Double (quasi-)Poisson algebras and their morphisms

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Algebra Seminar
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Plan for the talk

1. Motivation
2. Double brackets
3. Morphisms of double Poisson brackets
4. The “quasi-” case
Quiver varieties

(we do not consider stability parameter)

Fix: quiver $Q$, dimension vector $\alpha \in \mathbb{N}^I$, parameter $\lambda \in \mathbb{C}^I$

- Consider double $\bar{Q}$ (add $a^* : h \to t$ for each $a : t \to h$ in $Q$)
- Construct $\Pi^\lambda(Q) = \mathbb{C}\bar{Q}/\langle \sum_{a \in Q} [a, a^*] - \sum_{s \in I} \lambda_s e_s \rangle$
Quiver varieties

(we do not consider stability parameter)

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– Construct $\Pi^\lambda(Q) = \mathbb{C}\bar{Q}/\langle \sum_{a \in Q} [a, a^*] - \sum_{s \in I} \lambda_s e_s \rangle$

Get: A quiver variety:

$$\mathcal{M}_{\alpha, \lambda}^\Pi(Q) = \text{Rep}(\Pi^\lambda(Q), \alpha) // \text{GL}(\alpha)$$

which is a Poisson variety for

$$(\mathcal{X}(b) \in \text{Mat}_{\alpha_t(b) \times \alpha_h(b)}(\mathbb{C}) \ \forall b \in \bar{Q})$$

$$\{ \mathcal{X}(a)_{ij}, \mathcal{X}(a^*)_{kl} \}_P = (\text{Id}_{\alpha_{h(a)}})_{kj} (\text{Id}_{\alpha_{t(a)}})_{il}$$
Quiver varieties

(we do not consider stability parameter)

Fix: quiver $Q$, dimension vector $\alpha \in \mathbb{N}^I$, parameter $\lambda \in \mathbb{C}^I$

– Consider double $\tilde{Q}$ (add $a^* : h \to t$ for each $a : t \to h$ in $Q$)

– Construct $\Pi^\lambda(Q) = \mathbb{C}\tilde{Q}/\langle \sum_{a \in Q}[a, a^*] - \sum_{s \in I} \lambda_se_s \rangle$

Get: A quiver variety :

$$\mathcal{M}_{\Pi, \lambda}^\Pi(Q) = \text{Rep}(\Pi^\lambda(Q), \alpha)// \text{GL}(\alpha)$$

which is a Poisson variety for

$$(\mathcal{X}(b) \in \text{Mat}_{\alpha t(b) \times \alpha h(b)}(\mathbb{C}) \forall b \in \tilde{Q})$$

$$\{X(a)_{ij}, X(a^*)_{kl}\}_P = (\text{Id}_{\alpha h(a)})_{kj}(\text{Id}_{\alpha t(a)})_{il}$$

$\mathcal{M}_{\alpha, \lambda}^\Pi(Q)$ only depends on $\alpha, \lambda$ and $Q$ seen as an undirected graph, up to isomorphism of Poisson varieties (easy)
Multiplicative quiver varieties

Fix: quiver $Q$, dimension vector $\alpha \in \mathbb{N}^I$, parameter $q \in (\mathbb{C}^\times)^I$

– Algebra $A_Q$ is localisation of $\mathbb{C}\bar{Q}$ at all $1 + aa^*, 1 + a^*a$

– Construct $\Lambda^q(Q) = A_Q / \left< \prod_{\text{order}} (1 + aa^*)(1 + a^*a)^{-1} - \sum_{s \in I} q_s e_s \right>$
Fix: quiver $Q$, dimension vector $\alpha \in \mathbb{N}^I$, parameter $q \in (\mathbb{C}^\times)^I$

- Algebra $A_Q$ is localisation of $\mathbb{C} \bar{Q}$ at all $1 + aa^*, 1 + a^*a$

- Construct $\Lambda^q(Q) = A_Q / \left\langle \prod_{\text{order}} (1 + aa^*)(1 + a^*a)^{-1} - \sum_{s \in I} q_se_s \right\rangle$

Get: A multiplicative quiver variety [Crawley-Boevey - Shaw,04] :

$$\mathcal{M}_{\alpha,q}^\Lambda(Q) = \text{Rep}(\Lambda^q(Q), \alpha) // \text{GL}(\alpha)$$

These are Poisson varieties [Van den Bergh,08]
Multiplicative quiver varieties

Fix: quiver $Q$, dimension vector $\alpha \in \mathbb{N}^I$, parameter $q \in (\mathbb{C}^\times)^I$

- Algebra $A_Q$ is localisation of $\mathbb{C}\overline{Q}$ at all $1 + aa^*$, $1 + a^*a$

- Construct $\Lambda^q(Q) = A_Q / \left( \prod_{\text{order}} (1 + aa^*)(1 + a^*a)^{-1} - \sum_{s \in I} q_s e_s \right)$

Get: A \textit{multiplicative quiver variety} [Crawley-Boevey - Shaw,04] :

$$\mathcal{M}_{\alpha,q}^\Lambda(Q) = \text{Rep}(\Lambda^q(Q), \alpha) \rmod \text{GL}(\alpha)$$

These are Poisson varieties [Van den Bergh,08]

$\mathcal{M}_{\alpha,q}^\Lambda(Q)$ only depends on $\alpha, q$ and $Q$ \textit{seen as an undirected graph}, up to isomorphism of varieties [CBS,04]

A bit harder to prove: these isomorphisms preserve the Poisson structures
Goal for today

We will show that:

the isomorphisms of (multiplicative) quiver varieties hence obtained can be checked to preserve the Poisson structures directly at the level of the path algebras
Motivation

Double brackets

Morphisms of double Poisson brackets

The "quasi-" case

Plan for the talk
Following [Kontsevich, '93] and [Kontsevich-Rosenberg, '99]

associate C-algebra $\rightarrow$ commutative C-algebra

$A \rightarrow \mathbb{C}[\text{Rep}(A, n)]$

$\mathbb{C}[\text{Rep}(A, n)]$ is generated by symbols $a_{ij}, \forall a \in A, 1 \leq i, j \leq n.$
Rules: $1_{ij} = \delta_{ij}, (a + b)_{ij} = a_{ij} + b_{ij}, (ab)_{ij} = \sum_k a_{ik}b_{kj}.$

