# Double (quasi-)Poisson algebras

## and their morphisms

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Algebra Seminar
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# Plan for the talk

1. **Motivation**
2. Double brackets
3. Morphisms of double Poisson brackets
4. The "quasi-" case

# Quiver varieties

(we do not consider stability parameter)

Fix: quiver $Q$, dimension vector $\alpha \in \mathbb{N}^I$, parameter $\lambda \in \mathbb{C}^I$

– Consider double $\bar{Q}$ (add $a^* : h \to t$ for each $a : t \to h$ in $Q$)

– Construct $\Pi^\lambda(Q) = \mathbb{C}\bar{Q}/\langle \sum_{a \in Q}[a, a^*] - \sum_{s \in I} \lambda_s e_s \rangle$

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Get: A *quiver variety* :

$$\mathcal{M}^{\Pi}_{\alpha,\lambda}(Q) = \mathrm{Rep}(\Pi^\lambda(Q), \alpha)/\!/ \,\mathrm{GL}(\alpha)$$

which is a Poisson variety for $(\mathcal{X}(b) \in \mathrm{Mat}_{\alpha_{t(b)}\times\alpha_{h(b)}}(\mathbb{C})\ \forall b \in \bar{Q})$

$$\{\mathcal{X}(a)_{ij}, \mathcal{X}(a^*)_{kl}\}_{\mathrm{P}} = (\mathrm{Id}_{\alpha_{h(a)}})_{kj}\, (\mathrm{Id}_{\alpha_{t(a)}})_{il}$$

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$\mathcal{M}^{\Pi}_{\alpha,\lambda}(Q)$ only depends on $\alpha, \lambda$ and $Q$ *seen as an undirected graph*, up to isomorphism of Poisson varieties (easy)

# Multiplicative quiver varieties

Fix: quiver $Q$, dimension vector $\alpha \in \mathbb{N}^I$, parameter $q \in (\mathbb{C}^\times)^I$

– Algebra $A_Q$ is localisation of $\mathbb{C}\bar{Q}$ at all $1 + aa^*, 1 + a^*a$

– Construct $\Lambda^q(Q) = A_Q / \Big\langle \prod_{\text{order}} (1 + aa^*)(1 + a^*a)^{-1} - \sum_{s \in I} q_s e_s \Big\rangle$

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Get: A *multiplicative quiver variety* [Crawley-Boevey - Shaw,04] :

$$\mathcal{M}^{\Lambda}_{\alpha,q}(Q) = \mathrm{Rep}(\Lambda^q(Q), \alpha) /\!/ \mathrm{GL}(\alpha)$$

These are Poisson varieties [Van den Bergh,08]

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These are Poisson varieties [Van den Bergh,08]

$\mathcal{M}^{\Lambda}_{\alpha,q}(Q)$ only depends on $\alpha, q$ and $Q$ *seen as an undirected graph*, up to isomorphism of varieties [CBS,04]

A bit harder to prove : these isomorphisms preserve the Poisson structures

# Goal for today

We will show that :

the isomorphisms of (multiplicative) quiver varieties hence obtained can be checked to preserve the Poisson structures **directly at the level of the path algebras**

# Plan for the talk

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2. **Double brackets**
3. Morphisms of double Poisson brackets
4. The “quasi-” case

# Kontsevich-Rosenberg principle

Following [Kontsevich, '93] and [Kontsevich-Rosenberg, '99]

$$\begin{array}{ccc} \text{associative } \mathbb{C}\text{-algebra} & \to & \text{commutative } \mathbb{C}\text{-algebra} \\ A & \longrightarrow & \mathbb{C}[\mathrm{Rep}(A,n)] \end{array}$$

$\mathbb{C}[\mathrm{Rep}(A,n)]$ is generated by symbols $a_{ij}$, $\forall a \in A$, $1 \leq i,j \leq n$.
Rules : $1_{ij} = \delta_{ij}$, $(a+b)_{ij} = a_{ij} + b_{ij}$, $(ab)_{ij} = \sum_k a_{ik} b_{kj}$.

