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# HOMEOMORPHISMS OF PRISM MANIFOLDS

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## 1. Introduction

*The homeotopy group* $\mathscr{H}(X)$ of a space $X$ is the group of all self-homeomorphisms of $X$ modulo the subgroup consisting of those homeomorphisms which are isotopic to the identity. The purpose of this paper is to compute the homeotopy group of a certain family of 3-manifolds, called prism manifolds. In § 4, we will completely determine the homeotopy group of prism manifolds (Theorem 4.5). Furthermore, as an application of our theorem, we will determine the homeotopy group of the lens space $L(4n, 2n\pm 1)$ of type $(4n, 2n\pm 1)$. More precisely, we have

**Corollary 4.6.** $\mathscr{H}\{L(4n, 2n\pm 1)\}\cong\begin{cases} Z_2 & \text{if } n=1 \\ Z_2\times Z_2 & \text{if } n\neq 1 . \end{cases}$

In § 2, we will define a prism manifold. In § 3, the incompressible Klein bottles in the prism manifold will be discussed.

Throughout this paper we work in the piecewise linear category. For a subcomplex $X$ of a manifold $Y$, the regular neighbourhood of $X$ in $Y$ will be denoted by $N(X)$. The boundary and the interior of a manifold $Y$ will be denoted by Bd $Y$ and Int $Y$, respectively.

A surface $F$ properly embedded in a 3-manifold $M$ is said to be parallel to Bd $M$ if there exists an embedding $f$: $F\times I\to M$ such that $f(F\times\{0\})=F$ and $f\{(\mathrm{Bd}\, F)\times I\cup F\times\{1\}\}\subset \mathrm{Bd}\, M$, where $I$ denotes the unit interval [0, 1].

## 2. Prism manifolds

Let $p$; $N\to B$ be an $S^1$-bundle over a Möbius band $B$. Suppose that $N$ is orientable. Then $p^{-1}(a)$, where $a$ is a centerline of $B$, is a Klein bottle $K_0$. Let $c_0$ be a fiber on $K_0$ and let $c_1$ be an oriented simple closed curve on $K_0$ such that $c_0\cap c_1$ is a point $q$ and $p(c_1)\sim a$ on $B$. Then

$$\Pi_1(K_0, q)=\langle c_0, c_1; c_1^2=(c_1c_0)^2\rangle .$$

As $N$ is orientable, it is a regular neighbourhood of $K_0$ and can be considered as a line bundle over $K_0$. The restriction $B'$ of the line bundle $N$ to $c_1$ is homeomorphic to Möbius band and is a cross-section of the $S^1$-bundle $p: N \to B$. The boundary $c$ of $B'$ and a fiber $h$ on Bd $N$ form a system of generators of $H_1(\mathrm{Bd}\, N)$. By $V$ we denote a solid torus with a meridian $x$. Let $k_{\alpha\nu}$ be a homeomorphism from Bd $V$ to Bd $N$ which induces an isomorphism $k_{\alpha\nu}^*$: $H_1(\mathrm{Bd}\, V) \to H_1(\mathrm{Bd}\, N)$ such that $k_{\alpha\nu}^*(x) = \alpha c + \nu h$, where $\alpha$ and $\nu$ are relatively prime integers with $\alpha > 0$.

Let $M_{\alpha\nu}$ denote the 3-manifold obtained by gluing $N$ to $V$ via $k_{\alpha\nu}$. Let $b$ and $\beta$ be integers such that $b\alpha + \beta = \nu$ and $\alpha > \beta \geqq 0$. Then $M_{\alpha\nu}$ is homeomorphic to a Seifert fiber space with the invariants $\{b; (n_2, 1)\}$ or $\{b; (n_2, 1); (\alpha, \beta)\}$. If $(b, \alpha, \beta) \neq (0, 1, 0)$, we call $M_{\alpha\nu}$ *a prism manifold*. Using Van Kampen Theorem, we can show that

$$\Pi_1(M_{\alpha\nu}, q) = \langle c_0, c_1;\ c_1^2 = (c_1 c_0)^2,\ c_1^{2\alpha} c_0^{\nu} = 1 \rangle .$$

We denote $\Pi_1(M_{\alpha\nu}, q)$ by $G_{\alpha\nu}$. From now on we will assume that $(\alpha, \nu) \neq (1, 0)$.

