Topologie Algébrique II: Problem sheet 3

Problem 1. We showed that $F \times B \to B$ is a fibration. Show that the lifted homotopy $\widetilde{h}$ need not be unique.

Problem 2. Show that pullbacks of fibrations are fibrations.

Problem 3. Show that the composition of two fibrations is a fibration.

Problem 4. Let $\mathbb{F}$ be either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, in which cases let $d = 1, 2$ or 4 respectively i.e. d is the real dimension of $\mathbb{F}$. For any $n \geq 1$, there is a map

$$\begin{array}{rcl} S^{d(n+1)-1} & \to & \mathbb{F}P^n \\ (x_0, \ldots, x_n) & \mapsto & [x_0, \ldots, x_n]. \end{array}$$

Here points on the sphere are coordinates in $\mathbb{F}^{n+1}$ with $\sum_{i=0}^{n} |x_i|^2 = 1$, and the points of projective space are given in homogeneous coordinates. Show that these maps, called *Hopf fibrations* are fibrations. What are the fibres? Show that $\mathbb{F}P^1 \cong S^d$.

Problem 5. Consider the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Z \arrow[d] \\ Y \arrow[r] & P \arrow[r] & Q \end{tikzcd}$$

(i) Suppose that the two small squares are pullback squares. Show that the large rectangle with corners A, Z, Y and Q is also a pullback square.

(ii) Suppose that the large rectangle and the right hand square is a pullback square. Show that the left hand square is also a pullback square.