ALGEBRAIC TOPOLOGY IV ‖ MICHAELMAS 2019
PROBLEM SHEET 8

Please solve these problems during the break or in the first week of Epiphany. We will discuss them in class in weeks 2 or 3 of Epiphany term.

Problem 1. Consider the function

$$\begin{aligned} g\colon \mathbb{C} &\to \mathbb{C} \\ z &\mapsto z^n. \end{aligned}$$

This induces a function $\overline{g}\colon S^2 \to S^2$ on the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$. Show that

$$\deg(\overline{g}) = n.$$

Problem 2. Construct a CW complex Y with homology

$$H_0(Y) \cong \mathbb{Z}^2,\ H_1(Y) \cong \mathbb{Z}/3,\ H_2(Y) = 0,\ H_3(Y) \cong \mathbb{Z},\ \text{and } H_k(Y) = 0$$

for $k \geq 4$.

Problem 3. Let us prove that there is no locally flat embedding of $\mathbb{RP}^2$ into $\mathbb{R}^3$.

(a) Write down or compute the homology of $\mathbb{RP}^2$.

(b) Show that if $\mathbb{RP}^2 \subset \mathbb{R}^3$ is a locally flat subspace (meaning that for every $x \in \mathbb{RP}^2$ there is an open set $U \ni x$ in $\mathbb{R}^3$ with a homeomorphism of pairs $(U, U \cap \mathbb{RP}^2) \cong (D^3, D^2)$) then there is a neighbourhood V of $\mathbb{RP}^2$ with $V \simeq \mathbb{RP}^2$ and $\partial V \cong S^2$.

To do this, consider the thickened Möbius band and describe its boundary. Then glue a thickenned $D^2 \times I$ to it.

(c) Using Mayer-Vietoris, consider $\mathbb{R}^3 \cong X \cup_{S^2} V$ and obtain a contradiction.

(d) Describe an embedding of $\mathbb{RP}^2$ into $\mathbb{R}^4$.

Problem 4.

(a) Recall that for a finitely generated abelian group $G \cong \mathbb{Z}^r \oplus T$, where $|T| < \infty$, we say that G has rank r and write $\operatorname{rk} G = r \in \mathbb{N}_0$. Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of abelian groups. Show that $\operatorname{rk}(B) = \operatorname{rk}(A) + \operatorname{rk}(C)$.

(b) Let X be a finite CW complex. Let

$$c_i(X) := \mathrm{rk}(C_i^{CW}(X)),$$

which equals the number of i-cells. Let

$$\beta_i(X) := \mathrm{rk}(H_i(X)),$$

called the ith *Betti* number of X. Prove that

$$\sum_{i=0}^{\infty}(-1)^i c_i(X) = \sum_{i=0}^{\infty}(-1)^i \beta_i(X).$$

We call this sum the *Euler characteristic* of X and denote it $\chi(X)$. (Since X is a finite CW complex, these are in fact finite sums.)

Hint: write $C_n := C_n^{CW}(X)$, $Z_n := \ker d_n$, $B_n := \operatorname{Im} d_{n+1}$, and $H_n := Z_n/B_n \cong H_n^{CW}(X) \cong H_n(X)$. Use the short eact sequences

$$0 \to Z_n \to C_n \to B_{n-1} \to 0$$

and

$$0 \to B_n \to Z_n \to H_n \to 0$$

(c) Prove that $\sum_{i=0}^{\infty}(-1)^i c_i(X)$ is independent of the choice of finite CW structure on X, and that if Y is another CW complex homotopy equivalent to X, then $\chi(X) = \chi(Y)$.

(d) Compute $\chi(S^2)$, $\chi(T^2)$, $\chi(D^2)$, $\chi(\mathbb{RP}^2)$ and $\chi(\Sigma_3)$ (where Σ_3 is a genus 3 orientable surface).

(e) Let X be a CW complex and let Y, Z be subcomplexes with $Y \cup Z = X$ such that $A := Y \cap Z$ is also a subcomplex. Prove that

$$\chi(X) = \chi(Y) + \chi(Z) - \chi(A).$$

(f) Prove that for any finite CW complex X we have $\chi(X \times S^1) = 0$.

Problem 5. A *Platonic solid* has boundary S^2 divided into regular polygons (e.g. triangle, square, pentagon, etc.) all with the same number of sides. They meet along common edges and each vertex is part of the same number of faces. Using the Euler characteristic, show that there are exactly 5 possible Platonic solids: the tetrahedron, the cube, the octahedron, the icosahedron and the dodecahedron. How many vertices, edges, and faces does each of these have?

Problem 6. Let $X = A \cup B$ be a suitable Mayer-Vietoris decomposition, that is $X = \operatorname{Int} A \cup \operatorname{Int} B$. Suppose that $A \simeq B \simeq A \cap B \simeq S^1$. What are the possible homology groups of X? Must X be homotopy equivalent to S^1?