**Goal :** Find a property $P_{nc}$ on $A$ that gives the usual property $P$ on $\mathbb{C}[\mathrm{Rep}(A,n)]$ for all $n \in \mathbb{N}^\times$

# Double brackets

We follow [Van den Bergh, *double Poisson algebras*,'08]

$A$ denotes an arbitrary f.g. associative $\mathbb{C}$-algebra, $\otimes = \otimes_{\mathbb{C}}$

For $d \in A^{\otimes 2}$, set $d = d' \otimes d''(= \sum_k d'_k \otimes d''_k)$, and $\tau_{(12)}d = d'' \otimes d'$.
Multiplication on $A^{\otimes 2}$ : $(a \otimes b)(c \otimes d) = ac \otimes bd$.

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## Definition

A *double bracket* on $A$ is a $\mathbb{C}$-bilinear map $\{\!\{-,-\}\!\} : A \times A \to A^{\otimes 2}$ which satisfies

1. $\{\!\{a,b\}\!\} = -\tau_{(12)} \{\!\{b,a\}\!\}$ (cyclic antisymmetry)
2. $\{\!\{a,bc\}\!\} = (b \otimes 1) \{\!\{a,c\}\!\} + \{\!\{a,b\}\!\} (1 \otimes c)$ (outer derivation)
3. $\{\!\{ad,b\}\!\} = (1 \otimes a) \{\!\{d,b\}\!\} + \{\!\{a,b\}\!\} (d \otimes 1)$ (inner derivation)

# Preliminary result

(notation) $\rightsquigarrow \{\!\{a,b\}\!\} = \{\!\{a,b\}\!\}' \otimes \{\!\{a,b\}\!\}''$

## Lemma (Van den Bergh,'08)

*If $A$ has a double bracket $\{\!\{-,-\}\!\}$, then $\mathbb{C}[\mathrm{Rep}(A,n)]$ has a unique antisymmetric biderivation $\{-,-\}_{\mathrm{P}}$ satisfying*

$$\{a_{ij}, b_{kl}\}_{\mathrm{P}} = \{\!\{a,b\}\!\}'_{kj}\ \{\!\{a,b\}\!\}''_{il}\ . \tag{1}$$

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## Example

$A = \mathbb{C}[x]$, $\{\!\{x,x\}\!\} = x \otimes 1 - 1 \otimes x$ endows $\mathfrak{gl}_n(\mathbb{C}) = \mathbb{C}[\mathrm{Rep}(A,n)]$ with

$$\{x_{ij}, x_{kl}\}_{\mathrm{P}} = x_{kj} 1_{il} - 1_{kj} x_{il}$$

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# Double Poisson bracket

Recall $d = d' \otimes d'' \in A^{\otimes 2}$ (notation) $\rightsquigarrow \{\!\{a,b\}\!\} = \{\!\{a,b\}\!\}' \otimes \{\!\{a,b\}\!\}''$

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From a double bracket $\{\!\{-,-\}\!\}$, define $\{\!\{-,-,-\}\!\} : A^{\times 3} \to A^{\otimes 3}$

$$\begin{aligned} \{\!\{a,b,c\}\!\} = & \quad\ \ \big\{\!\!\big\{a, \{\!\{b,c\}\!\}'\big\}\!\!\big\} \otimes \{\!\{b,c\}\!\}'' \\ & + \tau_{(123)} \big\{\!\!\big\{b, \{\!\{c,a\}\!\}'\big\}\!\!\big\} \otimes \{\!\{c,a\}\!\}'' \\ & + \tau_{(132)} \big\{\!\!\big\{c, \{\!\{a,b\}\!\}'\big\}\!\!\big\} \otimes \{\!\{a,b\}\!\}'' , \quad \forall a,b,c \in A \end{aligned}$$

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**Definition**

A double bracket $\{\!\{-,-\}\!\}$ is *Poisson* if $\{\!\{-,-,-\}\!\} : A^{\times 3} \to A^{\otimes 3}$ vanishes. We say $(A, \{\!\{-,-\}\!\})$ is a *double Poisson algebra*.

