* $\mathcal{X}(a)$ *is a Lax matrix, i.e.* $\{\operatorname{tr} \mathcal{X}(a)^k, \operatorname{tr} \mathcal{X}(a)^l\} = 0$ *for any* $k, l \in \mathbb{N}$.

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$$\left\{\!\!\left\{ a^k, a^l \right\}\!\!\right\} = \sum_{\kappa=0}^{k-1} \sum_{\lambda=0}^{l-1} (a^\lambda \otimes a^\kappa) \{\!\{a, a\}\!\} (a^{k-\kappa-1} \otimes a^{l-\lambda-1})$$

$\Rightarrow \quad m \circ \left\{\!\!\left\{ a^k, a^l \right\}\!\!\right\}$ vanishes modulo commutators. $\square$

A non-example

Lemma (Weakest "Lax Lemma")

If $a \in A$ *satisfies* $\{\!\{a, a\}\!\} = 0$*, then* $\{\operatorname{tr} \mathcal{X}(a)^k, \operatorname{tr} \mathcal{X}(a)^l\} = 0\,.$

A non-example

Lemma (Weakest "Lax Lemma")

If $a \in A$ *satisfies* $\{\!\{a, a\}\!\} = 0$*, then* $\{\operatorname{tr} \mathcal{X}(a)^k, \operatorname{tr} \mathcal{X}(a)^l\} = 0\,.$

Example

$A = \mathbb{C}\langle x, y\rangle$, $\{\!\{x, x\}\!\} = 0 = \{\!\{y, y\}\!\}$, $\{\!\{x, y\}\!\} = 1 \otimes 1.$

Moment map : $\mu_A = xy - yx.$

$\Rightarrow$ for all $\lambda \in \mathbb{C}$, $(\operatorname{tr} \mathcal{X}(y)^k)_k$ Poisson commute on

$$\operatorname{Rep}(A/(\mu_A - \lambda), n)/\!/ \operatorname{GL}_n = \{(X, Y) \mid XY - YX = \lambda \operatorname{Id}_n\}/\!/ \operatorname{GL}_n$$

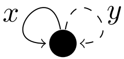
How to get something interesting?

[Van den Bergh,'08] $\longrightarrow$ a double Poisson bracket for any quiver

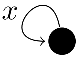
Example

$$A = \mathbb{C}\langle x, y\rangle = \mathbb{C}\bar{Q}_\circ$$

$\bar{Q}_\circ$



$Q_\circ$



The data $\{\!\{x, x\}\!\} = 0 = \{\!\{y, y\}\!\}$, $\{\!\{x, y\}\!\} = 1 \otimes 1$, $\mu_A = xy - yx$ is encoded in $Q_\circ$.

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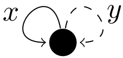

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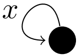
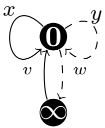


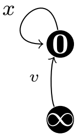
The data $\{\!\{x,x\}\!\} = 0 = \{\!\{y,y\}\!\}$, $\{\!\{x,y\}\!\} = 1 \otimes 1$, $\mu_A = xy - yx$ is encoded in $Q_\circ$.

We get an interesting IS by framing :

$\bar{Q}_1$



Q_1



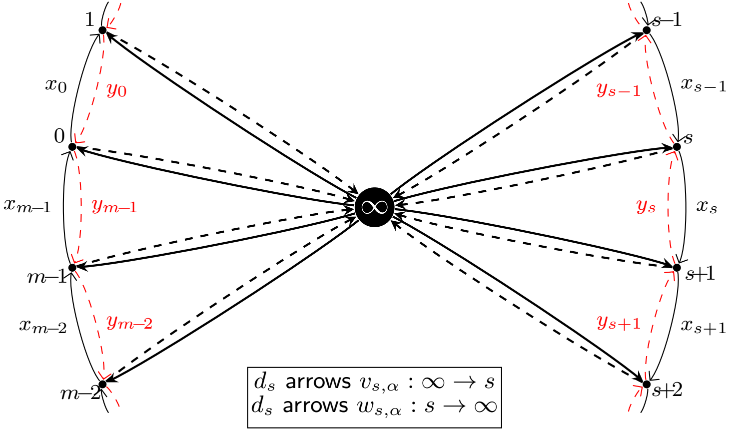
Attach $\mathbb{C}^n$ at 0, $\mathbb{C}$ at ∞