Goal: Find a property $P_{nc}$ on $A$ that gives the usual property $P$ on $\mathbb{C}[\text{Rep}(A, n)]$ for all $n \in \mathbb{N}^\times.$
Double brackets

We follow [Van den Bergh, *double Poisson algebras*, '08]

$A$ denotes an arbitrary f.g. associative $\mathbb{C}$-algebra, $\otimes = \otimes_\mathbb{C}$

For $d \in A^\otimes 2$, set $d = d' \otimes d'' (= \sum_k d_k' \otimes d_k'')$, and $\tau_{(12)} d = d'' \otimes d'$. Multiplication on $A^\otimes 2 : (a \otimes b)(c \otimes d) = ac \otimes bd$. 
Double brackets

We follow [Van den Bergh, *double Poisson algebras*, '08]

\(A\) denotes an arbitrary f.g. associative \(\mathbb{C}\)-algebra, \(\otimes = \otimes_\mathbb{C}\)

For \(d \in A \otimes^2\), set \(d = d' \otimes d'' (= \sum_k d'_k \otimes d''_k)\), and \(\tau_{(12)} d = d'' \otimes d'\).

Multiplication on \(A \otimes^2\): \((a \otimes b)(c \otimes d) = ac \otimes bd\).

Definition

A *double bracket* on \(A\) is a \(\mathbb{C}\)-bilinear map \(\{-, -\} : A \times A \to A \otimes^2\) which satisfies

1. \(\{a, b\} = -\tau_{(12)} \{b, a\}\) (cyclic antisymmetry)
2. \(\{a, bc\} = (b \otimes 1) \{a, c\} + \{a, b\} (1 \otimes c)\) (outer derivation)
3. \(\{ad, b\} = (1 \otimes a) \{d, b\} + \{a, b\} (d \otimes 1)\) (inner derivation)
Preliminary result

(definition) $\sim \{a, b\} = \{a, b\}' \otimes \{a, b\}''$

Lemma (Van den Bergh, '08)

If $A$ has a double bracket $\{\cdot, -\}$, then $\mathbb{C}[\text{Rep}(A, n)]$ has a unique antisymmetric biderivation $\{\cdot, -\}_P$ satisfying

$$\{a_{ij}, b_{kl}\}_P = \{a, b\}'_{kj} \{a, b\}''_{il}.$$  \hspace{1cm} (1)
(notation) \( \rightsquigarrow \{a, b\} = \{a, b\}' \otimes \{a, b\}'' \)

**Lemma (Van den Bergh,'08)**

*If* \( A \) *has a double bracket* \( \{\cdot, \cdot\} \), *then* \( \mathbb{C}[\text{Rep}(A, n)] \) *has a unique antisymmetric biderivation* \( \{\cdot, \cdot\}_P \) *satisfying*

\[
\{a_{ij}, b_{kl}\}_P = \{a, b\}'_{kj} \{a, b\}''_{il}.
\]  

(1)

**Example**

\( A = \mathbb{C}[x], \{x, x\} = x \otimes 1 - 1 \otimes x \) *endows* \( \mathfrak{gl}_n(\mathbb{C}) = \mathbb{C}[\text{Rep}(A, n)] \) *with*

\[
\{x_{ij}, x_{kl}\}_P = x_{kj} 1_{il} - 1_{kj} x_{il}
\]
(notation) $\rightsquigarrow \{a, b\} = \{a, b\}' \otimes \{a, b\}''$

**Lemma (Van den Bergh,'08)**

*If $A$ has a double bracket $\{-, -\}$, then $\mathbb{C}[\text{Rep}(A, n)]$ has a unique antisymmetric biderivation $\{-, -\}_P$ satisfying*

$$\{a_{ij}, b_{kl}\}_P = \{a, b\}'_{kj} \{a, b\}_i''.$$

(1)

**Example**

$A = \mathbb{C}[x]$, $\{x, x\} = x \otimes 1 - 1 \otimes x$ endows $\mathfrak{gl}_n(\mathbb{C}) = \mathbb{C}[\text{Rep}(A, n)]$ with

$$\{x_{ij}, x_{kl}\}_P = x_{kj}\delta_{il} - \delta_{kj}x_{il}.$$
Double Poisson bracket

Recall $d = d' \otimes d'' \in A^{\otimes 2}$ (notation) $\rightsquigarrow \{\{a, b\}\} = \{\{a, b\}' \otimes \{a, b\}'\}'$

From a double bracket $\{-, -\}$, define $\{-, -, -\} : A^{\times 3} \rightarrow A^{\otimes 3}$

$$\{\{a, b, c\}\} = \{\{a, \{b, c\}'\}\} \otimes \{b, c\}''' + \tau_{(123)} \{\{b, \{c, a\}'\}\} \otimes \{c, a\}''' + \tau_{(132)} \{\{c, \{a, b\}'\}\} \otimes \{a, b\}''' \quad \forall a, b, c \in A$$
Double Poisson bracket

Recall \( d = d' \otimes d'' \in A^\otimes 2 \) (notation) \( \leadsto \{ a, b \} = \{ a, b \}' \otimes \{ a, b \}'' \)

From a double bracket \( \{-, -\} \), define \( \{-, -, -\} : A^\times 3 \to A^\otimes 3 \)

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\{ a, b, c \} = \{ a, \{ b, c \}' \} \otimes \{ b, c \}'' + \tau_{(123)} \{ b, \{ c, a \}' \} \otimes \{ c, a \}'' + \tau_{(132)} \{ c, \{ a, b \}' \} \otimes \{ a, b \}'' , \quad \forall a, b, c \in A
\]

Definition

A double bracket \( \{-, -\} \) is Poisson if \( \{-, -, -\} : A^\times 3 \to A^\otimes 3 \) vanishes. We say \((A, \{-, -\})\) is a double Poisson algebra.
Double Poisson bracket

Recall \( d = d' \otimes d'' \in A^{\otimes 2} \) (notation) \( \Rightarrow \{a, b\} = \{a, b\}' \otimes \{a, b\}'' \)

From a double bracket \( \{-, -\} \), define \( \{-, -, -\} : A^{\times 3} \rightarrow A^{\otimes 3} \)

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**Definition**

A double bracket \( \{-, -\} \) is Poisson if \( \{-, -, -\} : A^{\times 3} \rightarrow A^{\otimes 3} \) vanishes. We say \((A, \{-, -\})\) is a double Poisson algebra.