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We say $(A, \{\!\{-, -\}\!\})$ is a *double Poisson algebra*.

## Example

1. $A = \mathbb{C}[x]$, $\{\!\{x, x\}\!\} = x \otimes 1 - 1 \otimes x$.
2. $A = \mathbb{C}\langle x, y\rangle$, $\{\!\{x, x\}\!\} = 0 = \{\!\{y, y\}\!\}$, $\{\!\{x, y\}\!\} = 1 \otimes 1$.

# A first result

(notation) $\rightsquigarrow \{\!\{a,b\}\!\} = \{\!\{a,b\}\!\}' \otimes \{\!\{a,b\}\!\}''$

## Proposition (Van den Bergh,'08)

*If $A$ has a double bracket $\{\!\{-,-\}\!\}$, then $\mathbb{C}[\mathrm{Rep}(A,n)]$ has a unique antisymmetric biderivation $\{-,-\}_{\mathrm{P}}$ satisfying*

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*If $\{\!\{-,-\}\!\}$ is Poisson, then $\{-,-\}_{\mathrm{P}}$ is a Poisson bracket.*

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$$\{x_{ij}, y_{kl}\}_{\mathrm{P}} = \delta_{kj}\delta_{il}\ , \quad \{x_{ij}, x_{kl}\}_{\mathrm{P}} = 0 = \{y_{ij}, y_{kl}\}_{\mathrm{P}}\ .$$

This is the canonical Poisson bracket on $T^*\mathfrak{gl}_n$.

# A first dictionary

**Algebra** $A$

double bracket $\{\!\{-,-\}\!\}$

double Poisson bracket $\{\!\{-,-\}\!\}$

**Geometry** $\mathbb{C}[\mathrm{Rep}(A,n)]$

anti-symmetric biderivation $\{-,-\}_{\mathrm{P}}$

Poisson bracket $\{-,-\}_{\mathrm{P}}$

# Hamiltonian reduction

## Definition

If $(A, \{\!\{-,-\}\!\})$ is a double Poisson algebra,
$\mu_A \in A$ is a *moment map* if $\{\!\{\mu_A, a\}\!\} = a \otimes 1 - 1 \otimes a$, $\forall a \in A$

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$\Rightarrow$ $\{-,-\}$ descends to a Lie bracket on the vector space $A_\lambda/[A_\lambda, A_\lambda]$ for $A_\lambda := A/\langle \mu_A - \lambda \rangle$

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## Proposition (Van den Bergh,'08)

*The Poisson structure* $\{-,-\}_{\mathrm{P}}$ *on* $\mathrm{Rep}(A,n)$ *descends to* $\mathrm{Rep}(A_\lambda, n)/\!/\,\mathrm{GL}_n$ *in such a way that*

$$\{\mathrm{tr}\,\mathcal{X}(a), \mathrm{tr}\,\mathcal{X}(b)\}_{\mathrm{P}} = \mathrm{tr}\,\mathcal{X}(\{a,b\})\,. \tag{3}$$

# Dictionary

## Algebra $A$

double bracket $\{\!\{-,-\}\!\}$

double Poisson bracket $\{\!\{-,-\}\!\}$

moment map $\mu_A$
$A_\lambda = A/(\mu_A - \lambda)$, $\lambda \in \mathbb{C}$
$(A_\lambda/[A_\lambda, A_\lambda], \{-,-\})$ is a Lie algebra

## Geometry $\mathbb{C}[\mathrm{Rep}(A,n)]$

anti-symmetric biderivation $\{-,-\}_{\mathrm{P}}$

Poisson bracket $\{-,-\}_{\mathrm{P}}$

moment map $\mathcal{X}(\mu_A)$
slice $S_\lambda := \mathcal{X}(\mu_A)^{-1}(\lambda\,\mathrm{Id}_n)$
$\{-,-\}_{\mathrm{P}}$ is Poisson on $\mathbb{C}[S_\lambda/\!/\,\mathrm{GL}_n]$

