HATCHER'S PROOF OF THE SMALE CONJECTURE

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Abstract. This is an overview of Hatcher's proof of the Smale conjecture. We have included extra examples and pictures as well as discussion regarding the 1-dimensional case.

Contents

Introduction

In 1959, Smale proved that the inclusion $O(3) \to \mathrm{Diff}(S^2)$ of rotations into all diffeomorphisms of the 2-sphere is a homotopy equivalence, [10]. Smale conjectured that the analogous statement was true for the 3-sphere; i.e., that $O(4) \to \mathrm{Diff}(S^3)$ is a homotopy equivalence. The conjecture, now referred to as the Smale conjecture, was proven by Hatcher in 1983, [6].

We discuss various results related to the Smale conjecture.

Progress toward the Smale conjecture. Since $\mathrm{Diff}(S^n) \simeq \mathrm{Diff}_\partial(\mathbb{D}^n) \times O(n+1)$, the Smale conjecture is equivalent to the statement that $\mathrm{Diff}_\partial(\mathbb{D}^3)$ is contractible. In 1968 [1], Cerf showed that $\pi_0(\mathrm{Diff}_\partial(\mathbb{D}^3)) = 0$.

Relation to exotic spheres. Besides being intrinsically interesting, the group $\mathrm{Diff}_\partial(\mathbb{D}^n)$, and hence $\mathrm{Diff}(S^n)$, is related to exotic spheres via the bijection $\pi_0\mathrm{Diff}_\partial(\mathbb{D}^n) \cong \Theta_{n+1}$ for $n \geq 5$. The existence of an exotic $(n+1)$-sphere therefore implies that $O(n+1) \to \mathrm{Diff}(S^n)$ is not a homotopy equivalence. In 1956, Milnor constructed smooth manifolds that are homeomorphic but not diffeomorphic to S^7, thus showing that the inclusion $O(7) \to \mathrm{Diff}(S^6)$ cannot be a homotopy equivalent, [9]. Recently thereafter, Kervaire and Milnor made an analysis of exotic spheres in all dimensions ≥ 5. In particular, they showed that exotic spheres exist in all odd dimensions $4n-3 > 5$, [8]. Thus $O(4n-3) \to \mathrm{Diff}(S^{4n-4})$ cannot be a homotopy equivalence for $n > 2$. More recently, Wang and Xu in [11] have shown that the only odd dimensions in which the sphere has a unique smooth structure are $n = 1, 2, 3, 5, 61$. Thus $O(2k+1) \to \mathrm{Diff}(S^{2k})$ is not a homotopy equivalence for $2k+1 \neq 1, 2, 3, 5, 61$.

Cerf's theorem also implies that $\pi_1 \mathrm{Diff}_\partial \mathbb{D}^n \to \pi_0 \mathrm{Diff}_\partial \mathbb{D}^{n+1}$ is surjective for $n \geq 5$. For example, $\pi_1 \mathrm{Diff}_\partial \mathbb{D}^5$ surjects onto $\Theta_7 \neq 0$ and is therefore nontrivial. Thus the space of diffeomorphisms of S^5 is not homotopy equivalent to $O(6)$. Similarly, $\pi_1 \mathrm{Diff}_\partial \mathbb{D}^{61}$ surjects onto $\Theta_{62} \neq 0$, and thus $O(n+1) \to S^n$ is not a homotopy equivalent for any $n \geq 5$.

Diffeomorphisms of 3-manifolds. Let M be a (smooth) 3-manifold. One can ask more generally if the group of diffeomorphisms $\mathrm{Diff}(M)$ deformation retracts onto a subgroup of diffeomorphisms that preserve extra structure. For example, the Smale conjecture says that $\mathrm{Diff}(S^3)$ deformation retracts onto the subgroup of orthogonal transformations. The Smale conjecture is known for many classes of 3-manifolds admitting geometric structure. In particular, the Smale conjecture is true for hyperbolic manifolds.

Consider a hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ where $\Gamma \subset \mathrm{Isom}(\mathbb{H}^3)$ is discrete and torsion free. One expects that $\mathrm{Diff}(M) \simeq \mathrm{Isom}(M)$. By Mostow rigidity, $\mathrm{Isom}(M) \simeq \mathrm{Out}(\pi_1 M)$. In 2001, Gabai proved the full Smale conjecture for hyperbolic manifolds, $\mathrm{Diff}(M) \simeq \mathrm{Isom}(M)$ in [3].

For $M = M_1 \# \cdots \# M_k$ a splitting of M into prime factors, César de Sá and Rourke reduced $\mathrm{Diff}(M)$ to $\mathrm{Diff}(M_j)$ plus the homotopy theory of certain configuration spaces and graph spaces, [2]. See also [7].

In the announcement of his result [4], Hatcher discusses these and other results relating to the linear structure of diffeomorphisms of 3-manifolds.

Outline of sections. We follow Hatcher's proof [6] of the Smale conjecture that $O(4) \to \mathrm{Diff}(S^3)$ is a homotopy equivalence. The version of the Smale conjecture that we will prove is as follow:

Theorem 0.1. *A smooth family of C^∞ embeddings $g_t \colon S^2 \to \mathbb{R}^3$, $t \in S^k$, extends to a smooth family of C^∞ embeddings $\bar{g}_t \colon \mathbb{D}^3 \to \mathbb{R}^3$, for any $k \geq 0$.*

When we state results from [6], we will indicate the corresponding number of the result in [6] in parenthesis. For example, **Proposition 3.1** (4.1) indicates Proposition 4.1 in [6].
An outline of the proof is roughly as follows:

Part 1: Reduction to Primitives.

Step 1: Use surgery to break $g_t(S^2)$ into simpler surfaces.

(i) Use projection onto the vertical axis $g_t(S^2) \to \mathbb{R}$ to break $g_t(S^2)$ into simpler "elementary" surfaces. Gluing horizontal disks along the boundary of an elementary surface creates a manifold (with corners) that is homeomorphic to a 2-sphere. Such spheres are called *primitive.* For $u \in [0,1]$, form a family of surfaces (with corners) Σ_{tu} with $\Sigma_{t1} = g_t(S^2)$ and Σ_{t0} a disjoint union of primitive spheres so that as u varies, Σ_{tu} changes by surgery on circles in $g_t(S^2)$ with $\varphi_t(c) = u$.
The primitive spheres in Σ_{tu} are called *factors.*

(ii) Stratify S^k so that for t in the interior of a fixed stratum, every space $\{\Sigma_{tu}\}$, $u \in [0,1]$ has the same number of components glued in the same way.

(iii) Define graphs Γ_{tu} capturing how factors are glued in Σ_{tu}.

To prove Theorem 0.1, we need to fill in $\Sigma_{t1} = g_t(S^2)$, $t \in S^k$, with embedded 3-disks, smoothly in t. By Alexander's theorem, each factor Σ of Σ_{tu} bounds a 3-manifold $\overline{\Sigma}$. We want to be able to deal with a single factor at a time and then glue back together, undoing the surgery process. To glue the pieces back together, we will model what happens during the surgery of $g_t(S^2)$ back in the domain S^2 of the embeddings.

Step 2: Model the family of spaces Σ_{tu} by a family of spaces S_{tu} in S^3.

(i) Construct families S_{tu} of and $\bar{S}_{tu}$ with $\partial\bar{S}_{tu} = S_{tu}$ that model the surgery process of $\{\Sigma_{tu}\}$ inside S^3.

(ii) Define polar foliations F_{tu} on $\bar{S}_{tu}$.

(iii) Construct continuous versions S^c_{tu} and $\overline{S}^c_{tu}$.

Once we have a model for what happens during surgery, we can reduce to working with primitive spheres. Since the models S_{tu} in S^3 are particularly nice, one knows how to extend embeddings $S_{tu} \to \mathbb{R}^3$, $u \in [0,1]$. to embeddings $\bar{S}_{tu} \to \mathbb{R}^3$.

Step 3: Reduce proving Theorem 0.1 to constructing a smooth map $\bar{g}_{t0} \colon \bar{S}_{t0} \to \overline{\Sigma}_{t0}$ that is a diffeomorphism on factors. This is the content of Proposition 3.1.

Fix t and let Σ be a factor of Σ_{t0}. The main idea for constructing $\bar{g}_{t0}$ is to define foliations on $\overline{\Sigma}$ whose leaves are paths from $\partial\overline{\Sigma}$ to either

(a) a pole in the interior of $\overline{\Sigma}$, or

(b) a line segment (called a face) in $\partial\overline{\Sigma}$.

corresponding to whether or not the foliation F_{t0} has a pole in the associated factor of $\bar{S}^c_{t0}$. The resulting foliation will be denoted by Φ_{t0} of $\overline{\Sigma}_{t0}$. Define a map $\bar{g}_{t0} \colon \bar{S}_{t0} \to \overline{\Sigma}_{t0}$ by sending:

- the boundary $S^2 \to g_t(S^2)$ by g_t,
- poles of F_{t0} to poles of Φ_{t0}, and
- sending leaves of F_{t0} to leaves of Φ_{t0}.

To define such a $\bar{g}_{t0}$ we therefore construct foliations Φ_{t0}. The foliations Φ_{t0} will be defined by the condition that their leaves are transverse to the stages of certain shrinkings of factors to either points or faces.

Part B: Construction on primitives.

Step 4: For a factor Σ of Σ_{tu}, let A denote the leaf quotient of the vertical foliation of $\overline{\Sigma}$. Call A the contour of Σ.

(i) Show that the quotient space A has what we will call a "disk with tongues" structure.

(ii) Show that shrinkings of contours lift to isotopies of Σ_{tu}.

One can successively shrink the tongues down to a disk. Lifting the shrinking of the contour of Σ to a shrinking of $\overline{\Sigma}$ results in primitive sphere whose contour is a disk.

Two problems arise.

(1) If Σ_{t0} contains multiple factors that are glued together along a face, shrinking Σ_{t0} by shrinking one factor may destroy the disk-with-tongues structure of another factor.

(2) The factor decomposition and disk-with-tongues structures are constant on strata of $\mathcal{S}_0$ but not globally on S^k.

Step 5: Fix the shrinkings to preserve the disk-with-tongues structures.

(i) Construct families of disk-with-tongues structures $P(\gamma)$ for γ a component of the graph Γ_{t0}.

Shrinking Γ_{t0} according to the $P(\gamma)$ structures will prevent connected factors from destroying each others disk-with-tongues structures.

(ii) For (2), triangulate S^k so that in the interior of simplicies, the disk with tongue patterns are constant. Take an associated handle decomposition of S^k and work inductively over the index of handles.

Step 6: Construct shrinkings of all factors at once.

(i) Construct n-parameter families of deformations of Σ_{tu}.

(ii) Refine the handle decomposition so that the n-parameter families of deformations of Σ_{t0} are defined on overlap of handles.

Step 7: Mimic the spherical models.

(i) Construct continuous versions of Σ_{t0}.

(ii) Define foliations Φ_{t0} modeling the polar foliations F_{t0} on $\overline{S}_{t0}$.

Step 8: Apply of Proposition 3.1

Throughout this note we will include examples and commentary on the analogous results 1-dimension down; i.e., for embedded circles in the plane. One can view these embedded circles as vertical slices of embedded 2-spheres.

