ALGEBRAIC TOPOLOGY IV ‖ MICHAELMAS 2019
HOMEWORK 1

Please hand in Problems 1 – 5 either at the lecture on Monday 21st October or by 5pm that day at CM233.

Problem 1. Describe the kernel, image and cokernel of each of the following homomorphisms. Recall, for a homomorphism $f\colon A \to B$ of groups or vector spaces, its *cokernel* is the quotient $B/\mathrm{Im}(f)$.

(1) $\mathbb{Z} \to \mathbb{Z}$ given by $n \mapsto 2n$;

(2) $\mathbb{Z}/r \to \mathbb{Z}/2r$ given by $n \mapsto 2n$;

(3) $\mathbb{Z} \oplus \mathbb{Z}/2 \to \mathbb{Z} \oplus \mathbb{Z}/4$ given by $(n, m) \mapsto (2n, 2m)$;

(4) $\mathbb{R} \to \mathbb{R}^2$ given by $x \mapsto (x, -x)$;

(5) $\mathbb{R}^2 \to \mathbb{R}^2$ given by $(a, b) \mapsto (a + b, a - b)$;

(6) $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ given by $(a, b) \mapsto (a + b, a - b)$

Problem 2. Compute the homology of the chain complex

$$0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{g} \mathbb{Z} \to 0$$

where the homomorphisms f and g are given by $f(n) = (0, 3n)$ and $g(y, z) = 2y$.

Problem 3.
In the following diagram, the rows represent chain complexes and the vertical maps a chain map f from the upper chain complex, C_* to the lower D_*. The nonzero groups are C_2, C_1, C_0 on the top row and D_2, D_1, D_0 on the bottom row. Check that both are indeed chain complexes and that f is a chain map. Compute the homology of each chain complex and the induced map in homology.

$$\begin{tikzcd}
C_* : & 0 \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "\alpha"] \arrow[d, "f_2"] & \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \arrow[r, "\beta"] \arrow[d, "f_1"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] \arrow[d, "f_0"] & 0 \\
D_* : & 0 \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "\gamma"] & \mathbb{Z} \arrow[r, "\delta"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

with homomorphisms given by

$$\alpha(x, y) = (x - y, x - y, x - y) \qquad \beta(x, y, z) = (z - x, 0)$$
$$f_2(x, y) = (2x, 2y) \qquad f_1(x, y, z) = 2x + 2y \qquad f_0(x, y) = y$$
$$\gamma(x, y) = 2x - 2y \qquad \delta(x) = 0\,.$$

Problem 4.

(1) Let X be a space with n path components. Prove that $H_0(X) \cong \mathbb{Z}^n$.

(2) Let $f\colon X \to Y$ be a continuous map. Describe the induced map $f_*\colon H_0(X) \to H_0(Y)$ in terms of the path components of X and Y.

Problem 5. We say that two singular n-chains $\sigma, \tau \in C_n(X)$ are *homologous* if there is an $(n+1)$-chain $\theta \in C_{n+1}(X)$ with $\partial\theta = \sigma - \tau$. Let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex. Recall that

$$\Delta^1 = \{(t, 1-t) \in \mathbb{R}^2 \mid t \in [0,1]\}.$$

Let $\overline{\sigma}\colon \Delta^1 \to X$ be given by

$$(t, 1-t) \mapsto \sigma(1-t, t).$$

Prove that $-\sigma$ and $\overline{\sigma}$ are homologous i.e. prove that $\sigma + \overline{\sigma}$ is a boundary. Also prove that $\sigma + \overline{\sigma}$ is a cycle.

Problem 6. Let $X \subseteq \mathbb{R}^N$ be a convex subset. That is, for any two points $x, y \in X$, the straight line joining x and y also lies in X. Prove that $H_1(X) = 0$. (In fact $H_n(X) \cong H_n(\mathrm{pt})$ for every $n \in \mathbb{N}_0$.)

Problem 7. For the ambitious: try to compute the homology groups of the circle $H_n(S^1)$, directly from the definition of singular homology.