Recall $\{-,-\} = \mathrm{m} \circ \{\!\{-,-\}\!\}$ descends to $A_\lambda/[A_\lambda, A_\lambda]$

# Examples from quivers

Fix quiver $Q$, with double $\bar{Q}$ (if $a \in Q$, $a : t \to h$, add $a^* : h \to t$)

**Theorem (Van den Bergh,'08)**

*The algebra $A = \mathbb{C}\bar{Q}$ has a double Poisson bracket given by*

$$\{\!\{a, a^*\}\!\} = e_{h(a)} \otimes e_{t(a)} \ \forall a \in Q\,, \quad \{\!\{a, b\}\!\} = 0 \text{ if } a \neq b^*, b \neq a^* \tag{4}$$

*and (non-commutative) moment map $\mu = \sum_{a \in Q} [a, a^*]$.*

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$\Longrightarrow$ $\operatorname{Rep}(\mathbb{C}\bar{Q}, \alpha)$ has a Poisson structure (with 'usual' moment map)

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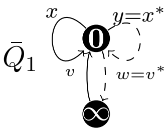
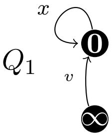
$\Longrightarrow$ $\operatorname{Rep}(\mathbb{C}\bar{Q}, \alpha)$ has a Poisson structure (with 'usual' moment map)
$\Longrightarrow$ Poisson structure on quiver varieties by Hamiltonian reduction on

$$\Big\{ \sum_{a\in Q} [\mathcal{X}(a), \mathcal{X}(a^*)] = \prod_{s\in I} \lambda_s \operatorname{Id}_{\alpha_s} \Big\} /\!/ \operatorname{GL}(\alpha) \simeq \underbrace{\operatorname{Rep}(\Pi^\lambda(Q), \alpha) /\!/ \operatorname{GL}(\alpha)}_{\mathcal{M}^{\Pi}_{\alpha,\lambda}(Q)}$$

# Nice example : CM spaces

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$$\{\!\{v, w\}\!\} = e_0 \otimes e_\infty$$

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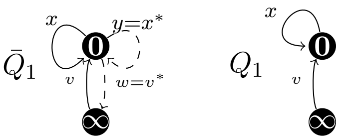


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1. Take $(\alpha_0, \alpha_\infty) = (n, 1)$, $n \geq 1$; Attach $\mathbb{C}^n$ at 0, $\mathbb{C}$ at $\infty$
2. $x, y, v, w \ \to \ X, Y \in \mathrm{Mat}_{n\times n}$, $V \in \mathrm{Mat}_{1\times n}$, $W \in \mathrm{Mat}_{n\times 1}$
3. $\{\!\{-,-\}\!\} \ \to \ \{X_{ij}, Y_{kl}\} = \delta_{kj}\delta_{il}\, , \{V_j, W_k\} = \delta_{kj}$
4. $\mu = [x, y] + [v, w]$ restricts to $[x, y] - wv \in e_0 \mathbb{C}\bar{Q}_1 e_0$
$\rightsquigarrow \ [X, Y] - WV$ is moment map for $\mathrm{GL}_n \curvearrowright \mathbb{C}^n \hookrightarrow \mathbb{C}^n \oplus \mathbb{C}$

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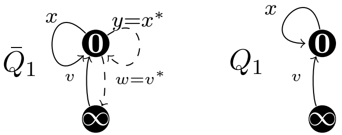


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   $\rightsquigarrow$ $[X, Y] - WV$ is moment map for $\mathrm{GL}_n \curvearrowright \mathbb{C}^n \hookrightarrow \mathbb{C}^n \oplus \mathbb{C}$

Hamiltonian reduction at $\mathrm{Id}_n$ : $\mathcal{C}_n = \{[X, Y] - WV = \mathrm{Id}_n\} /\!/ \mathrm{GL}_n$

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3. **Morphisms of double Poisson brackets**
4. The “quasi-” case

# Definition I

(mostly based on [F., 2008.01409] from now on)

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$A_1, A_2$ endowed with double brackets $\{\!\{-,-\}\!\}_1, \{\!\{-,-\}\!\}_2$.