$\Rightarrow$ Calogero-Moser space [Wilson,'98]

$(\operatorname{tr} \mathcal{X}(y)^k)_{k=1}^n$ define CM system

Generalisation

[Chalykh-Silantyev,'17] $\longrightarrow$ cyclic quivers give generalised CM systems

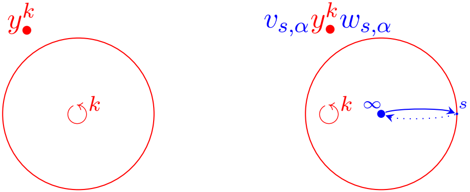


$y_\bullet = y_{m-1} \cdots y_1 y_0$: $\{\!\{y_\bullet, y_\bullet\}\!\} = 0$, so $\{\mathrm{tr}\,\mathcal{X}(y_\bullet)^k, \mathrm{tr}\,\mathcal{X}(y_\bullet)^l\} = 0$

Generalisation (bis)

[Chalykh-Silantyev,'17] $\longrightarrow$ cyclic quivers give generalised CM systems

We can visualise commuting elements :



True for any framing + maximally superintegrable [F.-Görbe, in prep.]

More in the parallel session !

$\rightsquigarrow$ details on rational CM system
$\rightsquigarrow$ elliptic CM system

$\rightsquigarrow$ double quasi-Poisson brackets and IS
$\rightsquigarrow$ trigonometric RS systems from cyclic quivers

Interested to know where double brackets pop up in maths? Check :
`www.maths.gla.ac.uk/~mfairon/DoubleBrackets.html` (soon updated!)

Maxime Fairon
Maxime.Fairon@glasgow.ac.uk

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Reminder

A has double Poisson bracket $\{\!\{-,-\}\!\}$
$\longrightarrow \operatorname{Rep}(A,n)/\!/\operatorname{GL}_n(\mathbb{C})$ (or Hamiltonian reduction) has Poisson bracket for

$$\{\operatorname{tr}\mathcal{X}(a), \operatorname{tr}\mathcal{X}(b)\}_{\mathrm{P}} = \operatorname{tr}\mathcal{X}(\{\!\{a,b\}\!\}'\,\{\!\{a,b\}\!\}'')\,,$$

Lemma

If the product $\{\!\{a,b\}\!\}'\,\{\!\{a,b\}\!\}''$ is a commutator, then the functions $\operatorname{tr}\mathcal{X}(a), \operatorname{tr}\mathcal{X}(b)$ Poisson commute

$\Rightarrow$ find "many" $(a_i)_{i\in I} \subset A$ such that $m \circ \{\!\{a_i, a_j\}\!\} \in [A,A]$

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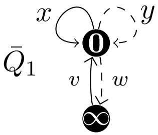
Framed Jordan quiver

$\bar{A} = \mathbb{C}\bar{Q}_1$ is a B-algebra, $B = \mathbb{C}e_0 \oplus \mathbb{C}e_\infty$



$$\{\!\{y, x\}\!\} = e_0 \otimes e_0,$$

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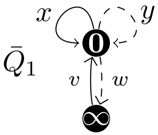

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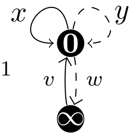
other double brackets are zero

Now $\mu = [x, y] + [v, w]$ is moment map.