**Example**

1. \( A = \mathbb{C}[x], \{x, x\} = x \otimes 1 - 1 \otimes x. \)
2. \( A = \mathbb{C}\langle x, y \rangle, \{x, x\} = 0 = \{y, y\}, \{x, y\} = 1 \otimes 1. \)
A first result

(notation) $\leadsto \{a, b\} = \{a, b\}' \otimes \{a, b\}''$

Proposition (Van den Bergh,'08)

If $A$ has a double bracket $\{\cdot, \cdot\}$, then $\mathbb{C}[\text{Rep}(A, n)]$ has a unique antisymmetric biderivation $\{\cdot, \cdot\}_P$ satisfying

$$\{a_{ij}, b_{kl}\}_P = \{a, b\}'_{kj} \{a, b\}''_{il}. \quad (2)$$

If $\{\cdot, \cdot\}$ is Poisson, then $\{\cdot, \cdot\}_P$ is a Poisson bracket.
A first result

(notation) \( \rightsquigarrow \{a, b\} = \{a, b\}' \otimes \{a, b\}'' \)

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\]

If \( \{-, -\} \) is Poisson, then \( \{-, -\}_P \) is a Poisson bracket.

Example

\( A = \mathbb{C}\langle x, y \rangle, \{x, x\} = 0 = \{y, y\}, \{x, y\} = 1 \otimes 1 \) endows \( \mathfrak{gl}_n(\mathbb{C})^\times 2 = \mathbb{C}[\text{Rep}(A, n)] \) with

\[
\{x_{ij}, y_{kl}\}_P = \delta_{kj} \delta_{il}, \quad \{x_{ij}, x_{kl}\}_P = 0 = \{y_{ij}, y_{kl}\}_P.
\]

This is the canonical Poisson bracket on \( T^* \mathfrak{gl}_n \).
### A first dictionary

<table>
<thead>
<tr>
<th>Algebra</th>
<th>Geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>( \mathbb{C}[\text{Rep}(A, n)] )</td>
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<tr>
<td>double bracket</td>
<td>anti-symmetric biderivation ( {-, -}_P )</td>
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<td>double Poisson</td>
<td>Poisson bracket ( {-, -}_P )</td>
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\[\text{double bracket } \{ -, - \} \]

\[\text{double Poisson bracket } \{ -, - \} \]
Definition

If \((A, \{\{-, -\}\})\) is a double Poisson algebra, \(\mu_A \in A\) is a moment map if \(\{\mu_A, a\} = a \otimes 1 - 1 \otimes a\), \(\forall a \in A\).
Hamiltonian reduction

Definition

If \((A, \{\{-, -\}\})\) is a double Poisson algebra, \(\mu_A \in A\) is a moment map if \(\{\mu_A, a\} = a \otimes 1 - 1 \otimes a, \forall a \in A\).

For any \(\lambda \in \mathbb{C}\), \(\{\mu_A - \lambda, a\} = 0\), where \(\{-, -\} = m \circ \{-, -\}\).
Hamiltonian reduction

Definition

If $(A, \{-, -\})$ is a double Poisson algebra, $\mu_A \in A$ is a moment map if $\{\mu_A, a\} = a \otimes 1 - 1 \otimes a, \forall a \in A$.

For any $\lambda \in \mathbb{C}$, $\{\mu_A - \lambda, a\} = 0$, where $\{-, -\} = m \circ \{-, -\}$.

$\Rightarrow \{-, -\}$ descends to a Lie bracket on the vector space $A_\lambda/[A_\lambda, A_\lambda]$ for $A_\lambda := A/\langle \mu_A - \lambda \rangle$. 
Hamiltonian reduction

Definition

If \((A, \{[-, -]\})\) is a double Poisson algebra, \(\mu_A \in A\) is a moment map if \(\{[\mu_A, a]\} = a \otimes 1 - 1 \otimes a\), \(\forall a \in A\).

For any \(\lambda \in \mathbb{C}\), \(\{\mu_A - \lambda, a\} = 0\), where \([-,-] = m \circ \{[-, -]\}\)

\(\Rightarrow \{-, -\}\) descends to a Lie bracket on the vector space \(A_\lambda/[[A_\lambda, A_\lambda]\) for \(A_\lambda := A/\langle \mu_A - \lambda \rangle\).

Proposition (Van den Bergh,’08)

The Poisson structure \(\{-, -\}_P\) on \(\text{Rep}(A, n)\) descends to \(\text{Rep}(A_\lambda, n)/\text{GL}_n\) in such a way that

\[\{\text{tr} \mathcal{X}(a), \text{tr} \mathcal{X}(b)\}_P = \text{tr} \mathcal{X}([a, b]).\] (3)
## Dictionary

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<td>double Poisson bracket ${ -, - }$</td>
<td>Poisson bracket ${ -, - }_P$</td>
</tr>
<tr>
<td>moment map $\mu_A$</td>
<td>moment map $\mathcal{X}(\mu_A)$</td>
</tr>
<tr>
<td>$A_\lambda = A/(\mu_A - \lambda), \lambda \in \mathbb{C}$</td>
<td>slice $S_\lambda := \mathcal{X}(\mu_A)^{-1}(\lambda \text{Id}_n)$</td>
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<tr>
<td>$(A_\lambda/[A_\lambda, A_\lambda], { -, - })$ is a Lie algebra</td>
<td>${ -, - }<em>P$ is Poisson on $\mathbb{C}[S</em>\lambda//\text{GL}_n]$</td>
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Recall $\{ -, - \} = m \circ \{ -, - \}$ descends to $A_\lambda/[A_\lambda, A_\lambda]$
Examples from quivers

Fix quiver $Q$, with double $\bar{Q}$ (if $a \in Q$, $a : t \to h$, add $a^* : h \to t$)

**Theorem (Van den Bergh,’08)**

The algebra $A = \mathbb{C}\bar{Q}$ has a double Poisson bracket given by

$$\{a, a^*\} = e_{h(a)} \otimes e_{t(a)} \quad \forall a \in Q, \quad \{a, b\} = 0 \text{ if } a \neq b^*, b \neq a^* \quad (4)$$

and (non-commutative) moment map $\mu = \sum_{a \in Q} [a, a^*]$. 
Fix quiver $Q$, with double $\bar{Q}$ (if $a \in Q$, $a : t \to h$, add $a^* : h \to t$)

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- Fix a dimension vector $\alpha \in \mathbb{N}^I$. Attach $\mathbb{C}^{\alpha_s}$ to vertex $s \in I$ of $\bar{Q}$
Examples from quivers

Fix quiver $Q$, with double $\bar{Q}$ (if $a \in Q$, $a : t \to h$, add $a^* : h \to t$)

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The algebra $A = \mathbb{C}\bar{Q}$ has a double Poisson bracket given by

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and (non-commutative) moment map $\mu = \sum_{a \in Q} [a, a^*]$.