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Part A. Reduction to Primitives

1. Surgery Process

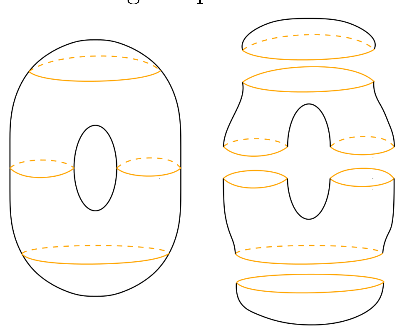
We construct a family of deformations Σ_{tu} for $t \in S^k$, $u \in [0,1]$ and a stratification $\mathcal{S}_0$ of S^k so that $\Sigma_{t1} = g_t(S^2)$ and Σ_{t0} forms a family of "primitive spheres" as t ranges over strata of $\mathcal{S}_0$.

To define a primitive sphere, we need the following preliminary definition.

Definition 1. Let $S_t \subset \mathbb{R}^3$ be a family of compact embedded surfaces, parameterized by t varying in a compact submanifold of S^k. Assume each S_t has a chosen orientation of its normal bundle, varying continuously in t. Say S_t is a *family of elementary surfaces* if the following hold:

(1) For each t, each component of ∂S_t lies in a horizontal plane.
(2) For each t, each vertical line in $\mathbb{R}^3$ meets S_t in a connected, possibly empty, set.
(3) If S_t^+ (S_t^-) denotes the subset of S_t where the positively oriented unit normal vector to S_t has strictly positive (negative) z-coordinate, then:
   (a) The closures in $\mathbb{R}^2 \times S^k$ of $\bigcup_t \pi(S_t^+)$ and $\bigcup_t \pi(S_t^-)$ are disjoint, where $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ is vertical projection to a horizontal $\mathbb{R}^2$.
   (b) The closures of $\bigcup_t S_t^+$ and $\bigcup_t S_t^-$ in $\mathbb{R}^3 \times S^k$ are disjoint from $\bigcup_t \partial S_t$.

Example 1. The standard torus T in $\mathbb{R}^3$ is not an elementary surface; however, if we break T into 4 components using the handle decomposition from the Morse height function, and straighten things out near the cuts, then each of the resulting components is an elementary surface.



Definition 2. A family of 2-spheres with corners $\Sigma_t \subset \mathbb{R}^3$ is called a *family of primitive 2-spheres* if there is a family of elementary surfaces $S_t \subset \Sigma_t$ such that, for each t, the closure of $\Sigma_t \setminus S_t$ consists of finitely many disjoint horizontal disks, called faces of Σ_t. The only corners occur at ∂S_t.

Example 2. Continuing with Example 1, if we attach horizontal disks along the boundary of each of the 4 components in the handle decomposition, we obtain 4 primitive spheres.

Non-Example 1. The standard unit sphere in $\mathbb{R}^3$ is not a primitive sphere, nor is it an elementary surface. The z-axis intersects S^2 in a disconnected set, so that condition (2) is not satisfied. If we straighten out S^2 near the equator so as to meet the equator orthogonally, and then cut S^2 apart along the equator, each of the two resulting straightened hemispheres are elementary surfaces.

As in our Example 1 and 2, we can break up any embedded 2-sphere in $\mathbb{R}^3$ into primitive spheres by surgery. The following proposition says that we can do this for families of embeddings $\{g_t\}$ where $t \in B_i$ varies over a cover $\{B_i\}$ of S^k.

Proposition 1.1 (1.1)**.** *Let $g_t \colon S^2 \to \mathbb{R}^3$, $t \in S^k$ be a family of embeddings. There exists a family of diffeomorphisms $f_t \colon \mathbb{R}^3 \to \mathbb{R}^3$ and a finite collection of closed k-balls $B_i \subset S^k$, each provided with a finite set of horizontal planes*

$$P_{ij} = \{(x, y, z) : z = z_{ij}\}$$

for $j = 1, \dots, n_i$ and $z_{i1} < \cdots < z_{in_i}$, such that
(i) $\bigcup_i \operatorname{int}(B_i) = S^k$,
(ii) $P_{ij} \neq P_{i'j'}$ if $(i, j) \neq (i', j')$,
(iii) for each pair (i, j) with $1 \leq j < n_i$,

$$f_t g_t(S^2) \cap \{(x, y, z) : z_{ij} \leq z \leq z_{i,j+1}\}$$

is a family of elementary surfaces as t ranges over B_i, and
(iv) for each i and each $t \in B_i$, $f_t g_t(S^2)$ lies between the planes P_{i1} and P_{in_i}.

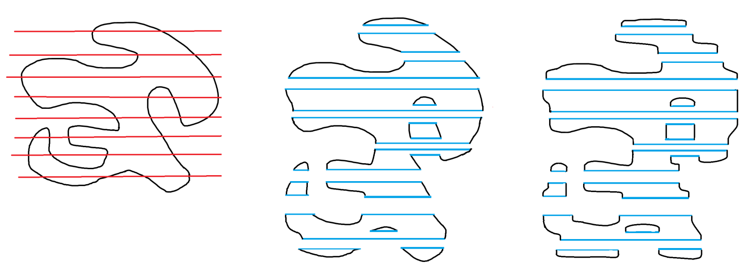
For planes P_{ij}, the index i indicates which ball B_i in the cover of S^k we are working over. For fixed i, the planes $P_{i1}, \dots, P_{in_i}$ are the various vertical levels of $f_t g_t(S^2)$ where we want to do surgery.

The diffeomorphisms f_t have the effect of straightening $g_t(S^2)$ near the cuts. We replace g_t with $f_t g_t$ from now on.

Example 3. Let $k = 1$ and take $g_t \colon S^2 \to \mathbb{R}^3$ to be the standard embedding of the unit sphere for all t. Take the cover of S^1 by two closed balls B_0 and B_1 that overlap (so that their interiors still form a cover of S^1 as in (i)). As we saw in Non-Example 1 above, it suffices to cut S^2 along the equator; however, we cannot cut in the same plane over B_0 and B_1 (condition (ii)). The following choice of planes works:

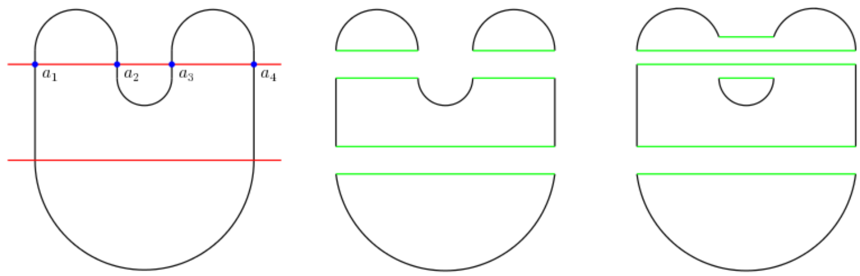
$$P_{01} = \{(x, y, -2)\} \qquad P_{11} = \{(x, y, -3)\}$$
$$P_{02} = \{(x, y, 0)\} \qquad P_{12} = \{(x, y, 1/2)\}$$
$$P_{03} = \{(x, y, 2)\} \qquad P_{13} = \{(x, y, -3)\}$$

Example 4. The first image in the following picture indicates an embedding of S^1 in $\mathbb{R}^2$. The red horizontal lines correspond to the planes P_{ij} (for fixed i since this is not a family of embeddings). The second picture indicates the embedded circle cut along the horizontal lines. Here the blue disks indicate the faces of the primitive spheres. In the third picture, the pieces of the cut embedded 1-sphere have been straightened near the horizontal disks so as to meet the faces orthogonally. The diffeomorphism from the second figure to the third is f in the above proposition.



For an embedding $S^2 \to \mathbb{R}^3$, a horizontal plane intersects the embedded sphere in a collection of horizontal circles. Each of these horizontal circles bounds a 2-disk. One dimension down, given an embedding $S^1 \to \mathbb{R}^2$, a horizontal line intersects the embedded circle in a collection of points. Each pair of points bounds a 1-disk. There is two choices of convention for how to pair these points: left to right or concentrically. The two choices are illustrated in the following example.

Example 5. The first figure indicates a circle embedded in the plane. The top horizontal line intersects the embedded circle at four points $a_1, \dots, a_4$. The second figure is the result of pairing a_1 with a_2 and a_3 with a_4. The third figure is the result of pairing a_1 with a_4 and a_2 with a_3.



Note that both conventions arise on an embedded circle coming from a vertical slice of an embedded sphere. More specifically, there exists embedded 2-spheres $S, S' \subset \mathbb{R}^3$ and a vertical plane Q so that $S \cap Q = S' \cap Q$ is the embedded circle in the first figure above. For each S, S' we can find horizontal planes P, P' so that $P \cap S$ is a disjoint union of 2 non-concentric circles and $P \cap S'$ is the disjoint union of 2 concentric circles. Cut S and S' along these circles, cutting inner most circles first. Let Σ and Σ' denote the resulting manifolds. Then $\Sigma \cap Q$ looks like the black part of the second figure above and $\Sigma \cap Q'$ looks like the third figure above.

Notation. Let C_t^{ij} be the collection of circles of $g_t(S^2) \cap P_{ij}$ for $t \in B_i$. Let

$$C_t^i = \bigcup_{j=1}^{n_i} C_t^{ij} \qquad\qquad C_t = \bigcup_{\{i:t\in B_i\}} = C_t^i.$$

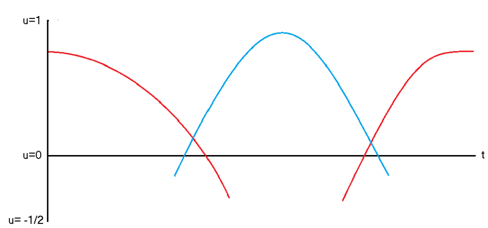
We can assume that $g_t^{-1}(C_t)$ are actual geometric circles on S^2 (c.f. [6, Prop. 1.2], this uses Smale's theorem).

We would like to form deformations $\{\Sigma_{tu}\}$ so that as u goes from 1 to 0, the spaces Σ_{tu} change by surgery along the circles in C_t. We need a way of choosing at what time u each circle gets surgered.

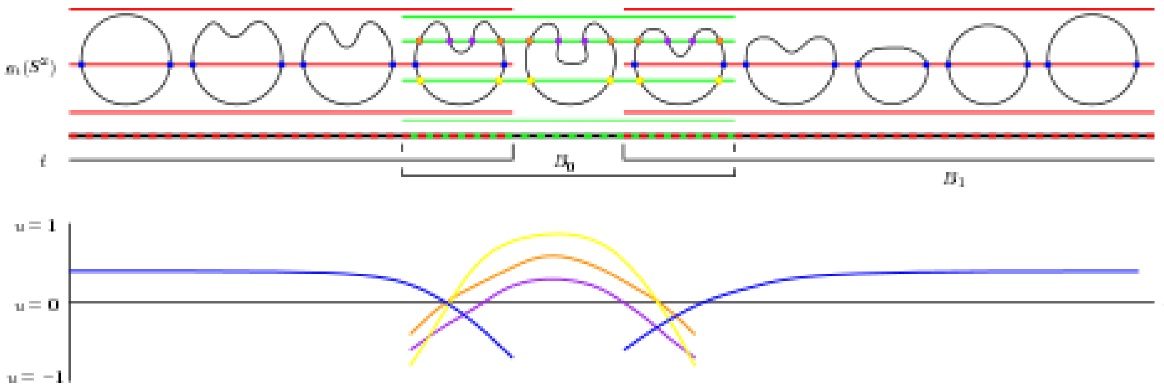
Choose a smooth family of functions $\varphi_t \colon C_t \to (-1, 1)$ for $t \in S^k$ so that

- φ_t is injective on C_t^{ij} giving a linear ordering on C_t^{ij} so that $\varphi_t(c) > \varphi_t(c')$ if c lies inside c' in the plane P_{ij}.
- $\varphi_t(C_t^i) > 0$ for $t \in B_i'$, where B_i' is a closed ball in $\mathrm{int}(B_i)$ with $\bigcup_i \mathrm{int}(B_i') = S^k$.
- $\varphi_t(C_t^i) < 0$ for $t \in \partial B_i$.