Get Lie bracket on $H_0(\bar{A}_\lambda) = \bar{A}_\lambda / [\bar{A}_\lambda, \bar{A}_\lambda]$
where $\bar{A}_\lambda = \bar{A}/([x, y] - wv = \lambda_0 e_0, vw = \lambda_\infty e_\infty)$, for $\lambda_0, \lambda_\infty \in \mathbb{C}$

Calogero-Moser space (1)

$\bar{Q}_1$



$\bar{A}_\lambda = \bar{A}/J_\lambda, \qquad J_\lambda = \langle [x,y] - wv = \lambda_0 e_0, vw = \lambda_\infty e_\infty \rangle$

For $\lambda_0 \in \mathbb{C}^\times, \lambda_\infty = -n\lambda_0$, 'Attach' $\mathbb{C}^n$ at 0 and $\mathbb{C}$ at ∞

$$\Rightarrow \text{get } \mathcal{M}_\lambda := \mathrm{Rep}(\bar{A}_\lambda, (1,n))$$

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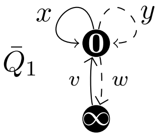

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For $g \cdot (X,Y,V,W) = (gXg^{-1}, gYg^{-1}, Vg^{-1}, gW)$, $g \in \operatorname{GL}_n$,

$$\mathcal{M}_\lambda // \operatorname{GL}_n = \operatorname{Spec}(\mathbb{C}[\operatorname{Rep}(\bar{A}_\lambda, (1,n))]^{\operatorname{GL}_n})$$

This is n-th Calogero-Moser space [Wilson, 98]

$(\operatorname{tr} Y^k)$ Poisson commute by the Lax lemma. They form an IS by counting

Calogero-Moser space (2)

$$\mathcal{M} := \{X, Y \in \mathfrak{gl}_n, V \in \mathrm{Mat}_{1\times n}, W \in \mathrm{Mat}_{n\times 1}\}$$
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GL_n action : $g \cdot (X, Y, V, W) = (gXg^{-1}, gYg^{-1}, Vg^{-1}, gW)$

On dense subset of $\mathcal{M}_\lambda /\!/ \mathrm{GL}_n$, choose

- $X = \mathrm{diag}(q_1, \dots, q_n)$
- $V = (1, \dots, 1)$

then for Darboux coordinates (q_i, p_i),

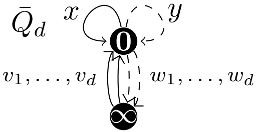
$$W = -\lambda_0 (1, \dots, 1)^T, \quad Y_{ij} = \delta_{ij} p_j - \delta_{(i\neq j)} \frac{\lambda_0}{q_i - q_j}$$

Calogero-Moser Hamiltonian :

$$\frac{1}{2} \mathrm{tr}\, Y^2 = \frac{1}{2} \sum_j p_j^2 - \sum_{i \neq j} \frac{\lambda_0^2}{(q_i - q_j)^2}$$

Spin Calogero-Moser space (1)

[Bielawski-Pidstrygach,'10; Tacchella,'15; Chalykh-Silantyev,'17]



$d \geq 2$. $\bar{A} = \mathbb{C}\bar{Q}_d$

$$\mu_0 = [x,y] - \textstyle\sum_\alpha w_\alpha v_\alpha$$
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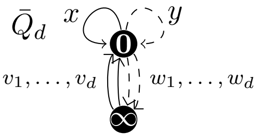

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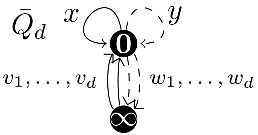

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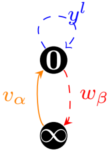
This is n-th Calogero-Moser space with d spins/degrees of freedom

$(\mathrm{tr}\, Y^k)$ Poisson commute but only n functionally independent...

