ALGEBRAIC TOPOLOGY IV – EPIPHANY
PROBLEM SHEET 2

Homework problems 2, 5, 6, 7, 9. Due in Wednesday 13th February.

Problem 1. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups and suppose that C is free abelian. Let G be an abelian group. Show that the dual sequence below is also short exact:

$$0 \to \mathrm{Hom}(C, G) \to \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G) \to 0$$

Problem 2. Given an example of a space X with $H^0(X;\mathbb{Z})$ and $H_0(X;\mathbb{Z})$ not isomorphic.

Problem 3. Calculate $\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}/n)$ for every $n \in \mathbb{N}$.

Problem 4.

(i) Calculate $\mathrm{Hom}(\mathbb{Q}, \mathbb{Q})$.

(ii) Show that $\mathrm{Hom}(\mathbb{Q}, \mathbb{Q})$ is isomorphic to $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}, \mathbb{Q})$ (the $\mathbb{Q}$-linear maps) as $\mathbb{Q}$ vector spaces.

(iii) Show that, as $\mathbb{C}$-vector spaces, $\mathrm{Hom}(\mathbb{C}, \mathbb{C})$ and $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$ (the $\mathbb{C}$-linear maps) are not isomorphic.

Problem 5. Let

$$G := \left\{ \frac{m}{2^n} \in \mathbb{Q} \,\middle|\, m, n \in \mathbb{Z} \right\}.$$

(i) Calculate $\mathrm{Hom}(G, G)$.

(ii) Show that the homomorphism $p_*\colon \mathrm{Hom}(G, G) \to \mathrm{Hom}(G, G/\mathbb{Z})$ induced by the projection $p\colon G \to G/\mathbb{Z}$ is not surjective.

Problem 6. Calculate $\mathrm{Ext}^1(\mathbb{Z}/n, \mathbb{Z}/m)$ for every $n, m \geq 2$.

Problem 7. Let (C_*, ∂) be the chain complex with $C_n = \mathbb{Z}$ for all $n \geq 0$ and $C_n = 0$ for all $n < 0$. Suppose that $\partial_{2n} = 0\colon C_{2n} \to C_{2n-1}$ and $\partial_{2n+1}\colon C_{2n+1} \to C_{2n}$ is given by multiplication by 3 for $n \geq 0$.

(i) Calculate $H_q(C_*)$ for all $q \geq 0$.

(ii) Calculate $H^q(\mathrm{Hom}(C_*, \mathbb{Z}/6))$ for all $q \geq 0$ directly.

(iii) Calculate $H^q(\mathrm{Hom}(C_*, \mathbb{Z}/6))$ for all $q \geq 0$ using the universal coefficient theorem.

Problem 8. Let X be a finite CW complex. The Euler characteristic of X is by definition the sum

$$\sum_{i\geq 0}(-1)^i \#\, i \text{ cells} = \sum_{i \geq 0}(-1)^i \dim_{\mathbb{Q}} C_i(X;\mathbb{Q}).$$

Show that

$$\chi(X) = \sum_{i\geq 0}(-1)^i \dim_{\mathbb{Q}} H_i(X;\mathbb{Q}).$$

Problem 9. Let M be a closed, oriented 3-dimensional manifold. Show that $\chi(M) = 0$.

Problem 10. Let

$$0 \to N \xrightarrow{i} G \xrightarrow{p} Q \to 0$$

be a short exact sequence of abelian groups. Let C_* be a chain complex of free abelian groups.

(i) Show that there is a long exact sequence of cohomology groups

$$\cdots \xrightarrow{p_*} H^{k-1}(C;Q) \xrightarrow{\beta} H^k(C;N) \xrightarrow{i_*} H^k(C;G) \xrightarrow{p_*} H^k(C;Q) \xrightarrow{\beta} \cdots$$

The map β is called a *Bockstein* homomorphism.

(ii) Consider the short exact sequence

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0.$$

Show that

$$\beta\colon H^1(\mathbb{RP}^2;\mathbb{Z}/2) \to H^2(\mathbb{RP}^2;\mathbb{Z}/2)$$

is nonzero.