ALGEBRAIC TOPOLOGY IV – EPIPHANY
PROBLEM SHEET 1

Problem 1. Compute the homology and the cohomology of $\mathbb{RP}^n$. Here are some steps to guide you through the computation via CW homology. The group $\mathbb{Z}/2$ acts on the sphere S^n by $T \cdot x := -x$, where T is the nontrivial element of $\mathbb{Z}/2$. The space $\mathbb{RP}^n$ is

$$(\mathbb{R}^{n+1} \setminus \{0\})/(x \sim \lambda x, \lambda \in \mathbb{R} \setminus \{0\}).$$

(a) Show that $\mathbb{RP}^n$ is homeomorphic to the quotient of S^n by the antipodal action $\mathbb{RP}^n \cong S^n/\langle T \rangle$.
(b) Construct a CW decomposition of the n-sphere S^n that has exactly two i-cells e_1^i and e_2^i in each dimension $0, 1, \ldots, n$, with the additional property that $T((S^n)^{(i)}) = (S^n)^{(i)}$, that is the i-skeleton is preserved by the action, and $T(e_1^i) = e_2^i$ and $T(e_2^i) = e_1^i$ for each $i = 0, 1, \ldots, n$.
(c) Show that $\mathbb{RP}^n$ admits a CW structure with one i cell in each dimension $0, 1, \ldots, n$ and hence write down the cellular chain groups $C_*^{CW}(\mathbb{RP}^n)$.
(d) Compute the boundary maps

$$\partial_i^{CW} : C_i^{CW}(\mathbb{RP}^n) \to C_{i-1}^{CW}(\mathbb{RP}^n)$$

for $i = 1, \ldots, n$. (Hint: thinking about the degree of the antipodal map $S^n \to S^n$ might help.)
(e) Write down the full CW chain complex and whence compute the homology and the cohomology of $\mathbb{RP}^n$. (The form of the answer should depend to some extent on the parity of n.)

Problem 2. For which $n \geq 1$ is $\mathbb{RP}^n$ orientable?

Problem 3. Compute the homology and the cohomology of $\mathbb{CP}^n$. Here are some steps to guide you through the computation via CW homology. The space $\mathbb{CP}^n$ is

$$(\mathbb{C}^{n+1} \setminus \{0\})/(z \sim \lambda z, \lambda \in \mathbb{C} \setminus \{0\}).$$

(a) Show that $\mathbb{CP}^n$ is homeomorphic to the quotient of S^{2n+1} by the action of $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ on

$$S^{2n+1} = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \mid |z_0|^2 + |z_1|^2 + \cdots |z_n|^2 = 1\}$$

given by

$$\lambda \cdot (z_0, z_1, \ldots z_n) = (\lambda \cdot z_0, \lambda \cdot z_1, \ldots, \lambda z_n),$$

so $\mathbb{CP}^n \cong S^{2n+1}/S^1$.
(b) Show that modulo the action of S^1, every element of S^{2n+1} can be written as

$$(w, \sqrt{1 - |w|^2}) \in \mathbb{C}^n \times C$$

for some $w \in \mathbb{C}^n$ with $|w| \leq 1$. When is this description unique? Hence describe $\mathbb{CP}^n$ as an $2n$-disc union a $(2n-2)$-dimensional CW-complex.
(c) Deduce that $\mathbb{CP}^n$ admits a CW structure with one i cell in each even dimension $0, 2, 4 \ldots, 2n$ and hence write down the cellular chain groups $C_*^{CW}(\mathbb{CP}^n)$.

(d) What are the boundary maps

$$\partial_i^{CW}: C_i^{CW}(\mathbb{CP}^n) \to C_{i-1}^{CW}(\mathbb{CP}^n)$$

for $i = 1, \ldots, 2n$?

(e) Write down the full CW chain complex and whence compute the homology and the cohomology of $\mathbb{CP}^n$.

Problem 4. For which $n \geq 1$ is $\mathbb{CP}^n$ orientable?

Problem 5. What is the cohomology of T^2? What is the cohomology of $\mathbb{K}$, the Klein bottle?

Problem 6. There is a CW structure on the lens space $L(p, q)$ with CW chain complex

$$C_3 \to C_2 \to C_1 \to C_0$$

given by:

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$

Compute the homology and the cohomology of $L(p, q)$.

Problem 7. Using the computations of homology and cohomology that you made in the problems thus far, try to guess the formulation of the universal coefficient theorem expressing the cohomology with $\mathbb{Z}$ coefficients in terms of the homology with $\mathbb{Z}$ coefficients. How do the ranks of the homology and cohomology compare? How do the torsion parts behave?

Problem 8. Let $f\colon M \to N$ and $g\colon N \to P$ be maps of closed, oriented n-manifolds. Show that the degree

$$\deg g \circ f = \deg g \cdot \deg f.$$

Problem 9. Let M be a closed, connected n-manifold. Construct a map $M \to S^n$ of degree k.

Problem 10. Let Σ_g be the orientable surface of genus g. Construct a degree one map $\Sigma_3 \to T^2$.

Problem 11. Let M be a compact, connected manifold of dimension n. Suppose that $\partial M \neq \emptyset$. Show that M is homotopy equivalent to an $(n-1)$-dimensional CW complex. Deduce that $H_{n-1}(M)$ is a free abelian group.

Problem 12. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups and suppose that C is free abelian. Let G be an abelian group. Show that the dual sequence below is also short exact:

$$0 \to \mathrm{Hom}(C, G) \to \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G) \to 0$$