- Fix a dimension vector $\alpha \in \mathbb{N}^I$. Attach $\mathbb{C}^{\alpha_s}$ to vertex $s \in I$ of $\bar{Q}$

$\implies \text{Rep}(\mathbb{C}\bar{Q}, \alpha)$ has a Poisson structure (with ‘usual’ moment map)
Examples from quivers

Fix quiver $Q$, with double $\tilde{Q}$ (if $a \in Q$, $a : t \to h$, add $a^* : h \to t$)

**Theorem (Van den Bergh, ’08)**

The algebra $A = \mathbb{C}\tilde{Q}$ has a double Poisson bracket given by

$$\{a, a^*\} = e_{h(a)} \otimes e_{t(a)} \quad \forall a \in Q, \quad \{a, b\} = 0 \text{ if } a \neq b^*, b \neq a^* $$  \hspace{1cm} (4)

and (non-commutative) moment map $\mu = \sum_{a \in Q} [a, a^*]$.

- Fix a dimension vector $\alpha \in \mathbb{N}^I$. Attach $\mathbb{C}^{\alpha_s}$ to vertex $s \in I$ of $\tilde{Q}$
- $\Rightarrow$ $\text{Rep}(\mathbb{C}\tilde{Q}, \alpha)$ has a Poisson structure (with ‘usual’ moment map)
- $\Rightarrow$ Poisson structure on quiver varieties by Hamiltonian reduction on

$$\left\{ \sum_{a \in Q} [X(a), X(a^*)] = \prod_{s \in I} \lambda_s \text{Id}_{\alpha_s} \right\} \parallel \text{GL}(\alpha) \cong \text{Rep}(\Pi^\lambda(Q), \alpha) \parallel \text{GL}(\alpha)$$

$$\mathcal{M}^{\Pi}_{\alpha, \lambda}(Q)$$
Nice example: CM spaces

\[
\{ x, y \} = e_0 \otimes e_0
\]

\[
\{ v, w \} = e_0 \otimes e_\infty
\]

(\{ a, b \} = 0 \text{ if } a \neq b^*, b \neq a^*)

\[
\mu = [x, y] + [v, w]
\]
Nice example: CM spaces

\[
\begin{align*}
\{x, y\} &= e_0 \otimes e_0 \\
\{v, w\} &= e_0 \otimes e_\infty \\
(\{a, b\} &= 0 \text{ if } a \neq b^*, b \neq a^*) \\
\mu &= [x, y] + [v, w]
\end{align*}
\]

1. Take \((\alpha_0, \alpha_\infty) = (n, 1), \ n \geq 1\); Attach \(\mathbb{C}^n\) at 0, \(\mathbb{C}\) at \(\infty\)
2. \(x, y, v, w \rightarrow X, Y \in \text{Mat}_{n \times n}, \ V \in \text{Mat}_{1 \times n}, \ W \in \text{Mat}_{n \times 1}\)
3. \(\{-, -\} \rightarrow \{X_{ij}, Y_{kl}\} = \delta_{kj}\delta_{il}, \ \{V_j, W_k\} = \delta_{kj}\)
4. \(\mu = [x, y] + [v, w]\ \text{restricts to } [x, y] - wv \in e_0\bar{Q}_1e_0\)
   \(\leadsto [X, Y] - WV\ \text{is moment map for } GL_n \curvearrowright \mathbb{C}^n \hookrightarrow \mathbb{C}^n \oplus \mathbb{C}\)
Nice example: CM spaces

1. Take $(\alpha_0, \alpha_\infty) = (n, 1), n \geq 1$; Attach $\mathbb{C}^n$ at 0, $\mathbb{C}$ at $\infty$

2. $x, y, v, w \rightarrow X, Y \in \text{Mat}_{n \times n}, V \in \text{Mat}_{1 \times n}, W \in \text{Mat}_{n \times 1}$

3. $\{-, -\} \rightarrow \{X_{ij}, Y_{kl}\} = \delta_{kj}\delta_{il}, \{V_j, W_k\} = \delta_{kj}$

4. $\mu = [x, y] + [v, w]$ restricts to $[x, y] - wv \in e_0 \mathbb{C} \bar{Q}_1 e_0$

   $\sim [X, Y] - WV$ is moment map for $\text{GL}_n \curvearrowright \mathbb{C}^n \hookrightarrow \mathbb{C}^n \oplus \mathbb{C}$

Hamiltonian reduction at $\text{Id}_n : \mathcal{C}_n = \{[X, Y] - WV = \text{Id}_n\} // \text{GL}_n
Plan for the talk

1. Motivation
2. Double brackets
3. **Morphisms of double Poisson brackets**
4. The “quasi-” case
Definition I

(mostly based on [F., 2008.01409] from now on)

\[ A_1, A_2 \text{ endowed with double brackets } \{\{-, -\}\}_1, \{\{-, -\}\}_2. \]

**Definition**

\( \phi : A_1 \rightarrow A_2 \) is a *morphism of double brackets* if it is an algebra homomorphism such that for any \( a, b \in A_1 \)

\[ \{\phi(a), \phi(b)\}_2 = (\phi \otimes \phi) \{a, b\}_1. \]
Definition I

(mostly based on [F., 2008.01409] from now on)

$A_1, A_2$ endowed with double brackets $\{\{ -, - \} \}_1, \{\{ -, - \} \}_2$.