Example 6. The following picture is a continuation of Example 3. The graphs of $\varphi_t(c)$ for the two circles C^{02} and C^{12} are drawn in red and blue, respectively. The vertical direction corresponds to $u \in [-1, 1]$. The horizontal direction corresponds to the circle S^1 with the two end points glued. The black horizontal line corresponds to $u = 0$.



Example 7. The following picture illustrates an example when a certain horizontal plane intersects $g_t(S^1)$ (fixed t) in multiple 0-circles. The first line of figures details $g_t(S^2)$ as t various over $S^1 = [0, 1]/\sim$. We have chosen a cover of S^1 by two open 1-balls B_0 and B_1, indicated on the black line by green and red dashes, respectively. On each 1-ball B_i, we have chosen horizontal lines in $\mathbb{R}^2$ as in Proposition 1.1 that cut $g_t(S^2)$ into elementary surfaces. Horizontal lines P_{ij} chosen for $t \in B_0$ (resp. $t \in B_1$) are indicated in green (resp. red). The various colored dots indicate families of 0-spheres where the horizontal lines intersect $g_t(S^2)$. The graphs of $\varphi_t(c)$, for the 0-spheres c, are drawn below in the color corresponding to the color of c. For example, near one of the overlaps of B_0 and B_1, we see a 0-sphere, say c, appearing when the top of the embedded circle begins to dip down. This 0-sphere is indicated in purple. The purple line in the graph corresponds to values of $\varphi_t(c)$.



Note that the green horizontal line containing c also intersects $g_t(S^1)$ in another 0-sphere, say c', indicated in orange. The requirement that $\varphi_t(c) > \varphi_t(c')$ is fulfilled in the drawn graph.

We will form Σ_{tu} so that as u decreases from $u > \varphi_t(c)$ to $u < \varphi_t(c)$, Σ_{tu} changes by surgery along c. More formally, let C_{tu} be the collection of circles $c \in C_t$ so that $u = \varphi_t(c)$ along with the two circles parallel to $c \in C_t$ above and below at distance $\delta(c) \cdot \min(s, 1)$ for $u = \varphi_t(c) - s\epsilon$, $s > 0$. Here $\delta > 0$ is such that $g_t(S^2)$ is vertical within distance δ of each plane P_{ij} with $t \in B_i$ and each P_{ij} is within distance greater than 2δ from any other $P_{i'j'}$. For $c \in C_t$, $\delta(c) \in (0, \delta)$ is chosen independently of t so that $\delta(c) > \delta(c')$ for c insides c' in P_{ij}. The constant $\epsilon > 0$ is a small number defined in below (Claim 4.4). The space Σ_{tu} is obtained from $g_t(S^2)$ by removing the open vertical annuli between pairs of parallel circles in C_{tu} and then attaching to each circle of C_{tu} the horizontal disk it bounds.

Definition 3. A *factor* of Σ_{tu} is a 2-sphere with corners contained in Σ_{tu} obtained from the closure of a component of $g_t(S^2) \setminus C_{tu}$ (other than a vertical annulus thrown away when doing surgery) by capping off its boundary circles.

Let $\overline{\Sigma}_{tu}$ be the space obtained from Σ_{tu} by gluing in the balls in $\mathbb{R}^3$ bounded by the factors of Σ_{tu}. The existence and uniqueness of such 3-manifolds (with corners) $\overline{\Sigma}_{tu}$ is a consequence of Alexander's theorem [5, Thm. 1.1].

Let Σ be a primitive sphere in Σ_{tu} and $\overline{\Sigma}$ the corresponding 3-manifold with corners that Σ bounds. The corners of $\overline{\Sigma}$ occur along a finite number of horizontal circles. After smoothing these corners, the resulting 3-manifold $\overline{\Sigma}'$ is abstractly diffeomorphic to $\mathbb{D}^3$. However, there is no canonical identification of $\overline{\Sigma}'$ with $\mathbb{D}^3$. One of the difficulties in the proof of Hatcher's theorem is creating such diffeomorphisms $\overline{\Sigma}' \cong \mathbb{D}^3$ in a uniform way that is continuous in the parameter direction $t \in S^k$.

Say a factor Σ is contained in another factor Σ'. When forming $\overline{\Sigma}_{tu}$, we glue on disjoint 3-balls bounded by Σ and Σ', respectively, only identifying their common horizontal faces.

We would like to say that $\{\Sigma_{t0}\}_{t \in S^k}$ is a family of primitive spheres, but this is not true over all of S^k. Indeed, as $t \in S^k$ moves out of a stratum of $\mathcal{S}_0$, the spaces Σ_{t0} change by surgery on horizontal circles in C_{t0}. In the next section, we define a stratification of S^k so that Σ_{t0} forms a family of primitives spheres when restricted to each of the strata.

1.1. **Stratification on Parameter Space.** The goal of this section is to define a stratification on S^k so that on stratum each factor of Σ_{tu} varies only by isotopy and the factors of Σ_{t0} form families of primitive 2-spheres. The stratification will be formed using the graphs of the functions $\varphi_t \colon C_t \to (-1, 1)$. Let

$$Z'(c) = \{(t, \varphi_t(c)) : t \in S^k\}$$

be the graph of φ_t,

$$Z(c) = Z'(c) \cap (S^k \times [0, 1])$$

the subset of the graph of nonnegative values, and

$$Z_0(c) = Z(c) \cap (S^k \times \{0\})$$

the values of $t \in S^k$ so that $\varphi_t(c) = 0$. Assume that the graphs $Z'(c)$ have generic intersections with each other and with $S^k \times \{0\}$. The intersections of various $Z(c)$'s give a stratification $\mathcal{S}$ of $S^k \times [0,1]$ which intersects $S^k \times \{0\}$ in a stratification $\mathcal{S}_0$ of $S^k \times \{0\}$.

1.2. **Surgery Graphs.** We define graphs Γ_{tu} that record the data of which primitives are glued to which other primitives during various stages of the surgery process of Σ_{tu}.

Definition 4. Let Γ_{tu} be the graph whose vertices correspond to the factors of Σ_{tu} and whose edges correspond to the common horizontal faces between factors.

Let e be an edge of Γ_{tu}. Then e corresponds to a face Δ with $(t,u) \in Z(\partial\Delta)$. Since Σ_{tu} changes by surgery on Δ as u decreases from $u > \varphi_t(\partial\Delta)$ to $u < \varphi_t(\partial\Delta)$, we have that for $u = \varphi_t(\partial\Delta) + s\epsilon$

- if $s > 0$, the edge e collapses to a single vertex of Γ_{tu}, and
- if $s < 0$, the edge e is deleted from Γ_{tu}.

Lemma 1.2. *The graphs Γ_{tu} satisfy the following properties.*

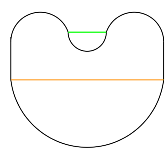
(1) The graphs Γ_{tu} are constant on strata of $\mathcal{S}$.

(2) The components of Γ_{tu} are trees.

(3) A component γ of Γ_{tu} corresponds to a connected component $\Sigma_{tu}(\gamma)$ of Σ_{tu} varying continuously with (t,u) in the given stratum of $\mathcal{S}$ over which γ is defined.

Definition 5. A common face Δ of two factors Σ_1 and Σ_2 of $\Sigma_{tu}(\gamma)$ is a *sum face* if Σ_1 and Σ_2 bound balls in $\mathbb{R}^3$ meeting only in Δ. Otherwise, one of the balls is contained in the other and we say Δ is a *difference face.*

Example 8. In the following picture the orange disk is a sum face and the green disk is a difference face.