## Definition

$\phi : A_1 \to A_2$ is a *morphism of double brackets* if it is an algebra homomorphism such that for any $a, b \in A_1$

$$\{\!\{\phi(a), \phi(b)\}\!\}_2 = (\phi \otimes \phi)\, \{\!\{a, b\}\!\}_1 \,.$$

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$\phi : A_1 \to A_2$ is a *morphism of double brackets* if it is an algebra homomorphism such that for any $a, b \in A_1$

$$\{\!\{\phi(a), \phi(b)\}\!\}_2 = (\phi \otimes \phi)\, \{\!\{a, b\}\!\}_1\ .$$

## Example

$A = \mathbb{C}\langle x, y\rangle$ can be endowed with

$$\{\!\{x, x\}\!\}_1 = 0 = \{\!\{x, x\}\!\}_2\ , \quad \{\!\{y, y\}\!\}_1 = 0 = \{\!\{y, y\}\!\}_2\ ,$$
$$\{\!\{x, y\}\!\}_1 = 1 \otimes 1,\ \{\!\{x, y\}\!\}_2 = -1 \otimes 1\,.$$

Automorphism $x \mapsto y, y \mapsto x$ defines an isomorphism of double brackets $(A, \{\!\{-,-\}\!\}_1) \to (A, \{\!\{-,-\}\!\}_2)$

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If $A_1, A_2$ admit moment maps $\mu_1, \mu_2$ and $\phi(\mu_1) = \mu_2$, we say that $\phi$ is a *morphism of Hamiltonian algebras*.

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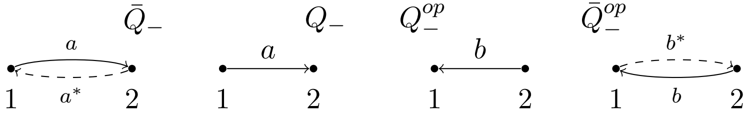
# Morphism for quivers : reversing the arrow



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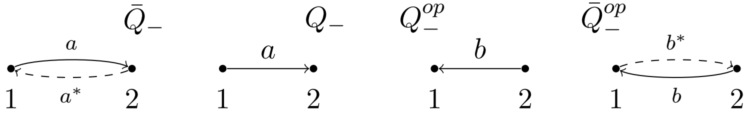


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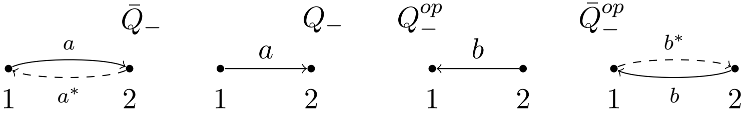


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$\phi : \mathbb{C}\bar{Q}_- \to \mathbb{C}\bar{Q}_-^{op}$ given by $\phi(a) = b^*$, $\phi(a^*) = -b$, is an isomorphism of Hamiltonian algebras.

$\phi$ lifts the isomorphism of preprojective algebras [Crawley-Boevey,Holland,'98]
$\rightsquigarrow$ Poisson isomorphism $\bar{\phi}_\alpha^\lambda : \mathcal{M}_{\alpha,\lambda}^\Pi(Q_-) \to \mathcal{M}_{\alpha,\lambda}^\Pi(Q_-^{op})$

# Fusion

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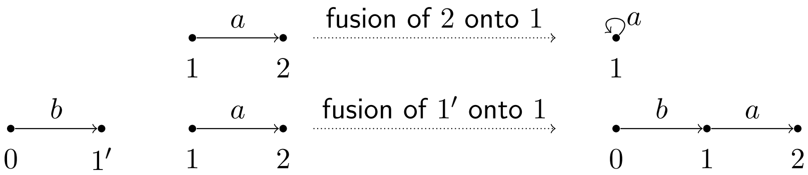