**Definition**

$\phi : A_1 \rightarrow A_2$ is a *morphism of double brackets* if it is an algebra homomorphism such that for any $a, b \in A_1$

$$\{\phi(a), \phi(b)\}_2 = (\phi \otimes \phi) \{a, b\}_1.$$  

**Example**

$A = \mathbb{C}\langle x, y \rangle$ can be endowed with

$$\{x, x\}_1 = 0 = \{x, x\}_2, \quad \{y, y\}_1 = 0 = \{y, y\}_2, \quad \{x, y\}_1 = 1 \otimes 1, \quad \{x, y\}_2 = -1 \otimes 1.$$  

Automorphism $x \mapsto y, y \mapsto x$ defines an isomorphism of double brackets $(A, \{\{ -, - \} \}_1) \rightarrow (A, \{\{ -, - \} \}_2)$
Definition II

$A_1, A_2$ endowed with double brackets $\{\{\cdot, \cdot\}\}_1, \{\{\cdot, \cdot\}\}_2$.

$\phi : A_1 \rightarrow A_2$ is a morphism of double brackets: $\{\{\phi(a), \phi(b)\}\}_2 = (\phi \otimes \phi) \{\{a, b\}\}_1$.
Definition II

$A_1, A_2$ endowed with double brackets $\{\{-, -\}_1, \{-, -\}_2\}$.
$\phi : A_1 \rightarrow A_2$ is a **morphism of double brackets**:
$\{\phi(a), \phi(b)\}_2 = (\phi \otimes \phi) \{a, b\}_1$

**Definition**

If $A_1, A_2$ are double Poisson algebras, $\phi$ is a **morphism of double Poisson algebras**.
If $A_1, A_2$ admit moment maps $\mu_1, \mu_2$ and $\phi(\mu_1) = \mu_2$, we say that $\phi$ is a **morphism of Hamiltonian algebras**.
Definition II

$A_1, A_2$ endowed with double brackets $\{\{-, -\}_1, \{-, -\}_2\}$.
$\phi : A_1 \to A_2$ is a morphism of double brackets:
$\{\phi(a), \phi(b)\}_2 = (\phi \otimes \phi) \{a, b\}_1$

**Definition**

If $A_1, A_2$ are double Poisson algebras, $\phi$ is a morphism of double Poisson algebras.

If $A_1, A_2$ admit moment maps $\mu_1, \mu_2$ and $\phi(\mu_1) = \mu_2$, we say that $\phi$ is a morphism of Hamiltonian algebras.

$\sim$ induces Poisson morphisms on Poisson varieties:

1. $\phi_n : \text{Rep}(A_1, n) \to \text{Rep}(A_2, n)$,
given by $\phi_n(\mathcal{X}(a)) = \mathcal{X}(\phi(a))$ s.t. $\phi_n(\mathcal{X}(\mu_1)) = \mathcal{X}(\mu_2)$
A_1, A_2 endowed with double brackets \{−, −\}_1, \{−, −\}_2.
φ : A_1 → A_2 is a morphism of double brackets : \{φ(a), φ(b)\}_2 = (φ \otimes φ) \{a, b\}_1.

Definition

If A_1, A_2 are double Poisson algebras, φ is a morphism of double Poisson algebras.
If A_1, A_2 admit moment maps µ_1, µ_2 and φ(µ_1) = µ_2, we say that φ is a morphism of Hamiltonian algebras.

⇝ induces Poisson morphisms on Poisson varieties:
1. φ_n : Rep(A_1, n) → Rep(A_2, n),
given by φ_n(\mathcal{X}(a)) = \mathcal{X}(φ(a)) s.t. φ_n(\mathcal{X}(µ_1)) = \mathcal{X}(µ_2)

2. \overline{φ}_n^λ : \mathcal{X}(µ_1)^{-1}(λ \text{Id}_n) // GL_n → \mathcal{X}(µ_2)^{-1}(λ \text{Id}_n) // GL_n
given by \overline{φ}_n^λ(\text{tr} \mathcal{X}(a)) = \text{tr} \mathcal{X}(φ(a))
Morphism for quivers: reversing the arrow

\[
\begin{align*}
\bar{Q}_- & \quad Q_- \quad Q^{op} \quad \bar{Q}^{op} \\
1 & \quad a \quad a^* & \quad a & \quad a \quad b & \quad b^* & \quad b \quad 2
\end{align*}
\]
**Morphism for quivers: reversing the arrow**

By [VdB,'08], $\mathbb{C}\bar{Q}_- / \mathbb{C}\bar{Q}_-^{op}$ is a Hamiltonian algebra for

$\bar{Q}_-$:
\[
\{ a, a \} = 0 = \{ a^*, a^* \} , \quad \{ a, a^* \} = e_2 \otimes e_1 , \quad \mu = [a, a^*] ,
\]

$\bar{Q}_-^{op}$:
\[
\{ b, b \} = 0 = \{ b^*, b^* \} , \quad \{ b, b^* \} = e_1 \otimes e_2 , \quad \mu' = [b, b^*] .
\]
By [VdB,'08], $\mathbb{C} \tilde{Q}_- / \mathbb{C} \tilde{Q}_-^{op}$ is a Hamiltonian algebra for

$\mathbb{C} \tilde{Q}_- : \{a, a\} = 0 = \{a^*, a^*\} , \{a, a^*\} = e_2 \otimes e_1 , \mu = [a, a^*] ,$

$\mathbb{C} \tilde{Q}_-^{op} : \{b, b\}' = 0 = \{b^*, b^*\}' , \{b, b^*\}' = e_1 \otimes e_2 , \mu' = [b, b^*].$

$\phi : \mathbb{C} \tilde{Q}_- \to \mathbb{C} \tilde{Q}_-^{op}$ given by $\phi(a) = b^* , \phi(a^*) = -b$, is an isomorphism of Hamiltonian algebras.

$\phi$ lifts the isomorphism of preprojective algebras [Crawley-Boevey,Holland,'98]

$\leadsto$ Poisson isomorphism $\tilde{\phi}_\alpha^\lambda : M_{\alpha, \lambda}(Q_-) \to M_{\alpha, \lambda}(Q_-^{op})$
Fusion

Any quiver $Q$ can be obtained by taking $|Q|$ copies of $\cdot \rightarrow \cdot$
Any quiver $Q$ can be obtained by taking $|Q|$ copies of $Q$: $\bullet \longrightarrow \bullet$. By "doubling", $\overline{Q}$ is obtained from $|Q|$ copies of $\overline{Q}$ by fusion of the idempotents corresponding to the identified vertices.
Fusion

Any quiver $Q$ can be obtained by taking $|Q|$ copies of $Q_-$: $\bullet \rightarrow \bullet$

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Fusion

Any quiver $Q$ can be obtained by taking $|Q|$ copies of $Q_-$ : $\bullet \rightarrow \bullet$

By “doubling”, $\bar{Q}$ is obtained from $|Q|$ copies of $\bar{Q}_-$. 