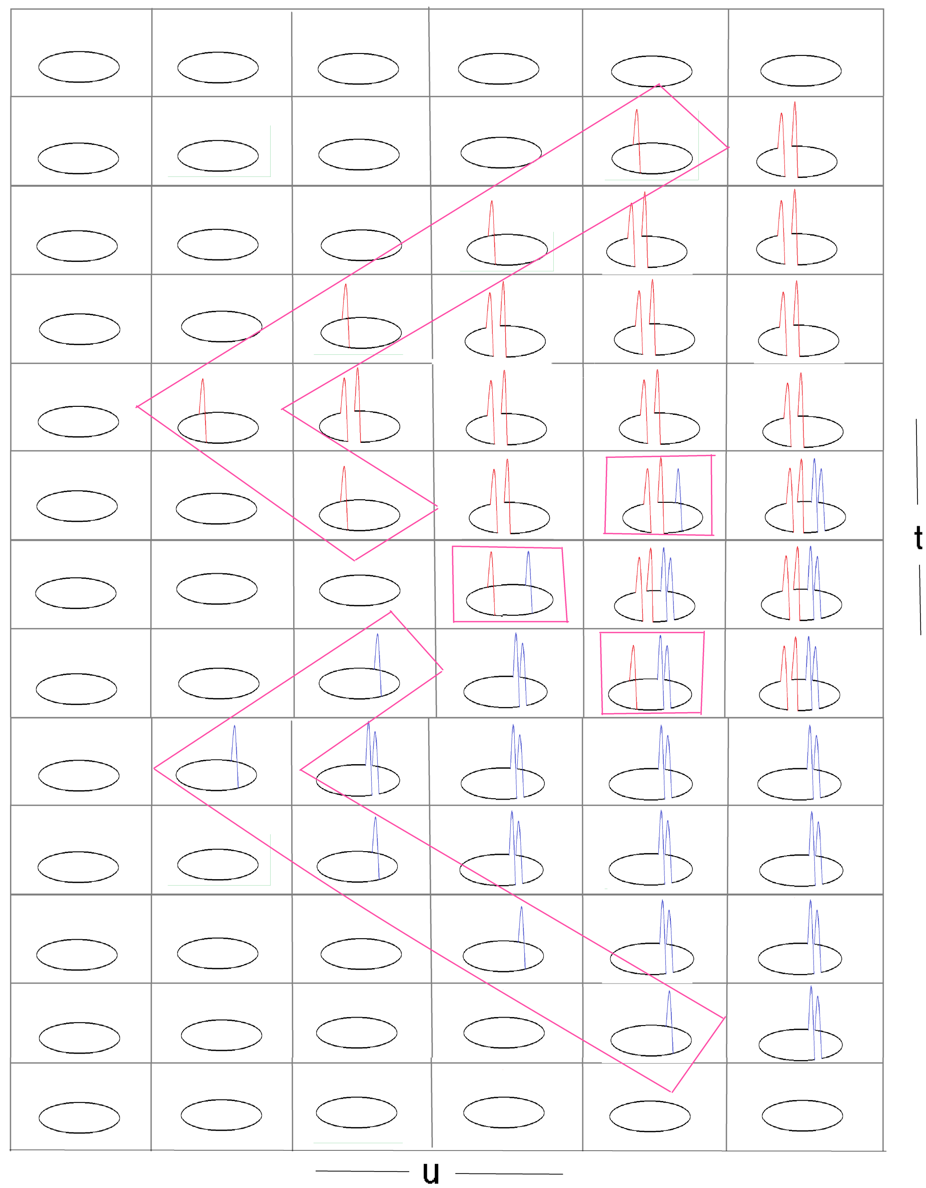
2. Spherical Models

We want to be able to deal with a single factor at a time and then glue back together, undoing the surgery process. To glue the pieces back together, we will model what is going on in the surgery process of $g_t(S^2)$ in the domain S^2. To do this, we construct families S_{tu} and $\bar{S}_{tu}$ modeling the surgery process of $\{\Sigma_{tu}\}$ inside S^3. We start with the trivial family $S^2 \times S^k$ and make the following modifications. For each circle $c \in C_{tu}$ glue a disk $\mathbb{D}^2$ to $g_t^{-1}(c)$ along its boundary. Remove the open vertical annuli in S^2 between circles that are removed during the surgery process in Σ_{tu}. To indicate whether surgery on c has the effect of removing a primitive sphere or gluing on a primitive sphere, we attach disks inside of different hemispheres of $S^3 \supset S^2$. Specifically, embed $S^2 \to S^3$ as the equator. Each circle $g_t^{-1}(c)$ has a corresponding 2-sphere S_c^2 in S^3 that meets S^2 orthogonally at $g_t^{-1}(c)$. Let B_c^+ and B_c^- denote the disks obtained from intersecting this orthogonal 2-sphere with the northern and southern hemispheres of S^3, respectively. We can detect whether surgery on c bounds a sum face or a difference face by whether an arrow form the interior of the disk to its boundary points outside or inside $\overline{g_t(S^2)}$.

- If c is a sum circle, glue B_c^+ to S^2 along $g_t^{-1}(S^2)$.
- If c is a difference circle, glue B_c^- to S^2 along $g_t^{-1}(S^2)$.

Let S_{tu} denote the resulting subset of S^3. Call the components of S_{tu} that are homotopy 2-spheres *factors* of S_{tu}. Under g_t, these correspond bijectively to the factors of Σ_{tu}.

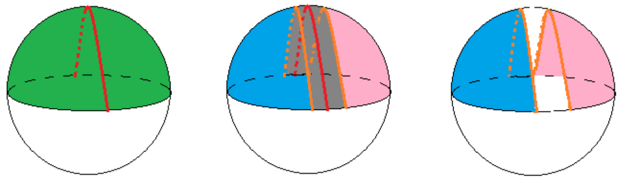
Example 9. In the following picture, the spaces S_{tu} are drawn for various $(t,u) \in S^1 \times [0,1]$. Horizontally, u varies from 1 to 0. Vertically t is varying across S^1, with the top and bottom rows being identified. The S_{tu} drawn here correspond to the graph of φ_t in Example 6. Strata are circled in pink.



Spaces $\bar{S}_{tu}$ are formed from S_{tu} by attaching 3-disks in S^3 bounded by components of S_{tu} to S_{tu}. Each factor S of S_{tu} bounds two 3-disks in S^3 by Alexander's theorem. These two 3-disks can be distinguished by whether an arrow from a point in the equator to the interior of the disk points into the northern or southern hemisphere of S^3. If for the corresponding factor Σ of Σ_{tu}, the normals to $\Sigma \cap g_t(S^2)$ pointing into $\overline{\Sigma}$ point into (resp. out of) the ball in $\mathbb{R}^3$ bounded by $g_t(S^2)$, attach the disk with normals pointing into the northern (resp. southern) hemisphere.

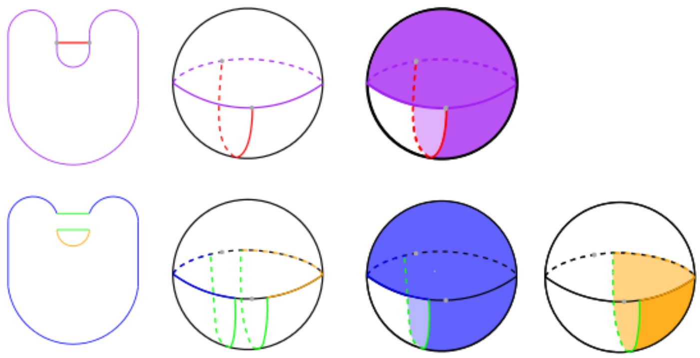
Remark 1. The spaces S_{tu} do not vary continuously in u; 2-disks are attached and removed suddenly. Also note that $\bar{S}_{tu}$ has discontinuities whenever a difference circle is surgered in Σ_{tu} but not during surgeries of attaching circles. Continuous versions of S_{tu} and $\bar{S}_{tu}$ will be constructed in Claim 7.1.

Example 10. The following picture illustrated $\bar{S}_{tu}$ as (t, u) goes from the boundary of a stratum to its interior. The red 1-disk corresponds to a sum face in Σ_{tu} along a 0-sphere in S^1. The green portion of the sphere indicates $\bar{S}_{tu}$ where $\varphi_t(c) = u$. As u decreases, the red 1-disk is replaced with two parallel orange disks and the "annulus" $S^0 \times \mathbb{D}^1$ (indicated in grey) is removed. The result is two factors, indicated in blue and grey, in S_{tu} for $u = \varphi_t(c) - \delta$ for some small $\delta > 0$.



Since the circle c corresponds to a sum face, there is no discontinuity in $\bar{S}_{tu}$.

Example 11. The following illustrates a difference face. The first column shows Σ_{tu} for two values of u. The second column shows the corresponding spherical models S_{tu} as colored subsets of S^3. In the third and fourth columns, there are pictures of $\bar{S}_{tu}$. The blue factor and orange factor are not glued in $\bar{S}_{tu}$. Notice that since the red face is a difference face, we see a discontinuity in $\bar{S}_{tu}$.



The following definitions will be useful in the next section.

Definition 6. FOr S_1, S_2 factors of S_{tu} say $S_1 \leq S_2$ if $\overline{S}_1 \subseteq \overline{S}_2$ in S^3.

Definition 7. A *maximal factor* of S_{tu} is a factor s not contained in any other factor.

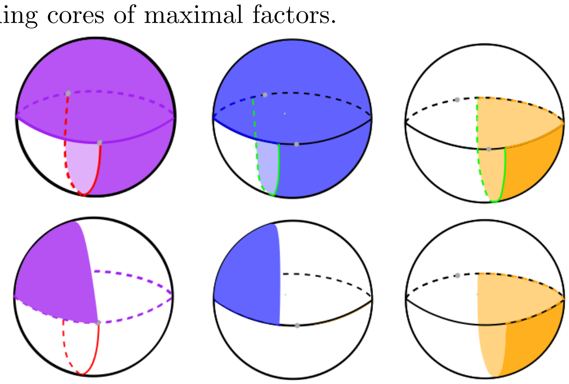
Definition 8. The *core* of a factor S of S_{tu} is

$$\overline{(\overline{S} \setminus \bigcup_i B_i)} \setminus S^2$$

where

- $\overline{S}$ is the 3-ball in S^3 bounded by S that is attached during the construction of $\bar{S}_{tu}$.
- $c_1, \ldots, c_n \subset S^2$ are the circles at which S has corners and $D_i \subset S$ is the 2-disk capping off c_i.
- $B_i \subset S^3$ is the 3-ball bounded by $S^2_{c_i}$ with $B_i \cap S = D_i$.

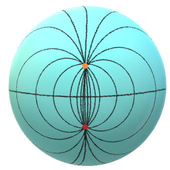
Example 12. Continuing with Example 7.1, the top row illustrates various $\overline{S}_{tu}$ and the bottom row ilustrates the corresponding cores of maximal factors.



2.1. **Polar Foliations.** We define singular foliations on S^3. From these we obtain foliations on $\bar{S}_{tu}$ by restricting the foliations to $\bar{S}_{tu} \to S^3$. Let γ be a component of the graph Γ_{tu}. Then γ is defined over a stratum of $\mathcal{S}$. We will define families of foliations $F_{tu}(\gamma)$ for (t, u) in the closure of the stratum over which γ is defined. For certain components γ, γ', we have $S_{tu}(\gamma) \subset S_{tu}(\gamma') \subset S^3$ (cf. Example 11). In this case, we require the foliations to agree, $F_{tu}(\gamma) = F_{tu}(\gamma')$.

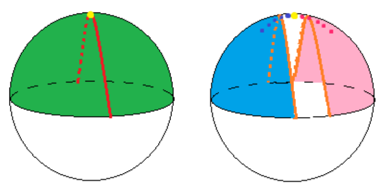
Observe that each point p in $S^3 \setminus S^2$ determines a polar foliation of S^3 whose leaves are arcs through p that meet S^2 orthogonally. Such foliations have two poles: one at p and one at a dual point p'.

Example 13. One dimension down, the longitude lines on S^2 define a polar foliation for the north pole. This foliation has poles at the north and south pole. The following picture indicates another polar foliation of S^2 with pole indicated with a red dot. The orange dot corresponds to the "dual" point.



For (t, u) in the interior of the stratum over which γ is defined, we would like each foliation $F_{tu}(\gamma)$ to have a pole in the core of a maximal factor of $S_{tu}(\gamma)$. Note that the core of a factor, by Definition 8, does not intersect S^2. Thus any point in the core of a maximal factor defines a polar foliation on S^3. Near the boundary of strata, we only require the pole to not cross the equator $S^2 \subset S^3$.

Example 14. In the following picture, we continue with the situation of Example 10. The yellow point in the first figure indicates the pole on S_{tu} for $\varphi_t(c) = u$. This corresponds to the foliation of the sphere by longitudinal lines. As (t, u) moves into the interior of the stratum, the corresponding graph Γ_{tu} splits into two components: one corresponding to the pink factor (say γ_p) and one corresponding to the blue factor (say γ_b). The foliations $F_{tu}(\gamma_p)$, for (t, u) moving from the boundary of the stratum into the interior, have poles given by the dark pink dots. Similarly, the foliations $F_{tu}(\gamma_b)$ have poles given by the dark blue dots.



The following lemma explains how the cores of maximal factors change over strata.

Lemma 2.1. *Let (t, u) vary over a stratum of $\mathcal{S}$ so that $S^2 \cap S_{tu}$ is constant along the stratum. Let (t_i, u_i), $i = 1, 2$ be two points in the stratum. Let U_i, $i = 1, 2$, denote the union of the cores of the maximal factors of $S_{t_i u_i}$. There is an inclusion $U_1 \subseteq U_2$ inducing a bijection on π_0.*

Note that 0-dimensional stratum of $\mathcal{S}$ are points (t_0, u_0) so that multiple 2-disks have been attached to S_{tu} along circles in $C_{t_0 u_0}$, but $S_{t_0 u_0}$ has not been split along these circles. This is analogous to deformations $\Sigma_{t_0 u_0}$ where horizontal disks have been attached in preparation for surgery, before the surgery has been done.