Spin Calogero-Moser space (2)

Using the double bracket, we can compute Poisson brackets on $\mathcal{M}_\lambda /\!/ \mathrm{GL}_n$ for $\operatorname{tr} Y^k$ and $t^l_{\alpha\beta} = V_\alpha Y^l W_\beta$.



$$\{\operatorname{tr} Y^k, \operatorname{tr} Y^l\}_{\mathrm{P}} = 0 = \{\operatorname{tr} Y^k, t^l_{\alpha\beta}\}_{\mathrm{P}}$$

$$\{t^k_{\alpha\beta}, t^l_{\gamma\epsilon}\}_{\mathrm{P}} = \delta_{\beta\gamma} t^{k+l}_{\alpha\epsilon} - \delta_{\alpha\epsilon} t^{k+l}_{\gamma\beta}$$

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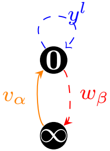

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Proposition

The commutative algebra generated by the elements $(\operatorname{tr} Y^k, t^k_{\alpha\alpha})$, $1 \leq \alpha \leq d$, *is a Poisson-commutative subalgebra of* $\mathbb{C}[\mathcal{M}_\lambda /\!/ \operatorname{GL}_n]$ *of dimension* nd.

Hence, we get Liouville integrability for any $\operatorname{tr} Y^k$.

What else from double Poisson bracket?

- As in plenary part of the talk $\Rightarrow$ framed cyclic quivers
- Can understand elliptic CM system [Chalykh-F., in prep.]

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Tool : non-commutative tangent space of an algebra A_0 has a double Poisson bracket [VdB,'08]

Remark : $A_0 = \mathbb{C}Q \rightsquigarrow A = \mathbb{C}\bar{Q}$ with previous $\{\!\{-,-\}\!\}$

Some details : end of Section 6.1 in [F., PhD thesis], ETHESES.WHITEROSE.AC.UK/24498/

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Method : apply this to $A_0 = \mathbb{C}[\mathcal{E}]$ for punctured elliptic curve $\mathcal{E}$

Remark : two punctures shifted by μ for the spectral parameter μ of Lax matrix of elliptic CM

Some details : end of Section 6.1 in [F., PhD thesis], ETHESES.WHITEROSE.AC.UK/24498/

Plan for the talk

Plenary talk :

1. Double brackets and associated structures
2. Relation to integrable systems

Parallel talk :

1. IS from double Poisson brackets
2. **IS from double quasi-Poisson brackets**

quasi-Dictionary

Recall $\{-,-\} = \mathrm{m} \circ \{\!\{-,-\}\!\}$ on $H_0(A) = A/[A,A]$

Algebra A	**Geometry** $\mathbb{C}[\mathrm{Rep}(A,n)]$
double bracket $\{\!\{-,-\}\!\}$	anti-symmetric biderivation $\{-,-\}_{\mathrm{P}}$
double quasi-Poisson bracket $\{\!\{-,-\}\!\}$	quasi-Poisson bracket $\{-,-\}_{\mathrm{P}}$
$(A/[A,A],\{-,-\})$ is Lie algebra	$\{-,-\}_{\mathrm{P}}$ is Poisson on $\mathbb{C}[\mathrm{Rep}(A,n)]^{\mathrm{GL}_n}$
multiplicative moment map Φ_A $A_q = A/(\Phi_A - q)$, $q \in \mathbb{C}^\times$ $(A_q/[A_q,A_q],\{-,-\})$ is Lie algebra	multiplicative moment map $\mathcal{X}(\Phi_A)$ slice $S_q := \mathcal{X}(\Phi_A)^{-1}(q\,\mathrm{Id}_n)$ $\{-,-\}_{\mathrm{P}}$ is Poisson on $\mathbb{C}[S_q/\!/\,\mathrm{GL}_n]$

$$\{\mathrm{tr}\,\mathcal{X}(a), \mathrm{tr}\,\mathcal{X}(b)\}_{\mathrm{P}} = \mathrm{tr}\,\mathcal{X}(\{\!\{a,b\}\!\}'\,\{\!\{a,b\}\!\}'')$$

Jordan quiver

There exists an algebra with a double *quasi*-Poisson bracket and a moment map associated to ANY quiver [VdB,'08]

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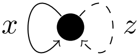
double Jordan quiver $\bar{Q}_0$



$$A = (\mathbb{C}\bar{Q}_0)_{x,z} = \mathbb{C}\langle x^{\pm 1}, z^{\pm 1} \rangle$$

$$\{\!\{z, z\}\!\} = \frac{1}{2}(1 \otimes z^2 - z^2 \otimes 1)$$

$+$ complicated bracket...

