

ALGEBRAIC TOPOLOGY IV – EPIPHANY
PROBLEM SHEET 3

Due in Monday 4th March (delay to allow you to focus on project presentations – problem class will be held that Monday).

Problem 1. With \mathbb{Z} coefficients:

- (i) Compute the cohomology of $S^n \times S^m$, for n, m positive integers.
- (ii) Compute the cohomology of the k -torus $T^k = (S^1)^k$.

Problem 2. Recall that the lens space $L(p, q)$ has a cell structure with CW chain complex

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

With \mathbb{Z} coefficients:

- (i) Compute the cohomology of $L(3, 1)$ and $L(5, 2)$.
- (ii) Compute $\text{Tor}_1(R, S)$ for $R, S \in \{\mathbb{Z}, \mathbb{Z}/3, \mathbb{Z}/5\}$.
- (iii) Use the Künneth theorem for cohomology to compute the cohomology of the product space $L(3, 1) \times L(5, 2)$, in each degree $0, 1, \dots, 6$.

Problem 3. Compute the cup product structure of $H^*(S^n; \mathbb{Z})$.

Problem 4. Complete the example done in class to compute the cup product structure of $H^*(T^2; \mathbb{Z})$. That is, compute $p^*(\theta) \cup p^*(\theta)$, $q^*(\theta) \cup p^*(\theta)$ and $q^*(\theta) \cup q^*(\theta)$, as well as the cup pairing $H^0(T^2; \mathbb{Z}) \times H^2(T^2; \mathbb{Z}) \rightarrow H^2(T^2; \mathbb{Z})$.

Problem 5. Compute the cohomology ring of the Klein bottle with $\mathbb{Z}/2$ coefficients $H^*(\mathbb{K}; \mathbb{Z}/2)$.