Consider the family of spherical models S_{tu} as (t, u) moves from a 0-dimensional stratum (t_0, u_0) of $\mathcal{S}$ into a 1-dimensional stratum. Either S_{tu} changes by splitting $S_{t_0 u_0}$ along one of the 2-disks attached in $S_{t_0 u_0}$ and the other 2-disks attached at stage (t_0, u_0) remain unchanged along the 1-dimensional stratum, or S_{tu} is constant along the interior of the stratum with one of the 2-disks attached and, at the boundary of the stratum, another 2-disk is attached to form $S_{t_0 u_0}$. This is true regardless of whether the corresponding face in Σ_{tu} is a sum or difference face. If the corresponding face in Σ_{tu} is a difference face, then $\bar{S}_{tu}$ can change more drastically as in Example 11.
Construction.

We construct $F_{tu}(\gamma)$ inductively over the strata of $\mathcal{S}$. For a 0-dimensional strata, we need to define a foliation on a single $\bar{S}_{tu}$. For each maximal factor of $\bar{S}_{tu}(\gamma)$, choose a point in the interior of the core of the maximal factor (a contractible space of choices). Let $F_{tu}(\gamma)$ be the corresponding polar foliation.

As (t, u) moves from a k-dimensional stratum ∂X along a $(k + 1)$-dimensional stratum X, two things can happen. Either $S^2 \cap S_{tu}$ remains unchanged or $S^2 \cap S_{tu}$ is split into more components. In the first case, S_{tu} has a certain number of 2-disks attached and at the boundary of the stratum an additional 2-disk is suddenly attached. In this case, there is an inclusion of the union of the cores of the maximal factors of $S_{t_0u_0}$ into the union of the cores of the maximal factors of S_{tu} for any (t,u) in the $(k+1)$-stratum. This inclusion induces a bijection on π_0. We can therefore define $F_{tu}(\gamma)$ for (t,u) in X to be the same as $F_{t_0u_0}(\gamma)$.

If $S^2 \cap S_{tu}$ changes as (t,u) varies over the $(k+1)$-dimensional stratum X, then there are factors S_1, S_2 of $S_{tu}(\gamma)$ so that $S_1 \leq S_2$ in $S_{t_0u_0}$ for $(t_0,u_0) \in \partial X$ but $S_1 \not\leq S_2$ for some $(t,u) \in X$. Assume S_2 was a maximal factor at stage (t_0,u_0) and that S_1 and S_2 are both maximal factors at stage (t,u). Then the foliation $F_{t_0u_0}(\gamma)$ contains a pole p in the core of S_2. As (t,u) moves away from (t_0,u_0) along X, the edge connecting S_1 and S_2 disappears. The graph Γ_{tu} is then a disjoint union of two graphs γ_1 and γ_2 where γ_i contains the vertex corresponding to S_i. Define $F_{tu}(\gamma_i)$ on S^3 by moving the pole p from the core of S_2 to the core of S_1 in some small amount of time.

Definition 9. Call a factor S of S_{tu} *polar* if the disk S bounds in $\bar{S}_{tu}$ contains a pole in the foliation F_{tu}. Otherwise, call S *facial*.

In §3 of [6], Hatcher uses the foliations F_{tu} to create continuous versions S^c_{tu} and $\bar{S}^c_{tu}$ of S_{tu} and $\bar{S}_{tu}$. The idea is to fold the attached 2-disks back into S^2 instead of deleting them. Since this is a technical point of the paper, we will not go into the details here.

Claim 2.2. *There exists continuous versions S^c_{tu} and $\bar{S}^c_{tu}$ of S_{tu} and $\bar{S}_{tu}$, respectively.*

3. Reduction to $u = 0$

The following proposition reduces the proof of Theorem 0.1 to constructing a smooth map $\bar{g}_{t0} \colon \bar{S}^c_{t0} \to \overline{\Sigma}^c_{t0}$ that is a diffeomorphism on factors.

Proposition 3.1 (4.1)**.** *Given a family of maps $\bar{g}_{t0} \colon \bar{S}^c_{t0} \to \mathbb{R}^3$ which restrict to embeddings on the factors of $\bar{S}^c_{t0}$ and which agree with g_t on $S^2 \cap S_{t0}$, there exists a family $\bar{g}_{tu} \colon \bar{S}^c_{tu} \to \mathbb{R}^3$, $u \in [0,1]$, extending $\bar{g}_{t0}$ which also restrict to embeddings on factors and agree with g_t on $S^2 \cap S_{tu}$.*

We will apply the proposition as follows. Continuous versions $\overline{\Sigma}^c_{t0}$ of $\overline{\Sigma}_{t0}$ will be created in a manner similar to how $\overline{S}^c_{t0}$ was created from $\overline{S}_{t0}$. The inclusions $\overline{\Sigma}_{t0} \subset \mathbb{R}^3$ will lead naturally to maps $\overline{\Sigma}^c_{tu} \to \mathbb{R}^3$. We will construct a smooth family of homeomorphisms $\bar{g}_{t0} : \overline{S}^c_{t0} \to \overline{\Sigma}^c_{t0}$ that are diffeomorphisms on factors. Composing with the maps to $\mathbb{R}^3$ gives a family of maps $\overline{S}^c_{t0} \to \mathbb{R}^3$ to which the proposition will be applied.

Part B. Construction on Primitives

4. Contours

For a factor Σ of Σ_{tu}, let A denote the leaf quotient of the vertical foliation of $\overline{\Sigma}$. Call A the contour of Σ. Below is an outline of this section. New terms used in the outline will be defined later in the section.

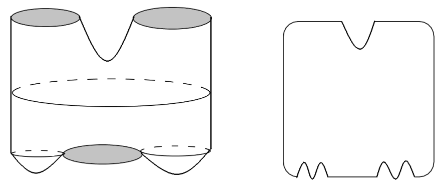
- The quotient space A has the structure of a disk with tongues attached.
- Within each strata of the stratification $\mathcal{S}_0$ of S^k, the disk-with-tongues structure of these contours is constant.
- We can successively shrink tongues down into the initial disk.
- Alterations can be made on Σ so that no point lying over the interior of a tongue has a vertical tangent.
- A shrinking of the contour of Σ lifts to a shrinking of $\overline{\Sigma}$ to a primitive sphere whose contour is a disk.

Our goal is to define a shrinking of $\overline{\Sigma}$ to a standard 3-disk. The process described here actually results in a shrinking of $\overline{\Sigma}$ to a cylinder with smoothed corners. We will achieve such a deformation by lifting shrinkings of the leaf space of the vertical foliation of $\overline{\Sigma}$ to a disk.

4.1. **Disk-with-tongues Structures.**

Definition 10. The *contour* of a factor Σ of Σ_{tu} is the leaf space of the vertical foliation of $\overline{\Sigma}$. Let $C(\Sigma)$ denote the contour of Σ and $\pi : \overline{\Sigma} \to C(\Sigma)$ the projection.

Example 15. If the contour of Σ is a disk D , then Σ has a particularly simple form. The preimage $\pi^{-1}(D)$ is a cylinder in $\mathbb{R}^3$. The space $\overline{\Sigma}$ is then a cylinder with various vertical modifications. We can linearly deform $\overline{\Sigma}$ into a cylinder with smooth corners. The picture below indicates an example of an embedded 2-sphere with contour a disk and an embedded 1-sphere with contour a line segment.



In general, the contour $C(\Sigma)$ will have the structure of a disk with tongues according to the following definition.

Definition 11. A *disk with tongues* is a space C which is expressible as the union of finitely many 2-disks, $C = \bigcup_{i=0}^{n} D_i$ where for each $i > 0$, the subspace

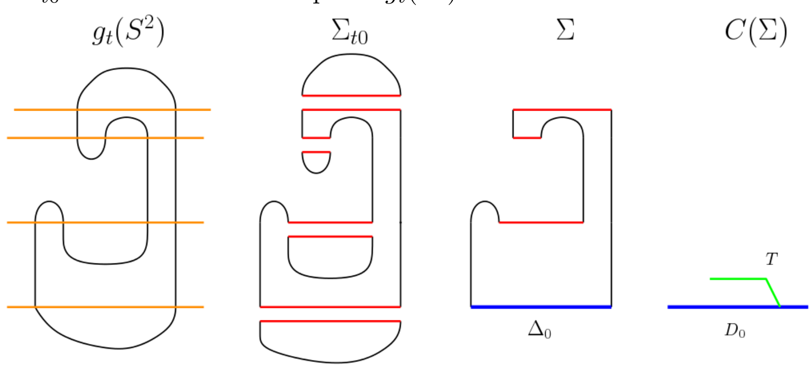
$$D_i \cap (\bigcup_{j<i} D_j)$$

is a subdisk d_i of D_i meeting ∂D_i in at least an arc. Furthermore, there exists a projection map $\pi : C \to \mathbb{R}^2$ which is an embedding on each D_i and such that $\pi(D_i)$ and $\pi(d_i)$ are smooth subdisks of $\mathbb{R}^2$.

Definition 12. A *disk-with-tongues structure* on a disk with tongues C is a decomposition of C into an *initial disk* D_0 and an (unordered) collection of *tongues* T_i with $T_i = D_i \setminus d_i$. The *free edge* of T_i is $\overline{\partial D_i \setminus \partial d_i}$ and the *attaching edge* of T_i is $\overline{\partial d_i \setminus \partial D_i}$.

Definition 13. A tongue T_i is of *Type I* if
(1) $\pi(\partial T_i) \cap \pi(\partial D_0) = \emptyset$, and
(2) $\pi(\partial T_i) \cap \pi(\partial T_j) = \emptyset$ for each tongue T_j, $j \neq i$.

Example 16. In the following picture, we have shown how a Type I tongue might arise in a contour of a factor Σ in Σ_{t0} for some embedded sphere $g_t(S^2)$.



Proposition 4.1 (5.1)**.** *The contours of a family of primitive 2-spheres $\Sigma_t \subset \mathbb{R}^3$ have the structure of a family of disks with Type I tongues.*

We would like to shrink the contour $C(\Sigma)$ to a disk. A disk-with-tongues structure on $C(\Sigma)$ preferences the initial disk. We will shrink the tongues of $C(\Sigma)$ into the initial disk. It would be useful to have more information about the initial disk of $C(\Sigma)$. In particular, Σ is a factor in a family of primitive spheres $\{\Sigma_{t0}\}_t$. We would like the initial disks of various factors $\Sigma_t \subset \Sigma_{t0}$ to vary nicely in t. By construction, factors like Σ contain specified horizontal disks (their faces, Δ_i) which have contours $C(\Delta_i)$ that are disks. It turns out that we may choose the initial disk of $C(\Sigma)$ to be the projection of a *large face* of Σ according to the following definition.