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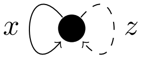

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$+$ complicated bracket...

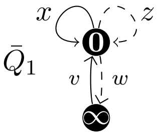
$\Rightarrow$ get $\{\operatorname{tr} \mathcal{X}(z)^k, \operatorname{tr} \mathcal{X}(z)^l\}_{\mathrm{P}} = 0$ on rep. spaces by the Lax lemma
$\Rightarrow$ $Z := \mathcal{X}(z)$ could be an interesting Lax matrix

Remark

Moment map : $\Phi = xzx^{-1}z^{-1}$
Get Lie bracket on $H_0(A_q)$ for $A_q := A/(xzx^{-1}z^{-1} - q)$, $q \in \mathbb{C}^\times$.

Ruijsenaars-Schneider space

Non-spin case / Case $d = 1$: [Chalykh-F.,'17 - 1704.05814]



$\bar{A} = \mathbb{C}\bar{Q}_1$ localised at $x, z, (e_0 + vw), (e_\infty + wv)$,

$\bar{A}$ has double quasi-Poisson bracket/moment map

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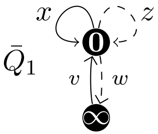

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$$\mathcal{M} := \{X, Z \in \mathrm{GL}_n, V \in \mathrm{Mat}_{1\times n}, W \in \mathrm{Mat}_{n\times 1}\}$$

$$\mathcal{M}_q /\!/ \mathrm{GL}_n := \{XZX^{-1}Z^{-1}(\mathrm{Id}_n + WV)^{-1} = q\,\mathrm{Id}_n\} /\!/ \mathrm{GL}_n$$

(q is *not* a n-th root of unity)

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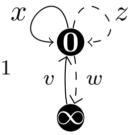

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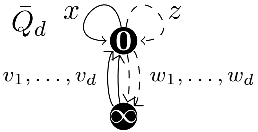
Z is Lax matrix for trigonometric Ruijsenaars-Schneider system
$\mathrm{tr}\, Z, \ldots, \mathrm{tr}\, Z^n$ form an integrable system by the Lax lemma.

(q is *not* a n-th root of unity)

Spin Ruijsenaars-Schneider space (1)

Spin case / Case $d \geq 2$ [Chalykh-F.,'20 / 1811.08727]



$\bar{A} = \mathbb{C}\bar{Q}_d$ suitably localised
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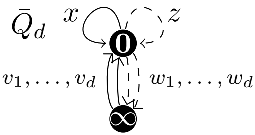

$\bar{A} = \mathbb{C}\bar{Q}_d$ suitably localised
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$$\mathcal{M} := \{X, Z \in \mathrm{GL}_n, V_\alpha \in \mathrm{Mat}_{1\times n}, W_\alpha \in \mathrm{Mat}_{n\times 1}\}$$

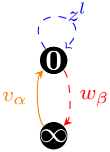
$$\Rightarrow \mathcal{C}_{n,q,d} := \{XZX^{-1}Z^{-1} \prod_{1\leq\alpha\leq d}^{\rightarrow} (\mathrm{Id}_n + W_\alpha V_\alpha)^{-1} = q\,\mathrm{Id}_n\} /\!/ \mathrm{GL}_n$$

Rearrange as $XZX^{-1} - qZ = q\mathcal{A}\mathcal{C}$, $\rightsquigarrow$ Z is spin trigo RS Lax matrix

(q is *not* a n-th root of unity)

Spin Ruijsenaars-Schneider space (2)

Introduce notation $t^l_{\alpha\beta} = V_\alpha Z^l W_\beta$.



