Topologie Algébrique II: Problem sheet 6

Solve the problems using what we have seen in class so far.

Problem 1. Let X be contractible. Show that $\pi_n(X) = 0$ for all $n \geq 0$.

Problem 2. Let X be a discrete space. Show that $\pi_n(X) = 0$ for $n \geq 1$.

Problem 3. Let $p\colon E \to B$ be a covering space. Show that $p_*\colon \pi_n(E) \to \pi_n(B)$ is an isomorphism for all $n \leq 2$.

Problem 4. Show that $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_n(S^1) = 0$ for $n \geq 2$.

Problem 5. Let $i \geq 2$. Show that $\pi_1(\mathbb{RP}^i) \cong \mathbb{Z}/2\mathbb{Z}$ and that $\pi_n(\mathbb{RP}^i) \cong \pi_n(S^i)$ for $n > 1$.

Problem 6. Show that $\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y)$ for any spaces X, Y and for $n \geq 0$.

Problem 7. Show that $\pi_i(S^n) = 0$ for $0 \leq i < n$. (This problem does not use fibrations; assume the fact that $f\colon S^i \to S^n$ is homotopic to a map such that there exists $p \in S^n$ with $p \notin f(S^i)$.)

Problem 8. For $n \geq 2$, show that $\pi_n(X \vee Y) \cong \pi_n(X) \oplus \pi_n(Y) \oplus \pi_{n+1}(X \times Y, X \vee Y)$.

Problem 9. Compute $\pi_n(\mathbb{RP}^n, \mathbb{RP}^{n-1})$ for $n \geq 2$. Whence show that the map of pairs $(\mathbb{RP}^n, \mathbb{RP}^{n-1}) \to (\mathbb{RP}^n/\mathbb{RP}^{n-1}, *)$ does not induce an isomorphism on π_n.

Problem 10. Use the Hopf bundle $S^3 \to S^7 \to S^4$ arising from the quaternions to show that $\pi_7(S^4)$ contains an element of infinite order. You may use that $\pi_n(S^n) \cong \mathbb{Z}$ for every $n \geq 1$, in particular for $n = 7$.

However you may not use other facts that we have not yet covered in the course, but which you might know. For example, $\pi_7(S^3)$ is finite, but do not use this. Instead, you could show that the composition $\pi_7(S^3) \xrightarrow{f \mapsto \Sigma f} \pi_8(S^4) \xrightarrow{\partial} \pi_7(S^3)$, where the second map is the boundary map in the long exact sequence of the Hopf fibration, is the identity.