Topologie Algébrique II: Problem sheet 1

Problem 1. Identify $S^n \wedge S^m$ with another well-known topological space.

Problem 2. Prove that homotopy determines an equivalence relation on the set of (based) maps $X \to Y$.

Problem 3. Prove that there are bijections

$$[A, X] \times [A, Y] \leftrightarrow [A, X \times Y]$$

and

$$[X, B] \times [Y, B] \leftrightarrow [X \vee Y, B]$$

for any based spaces X, Y, A, B.

Problem 4.

1. Prove that (based spaces, based maps) is a category (we call it Top or Top_* if we want to emphasise the base point.)
2. Prove that (based spaces, based homotopy classes of maps) is a category, the homotopy category of Top (we call it hTop or hTop_* if we want to emphasise the base point.)

Problem 5.* Show that the following inclusion is a deformation retract.

$$(S^{n-1} \times I) \cup (D^n \times \{0\}) \to D^n \times I.$$

Problem 6.* The *comb space* C is a subspace of $\mathbb{R}^2$ given by the union of $I \times \{0\}$, $\{0\} \times I$ and $\{1/n\} \times I$ for all $n \in \mathbb{N}$. Let $p_1 = (0, 0)$ and let $p_2 = (0, 1)$. Show that C is contractible when its basepoint is p_1 but when the basepoint is p_2, C is not contractible.

For this you can use the following fact: *If X is a contractible space, for every neighbourhood W of the basepoint p, there is a neighbourhood U of p such that $U \subseteq W$ and U is contractible inside W.*

It is possible to use a union of infinitely many copies of this space with itself to make a freely contractible space that is not contractible for any basepoint; see Question 6 of Chapter 0 of Hatcher, page 18.

Here are some exercises to remind you of algebraic topology I, that you should hopefully be able to do.

Problem 7. Apply the Seifert van Kampen theorem to compute:

(a) $\pi_1(\Sigma, *)$ where Σ is an orientable genus two surface.

(b) $\pi_1(\mathbb{K}, *)$, where $\mathbb{K}$ is the Klein bottle.

Problem 8. Compute the homology groups of the following spaces.

(i) S^n, for all n.

(ii) $(S^1)^n$, for all n, or if not for $n = 1, 2, 3$.

(iii) $X \vee Y$, for two based spaces X, Y, in terms of the homology groups of X and Y.

(iv) $\mathbb{RP}^n$, for all n.

Now some point set topology to finish.

Problem 9. Show that a compact subspace of a Hausdorff space is closed.

Problem 10.* Prove the adjoint theorem that for compactly generated spaces X, Y, Z, the adjoint map induces a homeomorphism

$$C(X \times Y, Z) \to C(X, C(Y, Z)).$$

Problem 11. Show that the identity map $k(X) \to X$ is continuous for every space X.

Problem 12.* Show that every metric space is compactly generated.