ALGEBRAIC TOPOLOGY IV ‖ EPIPHANY 2020
PROBLEM SHEET 2

Please solve these problems during week 13. Problems 2 and 3 will form part of the next homework.

Problem 1. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups and suppose that C is free abelian. Let G be an abelian group. Show that the dual sequence below is also short exact:

$$0 \to \mathrm{Hom}(C, G) \to \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G) \to 0.$$

Problem 2.

(a) Write down definitions of a cochain map and a cochain homotopy.
(b) Show that a cochain map induces a map on cohomology.
(c) Show that chain homotopic cochain maps induce the same map on cohomology.
(d) Show that a chain map gives rise to a cochain map on the dual complex, and that chain homotopic chain maps induce chain homotopic cochain maps.

Problem 3.

(a) Show that a short exact sequence of cochain complexes gives rise to a long exact sequence in cohomology.
(b) Write down the Mayer-Vietoris long exact sequence for cohomology groups. Using Problem 1, Problem 3(a), and the proof for homology from lectures, give a proof.
(c) Write down the long exact sequence of a pair in cohomology.
(d) State the excision theorem for cohomology groups of pairs.
(e) Give the definition of CW cohomology of a CW complex.

Problem 4. Compute the cohomology of the following spaces using (a) singular cohomology, and (b) CW cohomology:

(i) S^1;
(ii) $S^1 \times S^1$;
(iii) $\mathbb{RP}^2$;
(iv) $S^2 \times S^2$.

Problem 5. Give an example of a space X with $H^0(X; \mathbb{Z})$ and $H_0(X; \mathbb{Z})$ not isomorphic.