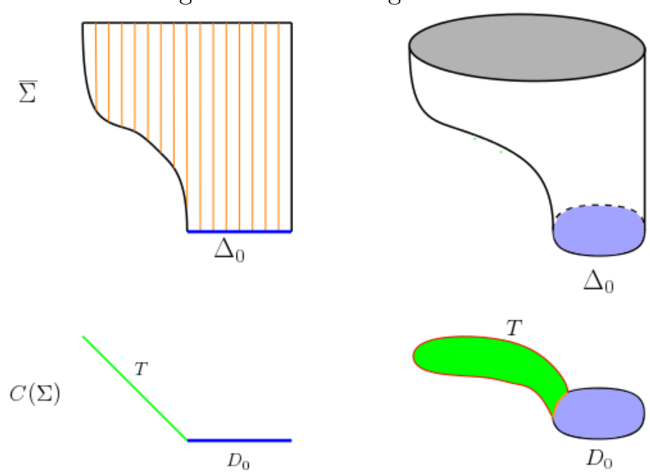
Definition 14. A face Δ_t of a family of primitives Σ_t is called *large* if, locally in t, $\pi(\Delta_t) \cap \partial\pi(\Sigma_t)$ contains a smoothly varying arc.

If we specify that $C(\Delta)$ is the initial disk of $C(\Sigma)$, we can no longer guarantee that all tongues will be of Type I. Additional tongues can be described as follows,

Definition 15. A tongue T_i is of *Type II* if

(1) The attaching edge of T_i lies in the initial disk D_0, and near its cups points lies in ∂D_0.
(2) The free edge of T_i projects disjointly from $\pi(D_0)$ except for its cusp points.
(3) $\pi(\partial T_i) \cap \pi(\partial T_j) = \emptyset$, for $j \neq i$.

Example 17. The pictures below show Type II tongues in the 1-dimensional and 2-dimensional cases. The top row shows $\overline{\Sigma}$ for a primitive 1-sphere (on the left) and a primitive 2-sphere (on the right). In the first image, we have also indicated the vertical foliations of $\overline{\Sigma}$ in orange. The bottom row shows the contours of these spheres. The initial disks D_0 are chosen to correspond to the faces Δ_0 in $\overline{\Sigma}$. In each case, the green tongue T is of Type II. For the primitive 2-sphere (on the right), the attaching edge of T is shown in orange and the free edge in red.



Proposition 4.2 (5.2)**.** *If Δ_t is a large face of the family of primitives Σ_t, then $C(\Sigma_t)$ has the structure of a disk with Type I and II tongues, with $C(\Delta_t)$ as the initial disk.*

Remark 2. Once the initial disk is specified, two disk-with-tongues structures on $C(\Sigma)$ with only Type I and II tongues have a canonical common subdivision obtained by taking tongues to be the intersection of the tongues in the two structures.

Notation. We let $C(\Sigma_t, \Delta_t)$ denote the disk with Type I and II tongues structure on the family of primitives Σ_t with initial disk $C(\Delta_t)$.

4.2. **Shrinking Contours.** We want to shrink the contour $C(\Sigma)$ down to its initial disk. The idea is to shrink the tongues, one at a time, into the initial disk. There are many ways to do this. For example we could change the order in which we shrink the tongues or the speed at which the tongues shrink. Formally, for C_t a family of disks with tongues, a shrinking of C_t will be any family $\{C_{ts}\}$, $s \in [0, 1]$, of disks with tongues satisfying the following:

(1) $C_{t0} = C_t$,
(2) $C_{ts} \subset C_{ts'}$ if $s > s'$,
(3) for T_t a tongue of C_t, $T_t \cap C_{ts}$ is a tongue of C_{ts}, and
(4) for D_t the initial disk of C_t, $D_t \cap C_{ts}$ is the initial disk of C_{ts}.

Given a family of 2-spheres with corners Σ_t and a family of shrinkings C_{ts} of their contours $C(\Sigma_t) =: C_t$, we would like to lift C_{ts} to a deformation Σ_{ts} of the spaces Σ_t. Naively, one could take $\Sigma_{ts} = \pi^{-1}(C_{ts})$ where $\pi : \Sigma_t \to C(\Sigma_t)$ is the projection. The spaces $\pi^{-1}(C_{ts})$ will not be smooth.

They will have corners at the horizontal disks at which Σ_t have corners, but will have other corners as well. Additionally, points of Σ_t with vertical tangents can cause the deformation Σ_{ts} to not be smooth in s.

Lemma 4.3 (6.1)**.** *Let $\Sigma_t \subset \mathbb{R}^3$ be a family of 2-spheres (with corners at horizontal disks) such that $C(\Sigma_t)$ is a family of disks with tongues. Then a shrinking C_{ts} of $C(\Sigma_t) = C_{t0}$ lifts to an isotopy Σ_{ts} of $\Sigma_t = \Sigma_{t0}$. Moreover, we can make Σ_{ts} smooth for $s > 0$.*

Explicitly, the statement that Σ_{ts} is a lift of C_{ts} means that $C(\Sigma_{ts}) = C_{ts}$ and $\overline{\Sigma}_{ts} \subset \overline{\Sigma}_{ts'}$ for $s' < s$.

Proof Idea. Using a partition of unity argument on S^k, we can assume that the tongues of C_t attach in a fixed order, independent of t. We construct the isotopies Σ_{ts} inductively, shrinking a single tongue at a time. We are reduced to showing that a shrinking of a single tongue of C_t or a shrinking of the initial disk D_0 lifts to an isotopy of Σ_t.

Assume a single tongue T_t is shrinking in $C(\Sigma_t)$. To avoid corners and problems from points with vertical tangents, we make a preliminary modification of Σ_t. Let $V \subset \Sigma_t$ be the set of points with vertical tangents that project to the interior of the tongue T_t that is shrinking. Partition $V = V_+ \sqcup V_-$ where V_+ (resp. V_-) is the subset of points $p \in V$ where an upward (resp. downward) pointing vertical line through p goes from inside $\overline{\Sigma}_t$ to outside $\overline{\Sigma}_t$. As Σ_t is a primitive sphere, Σ_t can be written as an elementary surface S together with a finite number of horizontal disks. Since S is an elementary surface, a vertical line intersects S on a connected set. Thus the sets V_+ and V_- cover V.

Claim 4.4. *There exists a smooth family of smooth vector fields v_t on $\mathbb{R}^3$ such that*
(i) $v_t|_{\Sigma_t}$ has support in $\pi^{-1}(\mathrm{int}(T_t))$
(ii) $v_t|_{\Sigma_t}$ is orthogonal to Σ_t, pointing into $\overline{\Sigma}_t$
(iii) $\frac{\partial}{\partial z}|v|$ is positive on V_+ and negative on V_-.

For an example of such a vector field, see Figure 6.2 of [6]. Form Σ'_{tr} for $r \in [0,1]$ from $\Sigma_t = \Sigma'_{t0}$ by flowing along v_t for $r \in [0,\epsilon]$ and staying still for $r \in [\epsilon, 1]$. Here ϵ is chosen so that no point $p \in V$ lies in the boundary of T_{tr} for $r < \epsilon$. Define $\overline{\Sigma}_{ts} = (\pi')^{-1}(C_{ts})$ where $\pi' : \Sigma'_{tr} \to C(\Sigma_t)$. The isotopy of Σ_t is then $\Sigma_{ts} := \partial\overline{\Sigma}_{ts}$.

Smooth the corners of Σ_{ts} according to Figure 6.1 of [6]. □

5. Choosing the Right Shrinking

Our plan is to shrink Σ to a standard disk by lifting shrinkings of contours. We therefore need the shrinkings of the tongues to be compatible with two things:

(1) the decomposition of $g_t(S^2)$ into primitive spheres; i.e., the surgery process, and
(2) the parameter $t \in S^k$.

More specifically, consider a connected component $\Sigma(\gamma) \subset \Sigma_{t0}$. Then $\Sigma(\gamma)$ is a union of a primitive surface Σ and a finite number of horizontal disks $\Delta_1, \dots, \Delta_n$. The horizontal disks break $\Sigma(\gamma)$ into primitive spheres $\Sigma_0, \dots, \Sigma_n$. For example, $\Sigma(\gamma)$ might be as in Figure 7.1 of [6]. We want a shrinking of $\Sigma(\gamma)$. So far we have constructed shrinkings of each Σ_i. As we glue together the Σ_i along the Δ_i, the shrinkings may not be compatible. For example, in Figure 7.1, Σ is a difference of two primitive spheres Σ_0 and Σ_1. As Σ_1 shrinks, Σ_0 expands. A more complicated problem arises if shrinking a factor Σ_i changes the disk-with-tongues structure of the contour of $\Sigma(\gamma)$. Section 7 of [6] describes an example when this happens.

Remark 3. For generic families of embeddings $g_t \colon S^2 \to \mathbb{R}^3$ for $t \in S^k$, $k \leq 2$ such problems do not arise. This is a consequence of the classification of singularities for generic families of maps $S^2 \to \mathbb{R}$. For $k = 0, 1, 2$, we can therefore skip this section and simplify define the shrinking of $\Sigma(\gamma)$ by shrinking each factor Σ_i (after assuming by genericity that the projection of the contour is at most 2 to 1).

Recall that the tree γ describes how the primitive factors $\Sigma_0, \dots \Sigma_n$ are attached along the faces $\Delta_1, \dots, \Delta_n$. We need to incorporate the data of γ into the chosen shrinkings of contours.
The outline of this section and the next is roughly as follows:

1. Describe a pattern of disk-with-tongue structures $P(\gamma)$ that takes into account the tree γ and varies continuously with small changes in $t \in S^k$.
2. Construct a triangulation $\mathcal{T}$ of S^k so that on each simplex σ of $\mathcal{T}$, the contours of $\Sigma_{t0}(\gamma)$ have the disk-with-tongue structure prescribed by $P(\gamma)$.
3. For $t \in S^k$ varying in a simplex σ of $\mathcal{T}$, combine the shrinkings of the primitive factors $\Sigma_0, \dots, \Sigma_n$ (using the disk-with-tongues structure from $P(\gamma)$) into n-parameter families of deformations $\Sigma_t^i(s_1, \dots, s_n)$, $s_j \in [0,1]$, where $\Sigma_t^n(0, \dots, 0, s_n)$ shrinks Σ_n to Δ_n and if $\Sigma_i < \Sigma_j$, then $\Sigma_t^j(s_1, \dots, s_n)$ shrinks Σ_i as well as Σ_j.
4. Show that the families $\Sigma_t^i(s_1, \dots, s_n)$ agree on overlap of simplices of $\mathcal{T}$.

These steps correspond to subsections 5.1.1, 5.2, 6, and 6.1.1 of this paper, respectively. In [6], these steps correspond to sections 9, 10, 11, and 12, respectively.

Remark 4. We lose smoothness at the second step. It will be regained in the step of taking transverse leaves.

5.1. **Tongue Patterns.** We begin by formalizing a way to compare disk-with-tongues structures.

Definition 16. A subset P of $\mathbb{R}^2$ is called a *tongue pattern* if it is the union of a finite number of disjoint subsets P_i called tongue blocks, each of which has the form

$$P_i = \bigcup_j \partial T_{ij}$$

where the T_{ij} are the tongues of a subdivision of a single tongue $T_i \subseteq \mathbb{R}^2$. A *family of tongue patterns* is defined to be a family of tongues T_i and T_{ij} as in §5.

Example 18. Let Σ be a primitive 2-sphere and $T_1, \dots, T_n$ the tongues of Type I of the contour $C(\Sigma)$ of Σ as in Proposition 4.1. Then $P(\Sigma) = \bigcup_i \pi(\partial T_i)$ is a tongue pattern.

