ALGEBRAIC TOPOLOGY IV ‖ EPIPHANY 2020
PROBLEM SHEET 7

The problems from this sheet and the previous sheet will be discussed in class on 16th March.

Apply your knowledge of cup and cap products and their naturality properties to obstruct the existence of maps between spaces with certain properties.

Problem 1. Given that the cohomology ring of $\mathbb{CP}^2$ is the truncated polynomial ring

$$H^*(\mathbb{CP}^2; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^3)$$

where $|x| = 2$, prove that $\mathbb{CP}^2$ and $S^2 \vee S^4$ are not homotopy equivalent.

Problem 2. Assume that the $\mathbb{Z}/2$-cohomology ring of $\mathbb{RP}^\ell$ is the truncated polynomial ring

$$H^*(\mathbb{RP}^\ell; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[y]/(y^{\ell+1})$$

where $|y| = 1$. Prove that $\mathbb{RP}^n$ is not a retract of $\mathbb{RP}^m$ if $n < m$.

For a subspace $X \subseteq Y$, with $i \colon X \to Y$ the inclusion map, a *retract* is a continuous map $r \colon Y \to X$ with $r \circ i = \mathrm{Id} \colon X \to X$.

Problem 3. Prove that there is no degree one map $S^1 \times S^2 \to \mathbb{RP}^3$.