Topologie Algébrique II: Problem sheet 9

Problem 1. Use the Künneth theorem to compute the homology groups of $\mathbb{RP}^m \times \mathbb{RP}^n$ and the homology groups of $L(p,1) \times L(q,1)$, where $L(p,1)$ is the lens space.

Problem 2. Compute the cup product $\cup\colon H^1(\mathbb{T}^2;\mathbb{Z}/2) \otimes H^1(\mathbb{T}^2;\mathbb{Z}/2) \to H^2(\mathbb{T}^2;\mathbb{Z}/2)$. Use a simplicial subdivision of the torus and the formula using the front p face and the back q face corresponding to the Alexander-Whitney diagonal chain approximation map.

Problem 3. Compute the cup product $\cup\colon H^1(\mathbb{RP}^2;\mathbb{Z}/2) \otimes H^1(\mathbb{RP}^2;\mathbb{Z}/2) \to H^2(\mathbb{RP}^2;\mathbb{Z}/2)$. There are only two options: show that it is nontrivial. Can you show that it is nontrivial without computing?

Problem 4. Let A and B be subspaces of X.

(i) Construct a relative cup product

$$H^p(X,A) \otimes H^q(X,B) \to H^{p+q}(X, A \cup B)$$

and show that the diagram below is commutative:

$$\begin{array}{ccc} H^p(X,A) \otimes H^q(X,B) & \longrightarrow & H^{p+q}(X, A\cup B) \\ \downarrow & & \downarrow \\ H^p(X) \otimes H^q(X) & \longrightarrow & H^{p+q}(X) \end{array}$$

(ii) Introduce a basepoint in $A \cap B$ (supposed nonempty), and show that the next diagram commutes:

$$\begin{array}{ccc} H^p(X,A) \otimes H^q(X,B) & \longrightarrow & H^{p+q}(X, A\cup B) \\ \downarrow & & \downarrow \\ \widetilde{H}^p(X) \otimes \widetilde{H}^q(X) & \longrightarrow & \widetilde{H}^{p+q}(X) \end{array}$$

(iii) Let $X = A \cup B$ and $A \cap B \neq \emptyset$. Show that the cup product

$$\widetilde{H}^p(X) \otimes \widetilde{H}^q(X) \to \widetilde{H}^{p+q}(X)$$

vanishes.

(iv) Let $X = \Sigma Y$ for some Y. Show that the cup product

$$\widetilde{H}^p(X) \otimes \widetilde{H}^q(X) \to \widetilde{H}^{p+q}(X)$$

vanishes.