Topologie Algébrique II: Problem sheet 4

Problem 1. Let $x_0, x_1 \in X$ be points. Show that the loop spaces satisfy $\Omega_{x_0} X \simeq \Omega_{x_1} X$.

Problem 2. Show that $X^I \simeq X$.

Problem 3. Show that $p \colon X^I \to X$, sending a free path $\gamma \colon I \to X$ to its endpoint $\gamma(1) \in X$ is a fibration.

Problem 4. Let X, Y be based spaces. Write $S^1 = I/\partial I$. Define $\Sigma X := X \wedge S^1$, the reduced suspension of X. Prove the adjunction of based maps $\mathcal{C}_*(\Sigma X, Y) \cong \mathcal{C}_*(X, \Omega Y)$. Deduce that there is a bijection $[\Sigma X, Y] \cong [X, \Omega Y]$.

Problem 4. Define an operation on $+\colon [\Sigma X, Y] \times [\Sigma X, Y] \to [\Sigma X, Y]$ by

$$(f+g)(x,t) := \begin{cases} f(x, 2t) & 0 \leq t \leq 1/2 \\ f(x, 2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

Show that this operation is well-defined and makes $[\Sigma X, Y]$ into a group. Define an operation on $[X, \Omega Y]$ that makes this set into a group, and show that your adjunction map from the previous problem is a group isomorphism.

Problem 5. Show that $[\Sigma^2 X, Y]$ is an *abelian* group for all X, Y.