# Topologie Algébrique II: Problem sheet 10

**Problem 1.** Let $F \to E \to B$ be a fibration of CW complexes and $B$ simply connected. Show that $\chi(E) = \chi(F) \cdot \chi(B)$.

**Problem 2.** Suppose that $S^k \to S^\ell \to S^m$ is a fibration. Show that $\ell = 2m-1$ and $k = m-1$.

**Problem 3.** This exercise aims to show that $\pi_4(S^3) \cong \mathbb{Z}/2$. We know from the Freudenthal suspension theorem that once we show this homotopy group is $\mathbb{Z}/2$, then $\pi_{n+1}(S^n) \cong \mathbb{Z}/2$ for all $n \geq 3$.

We know $\mathbb{Z} \cong H^3(S^3;\mathbb{Z}) \cong [S^3, K(\mathbb{Z},3)]$. Let $f\colon S^3 \to K(\mathbb{Z},3)$ be a generator. Consider the path space fibration

$$\Omega K(\mathbb{Z},3) \simeq K(\mathbb{Z},2) \to PK(\mathbb{Z},3) \simeq * \to K(\mathbb{Z},3)$$

and define a space $X$ to be the pullback

$$\begin{array}{ccc} X & \longrightarrow & S^3 \\ \downarrow & & \downarrow f \\ PK(\mathbb{Z},3) & \longrightarrow & K(\mathbb{Z},3). \end{array}$$

(i) Show that $X \to S^3$ is a fibration with fibre $K(\mathbb{Z},2)$ ($X$ is also the homotopy fibre of $f$).

(ii) Use the long exact sequence in homotopy groups associated to a fibration to prove that $\pi_k(X) = 0$ for $k \leq 3$ and $\pi_k(X) \cong \pi_k(S^3)$ for $k > 3$. It might help to also consider the map between fibration sequences induced by the pullback diagram above.

(iii) Deduce that $H_4(X;\mathbb{Z}) \cong \pi_4(S^3)$.

(iv) Consider the cohomology spectral sequence associated to the fibration

$$\Omega K(\mathbb{Z},2) \to PK(\mathbb{Z},2) \simeq * \to K(\mathbb{Z},2),$$

as in class, and use the cup product structure to complete the induction started in class to show compute the cohomology ring $H^*(K(\mathbb{Z},2);\mathbb{Z}) \cong \mathbb{Z}[c]$, with $\deg c = 2$.

(v) Write down the $E_2$ page of the cohomology spectral sequence for the fibration $K(\mathbb{Z},2) \to X \to S^3$, for $0 \leq p \leq 4$ and $0 \leq q \leq 6$. Let $i \in H^3(S^3)$ be a generator. Express the generators of the $E_2$ terms in the form $c^q$ or the form $ic^q$.

(vi) Show that $H^2(X) = H^3(X) = 0$, and deduce from the spectral sequence that $d_3(c) = i$.

(vii) Show that $d^3(c^2) = 2ci$.

(viii) Deduce from the spectral sequence that $H^4(X) = E_\infty^{0,4} = 0$ and $H^5(X) = E_\infty^{3,2} = \mathbb{Z}/2$.

(ix) Use the universal coefficient theorem to show that $H_4(X) = \mathbb{Z}/2$. This completes the computation of $\pi_4(S^3)$ and therefore shows that $\pi_1^S \cong \mathbb{Z}/2$.

**Problem 4.** The goal of this exercise is to compute $\pi_5(S^3)$. This will be sufficient to compute $\pi_2^S \cong \mathbb{Z}/2$. Consider the adjoint of the suspension map $\sigma\colon \Sigma S^2 \to S^3$, that is $s\colon S^2 \to \Omega S^3$. Let $F$ be the homotopy fibre of $s$. So we have a fibration sequence $F \to S^2 \to \Omega S^3$.

(i) Write down the long exact sequence in homotopy groups for this fibration sequence up to $k = 4$.

(ii) Show that $\pi_1(F) = \pi_2(F) = 0$ and deduce that $\pi_3(F) = H_3(F; \mathbb{Z})$.

(iii) Compute $H_4(\Omega S^3)$ (we did it in class).

(iv) Use the homology spectral sequence for the fibration $F \to S^2 \to \Omega S^3$ to show that the differential $H_4(\Omega S^3) \to H_3(F)$ is an isomorphism. Hence compute $\pi_3(F)$.

(v) Deduce from the long exact sequence from (i) that $\pi_4(S^2) \to \pi_5(S^3)$ is onto.

(vi) Show that $\pi_5(S^3)$ is either $\mathbb{Z}/2$ or 0. (Use the Hopf fibration $S^1 \to S^3 \to S^2$ and the outcome of the previous exercise to show that $\pi_4(S^2)$ is nontrivial.) We now need to show that the former possibility holds, i.e. we show that $\pi_5(S^3)$ is nontrivial, and we will then know that is is isomorphic to $\mathbb{Z}/2$.

(vii) Consider the space $X$ from the previous exercise, obtained from the pullback

$$\begin{array}{ccc} X & \longrightarrow & S^3 \\ \downarrow & & \downarrow{\scriptstyle f} \\ PK(\mathbb{Z},3) & \longrightarrow & K(\mathbb{Z},3). \end{array}$$

. We know that $\pi_4(X) = \pi_4(S^3) = \mathbb{Z}/2$. Let $f\colon X \to K(\mathbb{Z}/2, 4)$ be a map inducing an isomorphism on $\pi_4$, and let $Y$ be the homotopy fibre of $f$, $Y = Ff$. Show using the long exact sequence in homotopy groups of $Y \to X \to K(\mathbb{Z}/2, 4)$ and the Hurewicz theorem that $H_5(Y; \mathbb{Z}) \cong \pi_5(S^3)$.

(viii) Deduce that $\pi_5(S^3) \cong H^5(Y; \mathbb{Z}/2)$.

(ix) Write down the cohomology spectral sequence for the fibration $Y \to X \to K(\mathbb{Z}/2, 4)$, with $\mathbb{Z}/2$ coefficients.

(x) Show that

$$H_k(X;\mathbb{Z}) = \begin{cases} 0 & k \text{ odd} \\ \mathbb{Z} & k = 0 \\ \mathbb{Z}/n & k = 2n. \end{cases}$$

This is a generalisation of the cohomology spectral sequence computation from the end of the previous exercise. At least, compute the homology of $X$ up to degree 6.

(xi) Show that $H^6(X;\mathbb{Z}/2) = 0$.

(xii) Deduce that

$$d_6 \colon H^5(Y;\mathbb{Z}/2) = E_6^{0,5} \to E_6^{6,0} = H^6(K(\mathbb{Z}/2,4),\mathbb{Z}/2)$$

is onto.

(xiii) Assume that $H^6(K(\mathbb{Z}/2,4);\mathbb{Z}/2) \neq 0$ (this is part of the theory of Steenrod squares). Deduce that $\pi_5(S^3) \cong \mathbb{Z}/2$.

(xiv) Use the Hopf fibration $S^3 \to S^7 \to S^4$ to show that $\pi_6(S^4) \cong \mathbb{Z}/2$, and therefore that $\pi_2^S \cong \mathbb{Z}/2$.