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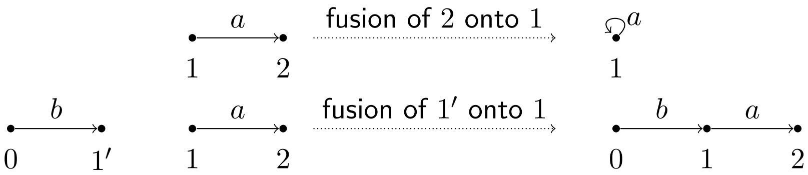


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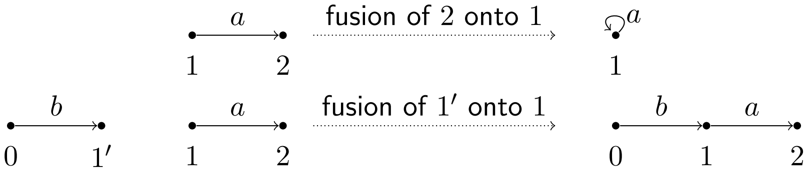


By "doubling", $\bar{Q}$ is obtained from $|Q|$ copies of $\bar{Q}_-$

The analogous construction at the algebra level is called fusion [VdB,'08]
It consists of identifying orthogonal idempotents in the algebra

$\rightsquigarrow$ can obtain $\mathbb{C}\bar{Q}$ from $|Q|$ copies of $\mathbb{C}\bar{Q}_-$ by fusion of the idempotents corresponding to the identified vertices

# Fusion and morphisms

### Lemma

*Fusion is compatible with morphisms of double (Poisson) brackets*

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Fusion of the idempotents in $\mathbb{C}\bar{Q}_-$ or $\mathbb{C}\bar{Q}_-^{op}$ results in the free algebra $\mathbb{C}\langle a, a^* \rangle$ or $\mathbb{C}\langle b, b^* \rangle$

Under identification with $\mathbb{C}\langle x, y \rangle$ through $a, b \leftrightarrow x$ and $a^*, b^* \leftrightarrow y$, we get the Hamiltonian algebra structure
$\{\!\{x, x\}\!\} = 0 = \{\!\{y, y\}\!\} \ , \quad \{\!\{x, y\}\!\} = 1 \otimes 1 \ , \ \mu = [x, y]$

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The isomorphism $\phi : \mathbb{C}\bar{Q}_- \to \mathbb{C}\bar{Q}_-^{op}$ given by $\phi(a) = b^*$, $\phi(a^*) = -b$ becomes an automorphism (of Hamiltonian algebras) on $\mathbb{C}\langle x, y \rangle$ given by $x \mapsto y$, $y \mapsto -x$.

# Isomorphic quiver varieties

## Proposition

*$\mathcal{M}^{\Pi}_{\alpha,\lambda}(Q)$ only depends on $\alpha, \lambda$ and $Q$ seen as an undirected graph, up to isomorphism of Poisson varieties*

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It suffices to get that the Hamiltonian algebra structure on $\mathbb{C}\bar{Q}$ given by Van den Bergh only depends on $Q$ *seen as an undirected graph*, up to isomorphism.

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This follows from the case of $Q_-$ by fusion. □

# A question

The morphism giving that "the Hamiltonian algebra structure on $\mathbb{C}\bar{Q}$ given by Van den Bergh only depends on $Q$ *seen as an undirected graph*" is obtained by lifting the corresponding isomorphism of (deformed) preprojective algebras $\Pi^\lambda(Q) = \mathbb{C}\bar{Q}/\langle \sum_a [a, a^*] - \lambda \rangle$ given in [CBH,'98]

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*When can we lift an automorphism of $\Pi^{\lambda}(Q)$ to an automorphism of Hamiltonian algebras on $\mathbb{C}\bar{Q}$? (for a fixed pair $(\{\!\{-,-\}\!\}, \mu)$)*

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## Example (first Weyl algebra $A_1 = \mathbb{C}\langle x, y\rangle/\langle xy - yx - 1\rangle$)