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Can we form an integrable system by extending $\operatorname{tr} Z, \dots, \operatorname{tr} Z^n$?

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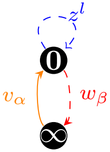

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Can we form an integrable system by extending $\operatorname{tr} Z, \dots, \operatorname{tr} Z^n$?

Problem: $\{t^k_{\alpha\beta}, t^l_{\gamma\epsilon}\}_{\mathrm{P}}$ is VERY complicated !

Spin Ruijsenaars-Schneider space (3)

Idea : on $\mathcal{C}_{n,q,d}$ we have from moment map

$$XZX^{-1} = qS_d\,, \quad S_d := (\mathrm{Id}_n + W_d V_d)\dots(\mathrm{Id}_n + W_1 V_1)Z$$

So $\operatorname{tr} S_d^k$ Poisson commute will all $t_{\alpha\beta}^l = V_\alpha Z^l W_\beta$ on $\mathcal{C}_{n,q,d}$.

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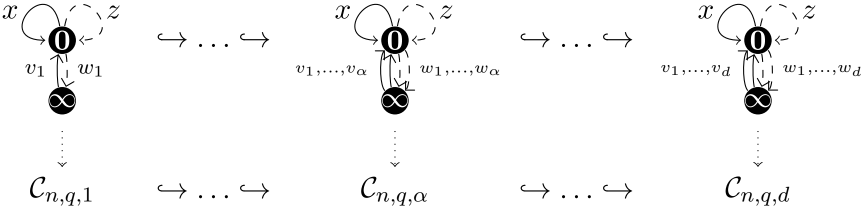


$\Rightarrow$ for $S_\alpha := (\mathrm{Id}_n + W_\alpha V_\alpha)\dots(\mathrm{Id}_n + W_1 V_1)Z$, $\mathrm{tr}\, S_\alpha^k \in \mathbb{C}[\mathcal{C}_{n,q,d}]$ and $\mathrm{tr}\, S_\alpha^k$ Poisson commute with any $t^l_{\gamma\beta} = V_\alpha Z^l W_\beta$, $1 \le \gamma, \beta \le \alpha$

Spin Ruijsenaars-Schneider space (4)

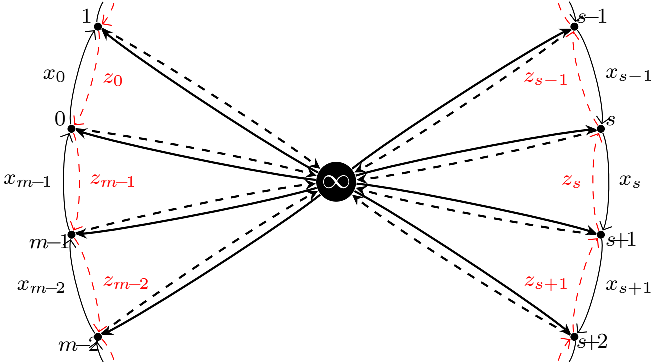
For $S_\alpha := (\mathrm{Id}_n + W_\alpha V_\alpha) \dots (\mathrm{Id}_n + W_1 V_1) Z$, $\operatorname{tr} S_\alpha^k \in \mathbb{C}[\mathcal{C}_{n,q,d}]$

Proposition (Chalykh-F.)

The elements $\operatorname{tr} S_\alpha^k$, $1 \leq \alpha \leq d$, $k = 1, \dots, n$, *form an integrable system.*

(Involutivity is checked with double brackets !)

Generalisation



Proposition

For suitable dimension vector and for generic parameters, there exists an integrable system containing $\operatorname{tr} Z_\bullet^k$, $k = 1, \ldots, n$, $z_\bullet = z_{m-1} \ldots z_1 z_0$.

[Chalykh-F.,'17] for one framing arrow, [F.,'19] for d framing arrows to one vertex

[F.,PhD thesis] (ETHESES.WHITEROSE.AC.UK/24498/) for general case

Thank you for your attention

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