Example 19. Let Σ be a primitive 2-sphere with Δ a large face of Σ. By Proposition 4.2, we have a disk with Type I and II tongue structures $C(\Sigma, \Delta)$ on Σ with initial disk Δ. Denote by $T_1, \dots, T_n$ the tongues of $C(\Sigma, \Delta)$ of Type I and II. Then $P(\Sigma, \Delta) = \bigcup_i \pi(\partial T_i)$ is a tongue pattern.

5.1.1. *Subdividing Tongues.* The goal of this section is to address the problem of disk-with-tongue structures being incompatible with the decomposition of Σ_{t0} into primitive spheres. More specifically, shrinking one factor of Σ_{t0} may induce a shrinking of Σ_{t0} that destroys the disk-with-tongues structure on another factor. Hatcher describes an instance of this problem in §7 of [6].

Recall that we have a family of graphs Γ_{tu} that record the data of which pieces of $g_t(S^2)$ have been surgered off at various times $u \in [0,1]$ in the surgery process. Let $\gamma \subset \Gamma_{tu}$ be a connected component and $\Sigma(\gamma)$ the primitive sphere corresponding to γ. Each subtree τ of γ corresponds to a primitive sphere $\Sigma_\tau \subset \Sigma(\gamma)$. Specifically, Σ_τ is obtained by taking the union of the factors Σ_i corresponding to vertices of τ and removing the interiors of the faces corresponding to edge of τ. Each τ represents a collection of pieces of $g_t(S^2)$ that are still left to be surgered in Σ_{tu} as u decreases.

We would like a compatibility relationship between the disk-with-tongues structures $C(\Sigma_\tau)$ as τ ranges over subtrees of γ.

Definition 17. For $\gamma \subset \Gamma_{tu}$ a connected component, define $P(\gamma)$ to be the union

$$P(\gamma) = \bigcup_{\tau \subset \gamma} (P(\Sigma_\tau) \cup P(\Sigma_\tau, \Delta_\tau))$$

where the union is taken over all subtrees τ of γ.

Proposition 5.1 (9.1)**.** *The tongue patterns $P(\gamma)$ define a family of tongue patterns.*

5.2. **Handle Decomposition of Parameter Space.** The goal of this section is to address the problem of disk-with-tongue structures changing in the parameter $t \in S^k$. We will do this by decomposing S^k into handles on which the disk-with-tongue structure changes in a manageable way. Moreover, we need our handle decomposition of S^k to respect the tongue patterns $P(\gamma)$ so that things continue to interact well with the surgery process for each fixed t. To be able to compare the chosen shrinkings of tongues, we introduce the following terminology:

Definition 18. The *associated tangent line field* to a shrinking of a tongue D is the section of the projectivization $\mathbb{P}(TD)$ of lines tangent to various stages of the shrinking.

Shrinkings of tongues with the same tangent line fields do not have to be the same shrinking, but there is always a path between them.

Lemma 5.2. *If a tongue T has two shrinkings T_s and T'_s with the same tangent line fields, then there exists a canonical path of shrinkings connecting T_s and T'_s.*

Warning. Even if the shrinking is C^∞, the tangent line field will only be C^0 near cusps of T.

The main result of this section is the following:

Proposition 5.3 (10.1)**.** *There is a triangulation $\mathcal{T}$ of S^k in which closed strata of $\mathcal{S}'_0$ are subcomplexes, such that for each simplex σ of $\mathcal{T}$ and each family $\Sigma_t(\gamma)$ defined for $t \in \sigma$, there exists:*

(i) families of tongue blocks $Q_r(\gamma)$, parameterized by $t \in \sigma$,

(ii) inclusion maps $P_q(\gamma) \hookrightarrow Q_r(\gamma)$ (with $r = r(q)$ depending on q) for the tongue blocks $P_q(\gamma)$ with $\coprod P_q(\gamma) = P(\gamma)$, and

(iii) families of shrinkings of the tongues of $Q_r(\gamma)$ such that if $r_1 \neq r_2$, the associated tangent line fields for the tongues of $Q_{r_1}(\gamma)$ meet those for the tongues of $Q_{r_2}(\gamma)$ transversely for all $t \in \sigma \setminus \partial\sigma$. Further, if σ' is a face of σ, $\Sigma_t(\gamma')$ is defined for $t \in \sigma'$, and $\Sigma_t(\gamma) < \Sigma_t(\gamma')$ for $t \in \sigma'$, then we have a diagram

$$\begin{array}{ccc} P_q(\gamma) & \longrightarrow & Q_r(\gamma) \\ \downarrow & & \downarrow \\ P_{q'}(\gamma') & \longrightarrow & Q_{r'}(\gamma') \end{array}$$

and the tangent line fields associated to the chosen shrinkings of the tongues $Q_{r'}(\gamma')$ restrict to those for the tongues of $Q_r(\gamma)$.

Remark 5. Our goal was to give a handle decomposition of S^k. We get a handle decomposition from the triangulation $\mathcal{T}$ whose i-handles $H^i = \mathbb{D}^i \times \mathbb{D}^{k-i}$ are ϵ_i-neighborhoods of i-simplicies σ^i of $\mathcal{T}$ with points in previously constructed handles of smaller index removed. Here the ϵ_i are chosen so that

$$\epsilon_0 \gg \epsilon_1 \gg \cdots \gg \epsilon_k$$

Lemma 5.4. *We may assume that g_t is constant on slices $\{x\} \times \mathbb{D}^{k-i}$ of the handles H^i in this handle decomposition of S^k.*

The proof is a consequence of an operation Hatcher calls "blowing up" where the ϵ_i in the construction of the handles H^i are decreased.

Proof. It suffices to construct a smooth family of C^∞ embeddings $\hat{g}_t \colon S^2 \to \mathbb{R}^3$ so that the maps $g, \hat{g} \colon S^k \to \mathrm{Emb}(S^2, \mathbb{R}^3)$ are homotopic. Define $\hat{g}_t = g_{h(t)}$ where $h \colon S^k \to S^k$ collapses each slice $\{x\} \times \mathbb{D}^{k-i}$ in a handle H^i to a point $h(x) \in \sigma^i$. □

6. Shrinking $\Sigma(\gamma)$ in Families

Let $\Sigma(\gamma)$ be a component of Σ_{t0}. Then $\Sigma(\gamma)$ is the union of a elementary surface Σ and finitely many horizontal disks $\Delta_1, \ldots, \Delta_n$ that split $\Sigma(\gamma)$ into primitive factors $\Sigma_0, \ldots, \Sigma_n$. Assume Σ_0 is a large factor. Reorder Σ_i so that Δ_i is a face of Σ_i. Each Δ_i splits Σ into two primitives Σ^i and ${}^i\Sigma$ so that

- Σ is either a sum or difference of Σ^i and ${}^i\Sigma$,
- $\Sigma^i \cup {}^i\Sigma = \Sigma \cup \Delta_i$, and
- $\Sigma^i \cap {}^i\Sigma = \Delta_i$.

Either Σ^i or ${}^i\Sigma$, but not both, intersects Σ_i at more than just Δ_i. Say Σ^i has this property and call Σ^i a cofactor of $\Sigma(\gamma)$. We want to say that we can shrink the contour $C(\Sigma^i)$ down to $C(\Delta_i)$. By Proposition 4.2, this can be done if Δ_i is a large face. In [6, Lem. 8.2], Hatcher shows that indeed Δ_i is a large face of Σ^i. We therefore have an isotopy from Σ^i to a smoothed preimage of the contour of the face, $\pi^{-1}(C(\Delta_i))$.

6.1. **Construction of n-Parameter Families.** We will use these deformations to construct deformations $\Sigma^i(s_1,\dots,s_n)$, $s_j \in [0,1]$ for each cofactor Σ^i of $\Sigma(\gamma)$. These deformations will have the following properties

- $\Sigma^i(s_1,\dots,s_n)$ is independent of $s_1,\dots,s_{i-1}$, and
- $\Sigma^i(0,\dots,0) = \Sigma^i$.

Our construction will proceed inductively starting with $\Sigma^i(0,\dots,s_n)$.

Base Case Construction.

Define $\Sigma^n(0,\dots,0,s_n)$ to be the shrinking of Σ^n to Δ_n so that

(a) $\Sigma^n(0,\dots,0) = \Sigma^n$,
(b) $\Sigma^n(0,\dots,0,1) = \Delta_n$,
(c) $\Delta_n \subset \Sigma^n(0,\dots,0,s_n)$ for all $s_n \in [0,1]$,
(d) $\Sigma^n(0,\dots,0,s_n) \setminus \operatorname{int}(\Delta_n)$ for $s_n \in (0,1)$ is a smooth disk bounded by $\partial\Delta_n$, which moves across $\overline{\Sigma}^n$ by monotone isotopy relative $\partial\Delta_n$ from $\Sigma^n \setminus \operatorname{int}(\Delta_n)$ to Δ_n as s_n goes from 0 to 1.

For $\Sigma^n < \Sigma^i$, $i \neq n$, we have $\Sigma^n \setminus \operatorname{int}(\Delta_n) \subset \Sigma^i$. Define $\Sigma^i(0,\dots,0,s_n)$ by

(a) $\Sigma^n(0,\dots,0,s_n) \setminus \operatorname{int}(\Delta_n) \subset \Sigma^i(0,\dots,0,s_n)$, and
(b) $\Sigma^i \setminus \Sigma^n \subset \Sigma^i(0,\dots,0,s_n)$.

If $\Sigma^n \not< \Sigma^i$, set $\Sigma^i(0,\dots,0,s_n) = \Sigma^i$.

Inductive Step.

Fix $j = 1,\dots,n-1$. Assume we have constructed families $\Sigma^i(0,\dots,0,s_{j+1},s_n)$. We wish to construct families $\Sigma^i(0,\dots,0,s_j,s_{j+1},\dots,s_n)$. Let $\Sigma^i(s_j)$ denote $\Sigma^i(0,\dots,s_j,\dots,s_n)$ where s_j, living in the jth spot, is the first nonzero parameter. To mimic the base case construction, we would like the following:

- $\Sigma^j(s_j)$ to be a shrinking of $\Sigma^j(0)$ to Δ_j.
- If $\Sigma^j < \Sigma^i$, we want $\Sigma^i(s_j)$ to be the induced shrinking of $\Sigma^i(0)$.
- If $\Sigma^j \not< \Sigma^i$, set $\Sigma^i(s_j) = \Sigma^i(0)$.

To preform the inductive step, we need the following conditions to hold:

(i) $\Sigma^j(0)$ is an embedded sphere containing Δ_j which is smooth except possibly at corners of Σ^j and at the circle $\partial\Delta_j \subset \Sigma^j$.
(ii) If $\Sigma^j < \Sigma^i$, $j \neq i$, then $\overline{\Sigma}^j(0) \cap \Sigma^i(0) = \Sigma^j(0) \setminus \operatorname{int}(\Delta_j)$.
(iii) The contour $C(\Sigma^j(0))$ is a disk with tongues with $C(\Delta_j)$ as the initial disk.
(iv) There exists a shrinking of $C(\Sigma^j(0))$ to $C(\Delta_j)$.

Using the disk with tongue structures prescribed by $P(\gamma)$, one can check that these conditions hold. This is done in §11 of [6]. In particular, $\Sigma^0(s_1,\dots,s_n)$ retains its disk with tongue structure as $(s_1,\dots,s_n)$ varies.