$A_1$ is isomorphic to $\Pi^1(Q_\circ)$ for $Q_\circ$ the one-loop quiver.
Following [Dixmier,'68], automorphisms of $A_1$ are generated by

$$\phi_{k,\gamma}(x) = x + \gamma y^k,\ \phi_{k,\gamma}(y) = y\,, \quad \phi'_{k,\gamma}(x) = x,\ \phi'_{k,\gamma}(y) = y + \gamma x^k$$

They can be lifted as Hamiltonian algebras automorphisms on $\mathbb{C}\bar{Q}_\circ$ (!)

# Plan for the talk

1. Motivation
2. Double brackets
3. Morphisms of double Poisson brackets
4. **The "quasi-" case**

# quasi-Dictionary

$\{-,-\} = \mathrm{m} \circ \{\!\{-,-\}\!\}$ descends to $A_q/[A_q, A_q]$

| Algebra $A$ | Geometry $\mathbb{C}[\mathrm{Rep}(A,n)]$ |
| --- | --- |
| double bracket $\{\!\{-,-\}\!\}$ | anti-symmetric biderivation $\{-,-\}_{\mathrm{P}}$ |
| double quasi-Poisson bracket $\{\!\{-,-\}\!\}$ | quasi-Poisson bracket $\{-,-\}_{\mathrm{P}}$ |
| multiplicative moment map $\Phi_A$<br>$A_q = A/(\Phi_A - q)$, $q \in \mathbb{C}^\times$<br>$(A_q/[A_q, A_q], \{-,-\})$ is Lie algebra | multiplicative moment map $\mathcal{X}(\Phi_A)$<br>slice $S_q := \mathcal{X}(\Phi_A)^{-1}(q\,\mathrm{Id}_n)$<br>$\{-,-\}_{\mathrm{P}}$ is Poisson on $\mathbb{C}[S_q//\,\mathrm{GL}_n]$ |

quasi-Poisson geometry after [Alekseev – Kosmann-Schwarzbach – Meinrenken,'02]

# Examples from quivers

Fix quiver $Q$. Let $A_Q = \mathbb{C}\bar{Q}_S$ localisation at $S = \{1 + aa^* \mid a \in \bar{Q}\}$

## Theorem (Van den Bergh,'08)

*The algebra $A_Q$ has a double quasi-Poisson bracket whose (non-commutative) multiplicative moment map is given by*

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$\Longrightarrow$ $\mathrm{Rep}(A_Q, \alpha)$ has quasi-Poisson structure (mult. mom. map $\mathcal{X}(\Phi)$)

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$\Longrightarrow$ $\mathrm{Rep}(A_Q, \alpha)$ has quasi-Poisson structure (mult. mom. map $\mathcal{X}(\Phi)$)

$\Longrightarrow$ Poisson structure on multiplicative quiver varieties by quasi-Hamiltonian reduction on

$$\Big\{\mathcal{X}(\Phi) = \prod_{s \in I} q_s \, \mathrm{Id}_{\alpha_s}\Big\} // \, \mathrm{GL}(\alpha) \simeq \underbrace{\mathrm{Rep}(\Lambda^q(Q), \alpha) // \, \mathrm{GL}(\alpha)}_{\mathcal{M}^{\Lambda}_{\alpha,q}(Q)}$$

# Fusion and Van den Bergh's proof

By fusion, we obtained $\mathbb{C}\bar{Q}$ from $|Q|$ copies of $\mathbb{C}\bar{Q}_-$
Similarly, we can obtain $A_Q$ from $|Q|$ copies of $A_{Q_-}$ ('localised' version)

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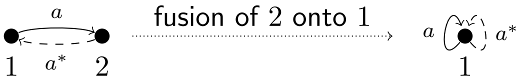
Drawback: the structure depends on the order of the fusion :
the fusions $e_i \to e_j$ or $e_j \to e_i$ give isomorphic algebras, but different double quasi-Poisson brackets!

