Topologie Algébrique II: Problem sheet 7

A connected space X is called an Eilenberg-Maclane space if there is a group G and an integer $n \geq 1$ such that $\pi_i(X) \cong G$ if $i = n$ and $\pi_i(X) = 0$ for $i \neq 1$ and $i \neq n$. We say that such a space is a $K(G, n)$-space. For $K(G, 0)$ we take G as a discrete space.

Problem 1. Show that for any group G there is a space $K(G, 1)$, and that for any abelian group G and $n \geq 2$ there is a space $K(G, n)$.

Problem 2. Show that any two CW complexes which are $K(G, n)$-spaces are homotopy equivalent.

Problem 3. Which spheres S^n are Eilenberg-Maclane spaces?

Problem 4. Show that S^∞ is contractible: this is the infinite sphere defined as the colimit of the inclusions $S^n \to S^{n+1}$ sending S^n to the equator of S^{n+1}.

Problem 5. Show that there is a fibration

$$S^1 \to S^\infty \to \mathbb{CP}^\infty.$$

Problem 6. Show that $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$-space.

Problem 7. Find a compact manifold that is a $K(\mathbb{Z}^n, 1)$.

Problem 8. Find a (infinite dimensional) manifold that is a $K(\mathbb{Z}/2\mathbb{Z}, 1)$.

Manifolds that represent $K(\pi, n)$ spaces are rather special.

Problem 9. What Eilenberg-Maclane space is the loop space ΩX of a $K(\pi, n)$-space X?

Definition The *reduced cohomology* of a space X with coefficients in an abelian group π is $\widetilde{H}^n(X; \pi) := [X, K(\pi, n)]$. In particular $\widetilde{H}^n(X; \mathbb{Z}) = [X, K(\mathbb{Z}, n)]$.

For context, note that there is an alternative definition using chain complexes,

$$\widetilde{H}^n(X; \mathbb{Z}) = H_{-n}(\mathrm{Hom}_{\mathbb{Z}}(\widetilde{S}_{-*}(X), \mathbb{Z})),$$

where $\widetilde{S}_*(X)$ is the reduced singular chain complex of X. We will discuss the homological algebra definition further, but you are now supposed to forget you I told you this definition for rest of the problem sheet.

Problem 9. Show that $\widetilde{H}^n(X;\pi)$ is an abelian group.

Problem 10. Compute the reduced cohomology $\widetilde{H}^n(S^k;\pi)$ for all $n, k \geq 0$.

Problem 11. Let $i\colon A \to X$ be a cofibration. For any coefficient abelian group π, omitted from the notation below, prove that there is an exact sequence

$$\begin{gathered}\widetilde{H}^0(X/A) \to \widetilde{H}^0(X) \to \widetilde{H}^0(A) \to \widetilde{H}^1(X/A) \to \widetilde{H}^1(X) \to \widetilde{H}^1(A) \to \cdots \\ \widetilde{H}^n(X/A) \to \widetilde{H}^n(X) \to \widetilde{H}^n(A) \to \widetilde{H}^{n+1}(X/A) \to \widetilde{H}^{n+1}(X) \to \widetilde{H}^{n+1}(A) \to \cdots\end{gathered}$$

Problem 12. Suppose that X is an n-dimensional closed oriented manifold. There is an isomorphism

$$PD\colon \widetilde{H}^{n-r}(X;\mathbb{Z}) \to H_r(X;\mathbb{Z})$$

for $0 \leq r \leq n-1$, called Poincaré duality. Think about how to prove that every $(n-1)$- and $(n-2)$-dimensional $\mathbb{Z}$-homology class in X is represented by a closed submanifold, and write down your outline. A key word is transversality.