

**ALGEBRAIC TOPOLOGY IV – EPIPHANY
PROBLEM SHEET 4**

Problems 1, 2, and 3, due in Monday 18th March.

Problem 1. Show that $S^2 \times S^2$ and $S^2 \vee S^2 \vee S^4$ are not homotopy equivalent.

Problem 2.

- (i) Is there a degree one map $S^1 \times S^2 \rightarrow \mathbb{R}P^2 \times S^1$?
- (ii) Is there a degree one map $L(p^2, 1) \rightarrow L(p, 1) \# L(p, 1)$?
- (iii) Is there a degree one map $T^3 \rightarrow S^1 \times S^2$?
- (iv) Is there a degree one map $S^2 \times S^2 \rightarrow \mathbb{C}P^2$?

Problem 3.

- (i) Compute the cohomology ring of $\mathbb{C}P^3$ with \mathbb{Z} coefficients.
- (ii) Let $\theta \in H^n(S^n; \mathbb{Z})$ be a generator, and let $p: S^n \times S^m \rightarrow S^n$ be the projection. Show that $p^*(\theta) \cup p^*(\theta) = 0$.
- (iii) Compute the cohomology ring of $S^2 \times S^4$.
- (iv) Show that $\mathbb{C}P^3$ and $S^2 \times S^4$ are not homotopy equivalent.

Problem 4. For $n < m$, show that $\mathbb{C}P^n$ is not a retract of $\mathbb{C}P^m$. Recall that given a subspace $i: X \hookrightarrow Y$, a retraction is a map $r: Y \rightarrow X$ with $r \circ i = \text{Id}$.

Problem 5. Consider the 3-torus $T^3 = S^1 \times S^1 \times S^1$. Let α, β and γ be generators of $H^1(T^3; \mathbb{Z}) \cong \mathbb{Z}^3$. Show that $\alpha \cup \beta, \beta \cup \gamma$, and $\alpha \cup \gamma$ generate $H^2(T^3; \mathbb{Z}) \cong \mathbb{Z}^3$. Show that $\alpha \cup \beta \cup \gamma$ generates $H^3(T^3; \mathbb{Z}) \cong \mathbb{Z}$.