The analogous construction at the algebra level is called fusion [VdB,’08]. It consists of identifying orthogonal idempotents in the algebra. $\sim$ can obtain $\mathbb{C}\bar{Q}$ from $|Q|$ copies of $\mathbb{C}\bar{Q}_-$ by fusion of the idempotents corresponding to the identified vertices.
Lemma

Fusion is compatible with morphisms of double (Poisson) brackets
Fusion and morphisms

Lemma

*Fusion is compatible with morphisms of double (Poisson) brackets*

Example \((Q_- : 1 \xrightarrow{a} 2, \quad Q_-^{op} : 2 \xrightarrow{b} 1)\)

Fusion of the idempotents in \(\mathbb{C}\bar{Q}_-\) or \(\mathbb{C}\bar{Q}_-^{op}\) results in the free algebra

\(\mathbb{C}\langle a, a^* \rangle\) or \(\mathbb{C}\langle b, b^* \rangle\)

Under identification with \(\mathbb{C}\langle x, y \rangle\) through \(a, b \leftrightarrow x\) and \(a^*, b^* \leftrightarrow y\), we get the Hamiltonian algebra structure

\[
\{x, x\} = 0 = \{y, y\}, \quad \{x, y\} = 1 \otimes 1, \quad \mu = [x, y]
\]
Fusion and morphisms

Lemma

*Fusion is compatible with morphisms of double (Poisson) brackets*

Example \((Q_- : 1 \xrightarrow{a} 2, \quad Q_-^{op} : 2 \xrightarrow{b} 1)\)

Fusion of the idempotents in \(\mathbb{C} \tilde{Q}_-\) or \(\mathbb{C} \tilde{Q}_-^{op}\) results in the free algebra \(\mathbb{C}\langle a, a^* \rangle\) or \(\mathbb{C}\langle b, b^* \rangle\)

Under identification with \(\mathbb{C}\langle x, y \rangle\) through \(a, b \leftrightarrow x\) and \(a^*, b^* \leftrightarrow y\), we get the Hamiltonian algebra structure
\[
\{x, x\} = 0 = \{y, y\}, \quad \{x, y\} = 1 \otimes 1, \quad \mu = [x, y]
\]

The isomorphism \(\phi : \mathbb{C} \tilde{Q}_- \rightarrow \mathbb{C} \tilde{Q}_-^{op}\) given by \(\phi(a) = b^*, \quad \phi(a^*) = -b\) becomes an automorphism (of Hamiltonian algebras) on \(\mathbb{C}\langle x, y \rangle\) given by \(x \mapsto y, \quad y \mapsto -x\).
Isomorphic quiver varieties

Proposition

\( \mathcal{M}_{\alpha, \lambda}^{\Pi}(Q) \) only depends on \( \alpha, \lambda \) and \( Q \) seen as an undirected graph, up to isomorphism of Poisson varieties.
Isomorphic quiver varieties

Proposition

\[ \mathcal{M}^{\Pi}_{\alpha,\lambda}(Q) \text{ only depends on } \alpha, \lambda \text{ and } Q \text{ seen as an undirected graph, up to isomorphism of Poisson varieties} \]

Proof.

It suffices to get that the Hamiltonian algebra structure on \( \mathbb{C}\tilde{Q} \) given by Van den Bergh only depends on \( Q \text{ seen as an undirected graph} \), up to isomorphism.
Isomorphic quiver varieties

Proposition

\[ \mathcal{M}^{\Pi}_{\alpha, \lambda}(Q) \text{ only depends on } \alpha, \lambda \text{ and } Q \text{ seen as an undirected graph, up to isomorphism of Poisson varieties} \]

Proof.

It suffices to get that the Hamiltonian algebra structure on \( \mathbb{C} \bar{Q} \) given by Van den Bergh only depends on \( Q \text{ seen as an undirected graph} \), up to isomorphism.

This follows from the case of \( Q_- \) by fusion.
A question

The morphism giving that “the Hamiltonian algebra structure on $\mathbb{C}\bar{Q}$ given by Van den Bergh only depends on $Q$ seen as an undirected graph” is obtained by lifting the corresponding isomorphism of (deformed) preprojective algebras $\Pi^\lambda(Q) = \mathbb{C}\bar{Q}/\langle \sum a[a, a^*] - \lambda \rangle$ given in [CBH,'98]
The morphism giving that “the Hamiltonian algebra structure on $\mathbb{C}\tilde{Q}$ given by Van den Bergh only depends on $Q$ seen as an undirected graph” is obtained by lifting the corresponding isomorphism of (deformed) preprojective algebras $\Pi^\lambda(Q) = \mathbb{C}\tilde{Q}/\langle \sum_a [a, a^*] - \lambda \rangle$ given in [CBH,'98]

**Question**

*When can we lift an automorphism of $\Pi^\lambda(Q)$ to an automorphism of Hamiltonian algebras on $\mathbb{C}\tilde{Q}$? (for a fixed pair $\langle \{-, -\}, \mu \rangle$)*
The morphism giving that “the Hamiltonian algebra structure on $\mathbb{C}\overline{Q}$ given by Van den Bergh only depends on $Q$ seen as an undirected graph” is obtained by lifting the corresponding isomorphism of (deformed) preprojective algebras $\Pi^\lambda(Q) = \mathbb{C}\overline{Q}/\langle \sum_a [a, a^*] - \lambda \rangle$ given in [CBH,'98]

**Question**

*When can we lift an automorphism of $\Pi^\lambda(Q)$ to an automorphism of Hamiltonian algebras on $\mathbb{C}\overline{Q}$? (for a fixed pair $(\{−, −\}, \mu)$)*

**Example** (first Weyl algebra $A_1 = \mathbb{C}\langle x, y \rangle/\langle xy - yx - 1 \rangle$)

$A_1$ is isomorphic to $\Pi^1(Q_\circ)$ for $Q_\circ$ the one-loop quiver.