Extend the deformations $\Sigma^i(s_1,\dots,s_n)$ over $s_j \in [1,2]$ by replacing $\Delta_j = \Sigma^j(1)$ with two parallel copies of $\Delta(j)$.

6.1.1. *Overlap of Handles.* We want to patch together the families $\Sigma^l_t(s_1,\dots,s_n)$ for $t \in \sigma$ constructed in the previous section to families defined on all of S^k. To do so, we need to check that our n-parameter deformations agree for t in the overlap of simplicies of $\mathcal{T}$.

Definition 19. By *specialization* we mean setting the appropriate variables s_m equal to 0 or 2, according to whether in passing from $\Sigma^{ij}_t(\gamma)$ to $\Sigma^{i'j'}_t(\gamma')$, the corresponding common face Σ_m of $\Sigma^{ij}_t(\gamma)$ is deleted from $\Sigma^{ij}_t(\gamma)$ or splits $\Sigma^{ij}_t(\gamma)$ into two components, respectively.

Proposition 6.1 (12.2)**.** *After contracting $\{H^{ij}\}$, we may construct deformations $\Sigma^l_t(s_1,\dots,s_n)$ for the cofactors Σ^l_t of all families $\Sigma^{ij}_t(\gamma)$, $t \in H^{ij}$, such that*

($\star$) *If $t \in H^{ij} \cap H^{i'j'}$ with $\sigma^{ij} \subset \partial\sigma^{i'j'}$ and $\Sigma^{l'}_t \subset \Sigma^{i'j'}_t(\gamma')$ corresponds to $\Sigma^l_t \subset \Sigma^{ij}_t(\gamma)$, then the deformation $\Sigma^{l'}_t(s_1,\dots,s_{n'})$ is obtained from $\Sigma^l_t(s_1,\dots,s_n)$ by specialization.*

7. Mimicking the Models

For fixed t, the main idea for constructing $\bar{g}_{t0}$ will be to define foliations on factors Σ of $\overline{\Sigma}$ where leaves are paths from $\partial\overline{\Sigma}$ to either

(a) a pole in the interior of $\overline{\Sigma}$, or
(b) a line segment (called a face) in $\partial\overline{\Sigma}$.
corresponding to whether the associated factor of $\bar{S}^c_{tu}$ is polar or facial. Let Φ_{t0} denote the resulting foliation of $\overline{\Sigma}_{t0}$. Define a map $\bar{S}_{t0} \to \overline{\Sigma}_{t0}$ by sending:
- the boundary $S^2 \to g_t(S^2)$ by g_t,
- poles of F_{tu} to poles of Φ_{tu}, and
- sending leaves of F_{tu} to leaves of Φ_{tu}.
The foliations Φ_{tu} will be defined by the condition that there leaves are transverse to stages of certain shrinkings of factors to either points or faces.

In §7.1, we will construct models Σ^c_{t0} from Σ_{t0} mimicking the construction of S^c_{t0} from S_{t0}. The new spaces Σ^c_{t0} will bound disks $\overline{\Sigma}^c_{t0}$. We then construct foliations on the $\overline{\Sigma}^c_{t0}$ using polar foliations as in In the next section we will flush out the details of how to apply Proposition 4.1 in this case.

7.1. **Continuous Versions of Σ_{t0}.** We will define Σ^c_{t0} for t in a handle H^{ij} of the handle decomposition of S^k described in §5.2. In §12 of [6], Hatcher proves that the description of Σ^c_{t0} does not depend on which handle H^{ij} we assumed t lived in.

We continue notation as above: $\Sigma_t(\gamma)$ is a component of Σ_{t0} with cofactors $\Sigma^0_t, \ldots, \Sigma^n_t$. First, re-parameterize the families $\Sigma^i_t(s_1, \ldots, s_n)$ for $s_j \in [1, 2]$ by setting $t_j = 1 - s_j$.

Let $t \in S^k$ be in a handle H^{ij} corresponding to a simplex σ^{ij} of $\mathcal{T}$. Recall that in §1.1, we defined the stratification $\mathcal{S}_0$ on S^2 using the graphs $Z_0(e_l)$ of functions φ_t.

Lemma 7.1. *For each edge e_l of γ, we have $\sigma^{ij} \subset Z_0(e_l) \times [-1, 1] \subset S^k$ which is a tubular neighborhood in S^k.*

Extend the product structure of $Z_0(e_l) \times [-1, 1]$ to $Z_0(e_l) \times \mathbb{R}$. Then $H^{ij} \subset Z_0(e_l) \times \mathbb{R}$. For $t \in Z_0(e_k) \times \{r\}$, let

$$a_k = \begin{cases} r & |r| \leq 1 \\ 1 & r > 1 \\ -1 & r < -1 \end{cases}$$

The a_k depend on ij and t. Define Σ^c_{t0} as follows. First, take the union

$$\Big(\coprod_{l=1}^{n} \Sigma^l_t(a_1, \ldots, a_n)\Big)/\sim$$

where common faces $\Delta_m(a_1, \ldots, a_n)$ are identified for $t_m \geq 0$. If γ has a base vertex, for "shock waves" (corresponding to hitting a pole), subdivide $\Sigma^l_t(t_1, \ldots, t_n)$ for $t_l \leq 0$ by adjoining the disk

$$\Delta_l(t_1, \ldots, \tau_l(t_l), t_{l+1}, \ldots, t_n)$$

where $\tau_l \colon [-1, 0] \to [0, 1]$ "chosen appropriately." If γ has a base edge, the face Δ_1 corresponds to the base edge. Subdivide $\Sigma^1(t_1, \ldots, t_n)$ for $t_1 \leq 0$ by adjoining the disk

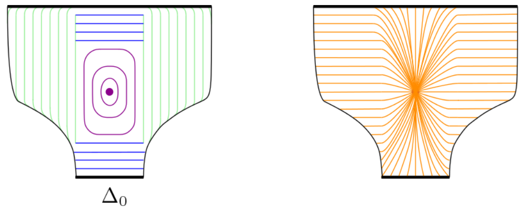
$$\Delta_1(\tau'_1(t_1), t_2, \ldots, t_n)$$

where $\tau'_1 \colon [-1, 0] \to [0, 1]$ deforms $\Delta_1(t_1, \ldots, t_n)$ as in Figure 3.3 of [6].

Remark 6. The construction of Σ^c_{t0} relied on C^1 shrinkings of various factors of Σ_{t0}. The spaces Σ^c_{t0}. $t \in S^k$, therefore only form a C^1 family. To apply Proposition 3.1, we need a smooth family Σ^c_{t0}. This lack of differentiability will be remedied in (Remark 8 below.

7.2. **Foliation on $\overline{\Sigma}^c_{t0}$.** Define foliations Φ_{t0} on $\overline{\Sigma}^c_{t0}$ as follows. On factors $\Sigma^l_t(t_1, \ldots, t_n)$ with $l > 0$, define Φ_{t0} to have leaves transverse to the stages of shrinkings of $\Sigma^l_t(t_1, \ldots, t_n)$ to its preferred face $\Delta_l(t_1, \ldots, _n)$. The contours of the factors $\Sigma^0_t(t_1, \ldots, t_n)$ retain their disk-with-tongues structures as $(t_1, \ldots, t_n)$ varies. Define Φ_{t0} on $\Sigma^0_t(t_1, \ldots, t_n)$ to have leaves transverse to the stages of the shrinking of $\Sigma^0_t(t_1, \ldots, t_n)$ to a point shrunk as follows: First, follow the shrinking lifted from a shrinking of $C(\Sigma^0_t(t_1, \ldots, t_n))$ to a small neighborhood of a point in its initial disk. This shrinks $\Sigma^0_t(t_1, \ldots, t_n)$ to something as in Example 15. Second, follow a linear vertical shrinking to a disk with rounded corners. Thirdly, take a standard radial shrinking of the disk with rounded corners to a point.

Example 20. The first figure below shows the stages of the shrinkings described above. The shrinking indicated in green is a lift of a shrinking of the contour to the initial disk Δ_0. The blue lines indicate the linear vertical foliation. Finally, the purple concentric circles indicate the radial shrinking to a point. The second figure indicates the flow lines of the correspond foliation with leaves transverse to the stages of the shrinkings in the first figure.



Remark 7. Note that $\Sigma_t^0(t_1, \ldots, t_n)$ corresponds to a polar factor of S_{t0}^c. Other factors $\Sigma_t^l(t_1, \ldots, t_n)$ for $l \neq 0$ correspond to facial factors of S_{t0}^c.

Remark 8. By definition 18, the foliations Φ_{t0} are transverse to a continuous family of tangent planes. We can choose such a Φ_{t0} to be smooth. The family Σ_{t0}^c can be perturbed into a smooth family, remaining transverse to Φ_{t0}.

8. Application of Proposition 3.1

We build a family of homeomorphisms $\bar{g}_{t0} \colon \bar{S}_{t0}^c \to \overline{\Sigma}_{t0}^c$, restricting to g_t on S^2 and diffeomorphisms on factors.

First, we extend g_t to a family of diffeomorphisms $S_{t0} \to \Sigma_{t0}$. Note that S_{t0} is formed from $S^2 \cap S_{t0}$ by attaching certain 2-disks in $S^3 \supset S^2$. Similarly, Σ_{t0} looks like $g_t(S^2)$ broken apart with certain horizontal 2-disks attached. To create the extension

$$\begin{array}{ccc} S^2 \cap S_{t0} & \xrightarrow{g_t} & g_t(S^2) \cap \Sigma_{t0} \\ \downarrow & & \downarrow \\ S_{t0} & \dashrightarrow & \Sigma_{t0} \end{array}$$

we need to define diffeomorphisms between the attached 2-disks, keeping their boundary in S^2 fixed. By Smale's theorem, $\mathrm{Diff}_\partial(\mathbb{D}^2)$ is contractible. We can therefore make this extension.

Next we extend the map $S_{t0} \to \Sigma_{t0}$ to a family of diffeomorphisms $g_{t0} \colon S_{t0}^c \to \Sigma_{t0}^c$.

We would like to extend g_{t0} to a map $\bar{g}_{t0} \colon \bar{S}_{t0}^c \to \overline{\Sigma}_{t0}^c$ by requiring $\bar{g}_{t0}$ to send leaves of F_{t0} to leaves of Φ_{t0}. Since $\mathrm{Diff}_\partial(\mathbb{D}^1)$ is contractible, the choice of how to send a particular leaf of F_{t0} to the corresponding leaf of Φ_{t0} does not matter. However, we need g_{t0} to send points connected by a leaf of F_{t0} to points connected by a leaf of Φ_{t0}. After possibly choosing a different Φ_{t0}, we can use Smale's theorem again to insure this occurs. The result is a homeomorphism $\bar{g}_{t0} \colon \bar{S}_{t0}^c \to \overline{\Sigma}_{t0}^c$ so that

- $\bar{g}_{t0}$ extends g_t,
- $\bar{g}_{t0}$ sends F_{t0} to Φ_{t0},
- $\bar{g}_{t0}$ is a diffeomorphism on factors.

Such a $\bar{g}_{t0}$ satisfies the conditions of Proposition 3.1. The Smale conjecture is therefore proven.

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