ALGEBRAIC TOPOLOGY IV ‖ EPIPHANY 2020
PROBLEM SHEET 4

Please solve these problems during week 15. These problems will form part of the next homework.

Problem 1. Let (C_*, ∂) be the chain complex with $C_n = \mathbb{Z}$ for all $n \geq 0$ and $C_n = 0$ for all $n < 0$. Suppose that $\partial_{2n} = 0\colon C_{2n} \to C_{2n-1}$ and $\partial_{2n+1}\colon C_{2n+1} \to C_{2n}$ is given by multiplication by 3 for $n \geq 0$.

(i) Calculate $H_q(C_*)$ for all $q \geq 0$.

(ii) Calculate $H^q(\mathrm{Hom}(C_*, \mathbb{Z}/6))$ for all $q \geq 0$ directly.

(iii) Calculate $H^q(\mathrm{Hom}(C_*, \mathbb{Z}/6))$ for all $q \geq 0$ using the universal coefficient theorem.

Problem 2. Let

$$0 \to N \xrightarrow{i} G \xrightarrow{p} Q \to 0$$

be a short exact sequence of abelian groups. Let C_* be a chain complex of free abelian groups.

(i) Show that there is a long exact sequence of cohomology groups

$$\cdots \xrightarrow{p_*} H^{k-1}(C;Q) \xrightarrow{\beta} H^k(C;N) \xrightarrow{i_*} H^k(C;G) \xrightarrow{p_*} H^k(C;Q) \xrightarrow{\beta} \cdots$$

The map β is called a *Bockstein* homomorphism.

(ii) Consider the short exact sequence

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0.$$

Show that

$$\beta\colon H^1(\mathbb{RP}^2;\mathbb{Z}/2) \to H^2(\mathbb{RP}^2;\mathbb{Z}/2)$$

is nonzero.