Following [Dixmier,'68], automorphisms of $A_1$ are generated by

$$\phi_{k,\gamma}(x) = x + \gamma y^k, \quad \phi_{k,\gamma}(y) = y, \quad \phi'_{k,\gamma}(x) = x, \quad \phi'_{k,\gamma}(y) = y + \gamma x^k$$

They can be lifted as Hamiltonian algebras automorphisms on $\mathbb{C}\overline{Q}_\circ$ (!)
Plan for the talk

1. Motivation
2. Double brackets
3. Morphisms of double Poisson brackets
4. The “quasi-” case
Motivation

Double brackets Morphisms quasi-...

quasi-Dictionary

\[ \{ - , - \} = m \circ \{ - , - \} \] descends to \( A_q/[A_q, A_q] \)

Algebra \( A \)

double bracket \( \{ - , - \} \)

double quasi-Poisson bracket \( \{ - , - \} \)

multiplicative moment map \( \Phi_A \)
\( A_q = A/(\Phi_A - q) \), \( q \in \mathbb{C}^\times \)
\( (A_q/[A_q, A_q], \{-, -\}) \) is Lie algebra

Geometry \( C[\text{Rep}(A, n)] \)

anti-symmetric biderivation \( \{-, -\}_P \)

quasi-Poisson bracket \( \{-, -\}_P \)

multiplicative moment map \( \mathcal{X}(\Phi_A) \)
slice \( S_q := \mathcal{X}(\Phi_A)^{-1}(q \text{ Id}_n) \)
\( \{-, -\}_P \) is Poisson on \( C[S_q//\text{GL}_n] \)

quasi-Poisson geometry after [Alekseev – Kosmann-Schwarzbach – Meinrenken,’02]
Examples from quivers

Fix quiver $Q$. Let $A_Q = \mathbb{C}\bar{Q}_S$ localisation at $S = \{1 + aa^* \mid a \in \bar{Q}\}$

**Theorem (Van den Bergh,’08)**

The algebra $A_Q$ has a double quasi-Poisson bracket whose (non-commutative) multiplicative moment map is given by

$$\Phi = \prod_{\text{order}} (1 + aa^*)(1 + a^*a)^{-1}$$

(It depends on an order !)
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- Fix a dimension vector $\alpha \in \mathbb{N}^I$, attach $\mathbb{C}^{\alpha_s}$ to vertex $s \in I$ of $\overline{Q}$

$\Rightarrow$ Rep($A_Q, \alpha$) has quasi-Poisson structure (mult. mom. map $\mathcal{V}(\Phi)$)
Examples from quivers

Fix quiver $Q$. Let $A_Q = \mathbb{C} \bar{Q}_S$ localisation at $S = \{1 + aa^* \mid a \in \bar{Q}\}$

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$\Rightarrow$ Rep($A_Q, \alpha$) has quasi-Poisson structure (mult. mom. map $\mathcal{X}(\Phi)$)

$\Rightarrow$ Poisson structure on multiplicative quiver varieties by quasi-Hamiltonian reduction on

$$\left\{ \mathcal{X}(\Phi) = \prod_{s \in I} q_s \text{Id}_{\alpha_s} \right\} \text{ // } \text{GL}(\alpha) \simeq \text{Rep}(\Lambda^q(Q), \alpha) \text{ // } \text{GL}(\alpha)$$

$$\mathcal{M}^\Lambda_{\alpha,q}(Q)$$
Fusion and Van den Bergh’s proof

By fusion, we obtained $\mathbb{C} \bar{Q}$ from $|Q|$ copies of $\mathbb{C} \bar{Q}$
Similarly, we can obtain $A_Q$ from $|Q|$ copies of $A_Q$ (‘localised’ version)
Fusion and Van den Bergh’s proof

By fusion, we obtained \( \mathbb{C} \mathbb{Q} \) from \( |Q| \) copies of \( \mathbb{C} \mathbb{Q}_- \)

Similarly, we can obtain \( A_Q \) from \( |Q| \) copies of \( A_Q_- \) (‘localised’ version)

**Problem:** If \( \{ -, - \} \) is a double quasi-Poisson bracket on \( A \), its image after fusion is not a double quasi-Poisson bracket
Fusion and Van den Bergh’s proof

By fusion, we obtained $\mathbb{C} \tilde{Q}$ from $|Q|$ copies of $\mathbb{C} \tilde{Q}$
Similarly, we can obtain $A_Q$ from $|Q|$ copies of $A_Q$ ('localised' version)

**Problem:** If $\{\{-,-\}\}$ is a double quasi-Poisson bracket on $A$, its image after fusion is not a double quasi-Poisson bracket

**Solution:** add an extra part to the double bracket
See [VdB,’08] and in general [F.,AlgRepTh’??]; NC versions of [AKSM,’02]
Fusion and Van den Bergh’s proof

By fusion, we obtained $\mathbb{C}\bar{Q}$ from $|Q|$ copies of $\mathbb{C}\bar{Q}$
Similarly, we can obtain $\mathbb{A}_Q$ from $|Q|$ copies of $\mathbb{A}_Q$ (‘localised’ version)

**Problem:** If $\{\{-,-\}\}$ is a double quasi-Poisson bracket on $A$, its image after fusion is not a double quasi-Poisson bracket

**Solution:** add an extra part to the double bracket
See [VdB,’08] and in general [F.,AlgRepTh’??]; NC versions of [AKSM,’02]

**Drawback:** the structure depends on the order of the fusion: the fusions $e_i \to e_j$ or $e_j \to e_i$ give isomorphic algebras, but different double quasi-Poisson brackets!