$\rightsquigarrow$ get Van den Bergh's result from basic case $A_{Q_-}$;
the order of fusion is the one used in the multiplicative moment map

# Fusion in the quasi-case : example




LHS : $A_{Q_-} = (\mathbb{C}\bar{Q}_-)_{1+aa^*,1+a^*a}$ has double quasi-Poisson bracket

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RHS : $A_{Q_\circ} \simeq \mathbb{C}\langle a,a^*\rangle_{1+aa^*,1+a^*a}$ has double quasi-Poisson bracket

$$\{\!\{a,a\}\!\} = \frac{1}{2}(a^2 \otimes 1 - 1 \otimes a^2)\,,$$

$$\{\!\{a^*,a^*\}\!\} = -\frac{1}{2}((a^*)^2 \otimes 1 - 1 \otimes (a^*)^2)\,,$$

$$\{\!\{a,a^*\}\!\} = 1 \otimes 1 + \frac{1}{2}a^*a \otimes 1 + \frac{1}{2}1 \otimes aa^* + \frac{1}{2}(a^* \otimes a - a \otimes a^*)$$

(this corresponds to the ordering $a < a^*$ in [VdB,'08])

# Fusion and morphisms I

Recall the drawback of fusion in an algebra with a double quasi-Poisson bracket : performing the fusions $e_i \to e_j$ or $e_j \to e_i$ gives isomorphic algebras, but different double quasi-Poisson brackets!

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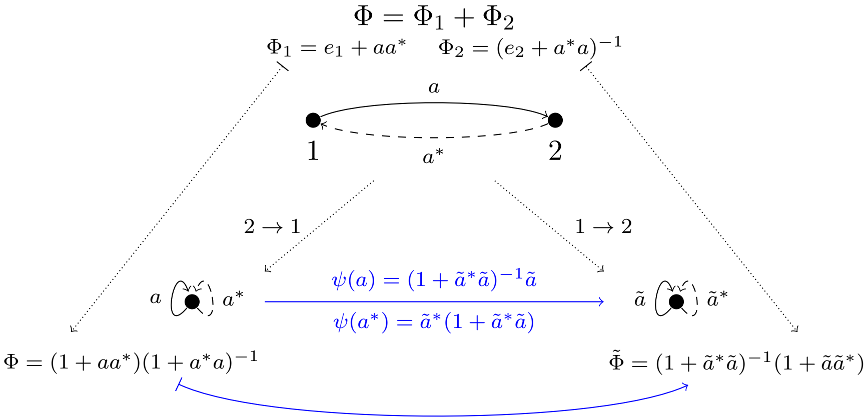
• We need the multiplicative moment map to define the isomorphism, contrary to the Hamiltonian case (this can be slightly relaxed)

• Non-commutative version of [AKSM,'02]

# Isomorphism and fusion in the quasi-case : example




$\psi : A_{Q_\circ} \to A_{Q_\circ}$ is an isomorphism of quasi-Hamiltonian algebras for the two *distinct* structures induced by fusion

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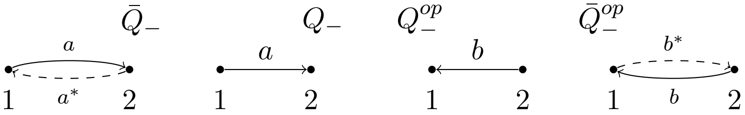


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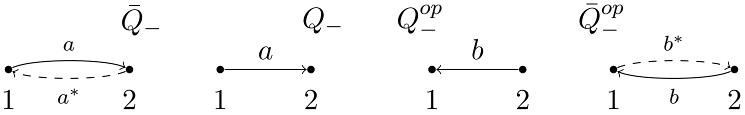


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$\psi$ lifts the isomorphism of multiplicative preprojective algebras defined in [Crawley-Boevey – Shaw,'06]

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This follows from the case of $Q_-$ by fusion. □

Main technicality here : changing the order of fusions yields a non-trivial isomorphism, so it seems quite cumbersome to explicitly write down this map in general.

# Thank you for your attention

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