$\leadsto$ get Van den Bergh’s result from basic case $\mathbb{A}_Q$; the order of fusion is the one used in the multiplicative moment map
Fusion in the quasi-case: example

LHS: $A_{Q^-} = (\mathbb{C} \bar{Q}^-)_{1+aa^*, 1+a^*a}$ has double quasi-Poisson bracket

\[
\{a, a\} = 0 = \{a^*, a^*\}, \quad \{a, a^*\} = e_2 \otimes e_1 + \frac{1}{2} a^* a \otimes e_1 + \frac{1}{2} e_2 \otimes aa^*
\]

RHS: $A_{Q^\circ} \simeq \mathbb{C} \langle a, a^* \rangle_{1+aa^*, 1+a^*a}$ has double quasi-Poisson bracket

\[
\{a, a\} = \frac{1}{2} (a^2 \otimes 1 - 1 \otimes a^2),
\]

\[
\{a^*, a^*\} = -\frac{1}{2} ((a^*)^2 \otimes 1 - 1 \otimes (a^*)^2),
\]

\[
\{a, a^*\} = 1 \otimes 1 + \frac{1}{2} a^* a \otimes 1 + \frac{1}{2} 1 \otimes aa^* + \frac{1}{2} (a^* \otimes a - a \otimes a^*)
\]

(this corresponds to the ordering $a < a^*$ in [VdB,'08])
Recall the drawback of fusion in an algebra with a double quasi-Poisson bracket: performing the fusions $e_i \rightarrow e_j$ or $e_j \rightarrow e_i$ gives isomorphic algebras, but different double quasi-Poisson brackets!
Recall the drawback of fusion in an algebra with a double quasi-Poisson bracket: performing the fusions $e_i \rightarrow e_j$ or $e_j \rightarrow e_i$ gives isomorphic algebras, but different double quasi-Poisson brackets!

Let $\psi : A_1 \rightarrow A_2$ be a morphism of double brackets.

**Definition**

If $A_1, A_2$ are double quasi-Poisson algebras, $\psi$ is a *morphism of double quasi-Poisson algebras*. If $A_1, A_2$ admit multiplicative moment maps $\Phi_1, \Phi_2$ and $\psi(\Phi_1) = \Phi_2$, we say that $\psi$ is a *morphism of quasi-Hamiltonian algebras*. 
Recall the drawback of fusion in an algebra with a double quasi-Poisson bracket: performing the fusions $e_i \to e_j$ or $e_j \to e_i$ gives isomorphic algebras, but different double quasi-Poisson brackets!

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If $A_1, A_2$ are double quasi-Poisson algebras, $\psi$ is a *morphism of double quasi-Poisson algebras*.
If $A_1, A_2$ admit multiplicative moment maps $\Phi_1, \Phi_2$ and $\psi(\Phi_1) = \Phi_2$, we say that $\psi$ is a *morphism of quasi-Hamiltonian algebras*.

**Proposition**

If $A$ is a quasi-Hamiltonian algebra, the algebras $A_{i\to j}$ and $A_{j\to i}$ obtained by fusion of $e_i \to e_j$ and $e_j \to e_i$ are isomorphic as quasi-Hamiltonian algebras (with their structure induced by fusion).
Fusion and morphisms II

Proposition

If $A$ is a quasi-Hamiltonian algebra, the algebras $A_{i \rightarrow j}$ and $A_{j \rightarrow i}$ obtained by fusion of $e_i \rightarrow e_j$ and $e_j \rightarrow e_i$ are isomorphic as quasi-Hamiltonian algebras (with their structure induced by fusion).

- We need the multiplicative moment map to define the isomorphism, contrary to the Hamiltonian case (this can be slightly relaxed).
- Non-commutative version of [AKSM,’02]
Isomorphism and fusion in the quasi-case: example

\[ \Phi = \Phi_1 + \Phi_2 \]
\[ \Phi_1 = e_1 + aa^* \quad \Phi_2 = (e_2 + a^*a)^{-1} \]
\[ \tilde{\Phi} = (1 + \tilde{a}^*\tilde{a})^{-1}(1 + \tilde{\tilde{a}}a^*) \]

\[ \psi : A_{Q_1} \to A_{Q_2} \]
\[ \psi(a) = (1 + \tilde{a}^*\tilde{a})^{-1}\tilde{a} \]
\[ \psi(a^*) = \tilde{a}^* (1 + \tilde{a}^*\tilde{a}) \]

\[ \psi : A_{Q_1} \to A_{Q_2} \] is an isomorphism of quasi-Hamiltonian algebras for the two distinct structures induced by fusion.
Morphism for quivers: reversing the arrow

By [VdB,'08], $A_{Q_-}/A_{Q_-}^{op}$ is a quasi-Hamiltonian algebra for

$$A_{Q_-} : \{ -, - \} = \ldots , \Phi = (1 + aa^*)(1 + a^*a)^{-1} ,$$
$$A_{Q_-}^{op} : \{ -, - \}' = \ldots , \Phi' = (1 + bb^*)(1 + b^*b)^{-1} .$$
Morphism for quivers: reversing the arrow

\[ Q_1 \quad a \quad Q_2 \quad Op \quad b \quad Q_3 \quad b^* \]

By [VdB,'08], \( A_{Q_-}/A_{Q_-^{op}} \) is a quasi-Hamiltonian algebra for

\[
A_{Q_-} : \quad \{ -,- \} = \ldots , \quad \Phi = (1 + aa^*)(1 + a^*a)^{-1}
\]

\[
A_{Q_-^{op}} : \quad \{ -,- \}' = \ldots , \quad \Phi' = (1 + bb^*)(1 + b^*b)^{-1}
\]

\( \psi : A_{Q_-} \to A_{Q_-^{op}} \) given by \( \phi(a) = b^* \), \( \phi(a^*) = -(1 + bb^*)^{-1}b \), is an isomorphism of quasi-Hamiltonian algebras.

\( \psi \) lifts the isomorphism of multiplicative preprojective algebras defined in [Crawley-Boevey – Shaw,’06]
Proposition

\[ \mathcal{M}_{\alpha,q}^\Lambda(Q) \] only depends on \( \alpha, q \) and \( Q \) seen as an undirected graph, up to isomorphism of Poisson varieties
Isomorphic multiplicative quiver varieties

Proposition

\[ M_{\alpha,q}^\Lambda(Q) \] only depends on \( \alpha, q \) and \( Q \) seen as an undirected graph, up to isomorphism of Poisson varieties

Proof.

It suffices to get that the quasi-Hamiltonian algebra structure on \( A_Q \) given by Van den Bergh only depends on \( Q \) seen as an undirected graph, up to isomorphism.

This follows from the case of \( Q_- \) by fusion.

Main technicality here: changing the order of fusions yields a non-trivial isomorphism, so it seems quite cumbersome to explicitly write down this map in general.
Thank you for your attention

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