**Lemma 2.1.** *Each element of $G_{\alpha\nu}$ can be represented uniquely by the word $c_1^{\gamma} c_0^{\delta}$, where $0 \leqq \gamma < 2\alpha$ and $0 \leqq \delta < |2\nu|$.*

**Proof.** Applying the first relation in $G_{\alpha\nu}$, each element of $G_{\alpha\nu}$ can be represented by the word $c_1^{\lambda} c_0^{\mu}$. Since $c_1^{-1} c_0 c_1 = c_0^{-1}$, we have $c_0^{-\nu} = c_1^{-1} c_1^{2\alpha} c_1 = c_1^{-1} c_0^{-\nu} c_1 = c_0^{\nu}$. Thus we can reduce the word $c_1^{\lambda} c_0^{\mu}$ so that $0 \leqq \lambda < 2\alpha$ and $0 \leqq \mu < |2\nu|$. It follows from [3] that $G_{\alpha\nu}$ has a finite order $|4\alpha\nu|$. This implies that $c_1^{\gamma} c_0^{\delta}$ is uniquely represented.

The word $c_1^{\gamma} c_0^{\delta}$, $0 \leqq \gamma < 2\alpha$ and $0 \leqq \delta < |2\nu|$, is called *the normal form* of an element of $G_{\alpha\nu}$.

## 3. Klein bottles in $M_\alpha$

A surface $F$ in the 3-manifold $Q$ is said to be *compressible in $Q$*, if

(1) there exists a disk $D$ in $Q$ such that $D \cap F = \mathrm{Bd}\, D$ and Bd $D$ is essential in $F$, or

(2) there exists a 3-ball $E$ in $Q$ such that Bd $E = F$. We say that $F$ is *incompressible* in $Q$, if $F$ is not compressible in $Q$. In this section we will show that $K_0$ is incompressible in $M_{\alpha\nu}$ and will classify the incompressible Klein bottles in $M_{\alpha\nu}$ up to ambient isotopy.

Let $c_2$ be a simple closed curve on $K_0$ with $c_2 \cap c_0 = c_1 \cap c_0 = q$ such that $c_2$ represents $c_1 c_0$ in $\Pi_1(K_0, q)$. Then any one-sided curves on $K_0$ is ambient isotopic to $c_1$, $c_2$, $c_1^{-1}$ or $c_2^{-1}$, where $c_i^{-1}$ is the same curve as $c_i$ with opposite orientation. Furthermore any essential two-sided curve on $K_0$ can be deformed so that it coincides with either $c_0$ or the boundary of a regular neighbourhood of $c_1$ on $K_0$.

The following lemma has been proved by T. M. Price [5] for the case $(\alpha, \nu)=(1, 2)$.

**Lemma 3.1.** *$K_0$ is incompressible in $M_{\alpha\nu}$.*

**Proof.** Assume that $K_0$ is compressible in $M_{\alpha\nu}$. Then there exists a disk $D$ in $M_{\alpha\nu}$ such that $D\cap K_0=\mathrm{Bd}\, D$ and $\mathrm{Bd}\, D$ is essential on $K_0$. As $N$ is a regular neighbourhood of $K_0$, we can deform $D$ so that $D\cap V$ is a disk. Since the homomorphism from $\Pi_1(\mathrm{Bd}\, V)$ into $\Pi_1(N)$ induced by the inclusion is injective, $\mathrm{Bd}\, V\cap D$ is essential in $\mathrm{Bd}\, V$. Thus $V\cap D$ is a meridian disk of $V$ and $\Pi_1(M_{\alpha\nu}, q)\cong \Pi_1(K_0\cup D, q)$. As $\mathrm{Bd}\, D$ is a two-sided curve on $K_0$, it can be deformed so that it coincides with $c_0$ or the boundary of a regular neighburhood of $c_1$ on $K_0$. Hence $\Pi_1(K_0\cup D, q)$ is isomorphic to $\langle c_0, c_1;\ c_1^2=(c_1c_0)^2=1\rangle\cong Z_2*Z_2$ or $\langle c_0, c_1;\ c_1^2=(c_1c_0)^2, c_0=1\rangle\cong Z$, where $*$ denotes the free product of two groups. While $\Pi_1(M_{\alpha\nu}, q)$ has a finite order, both $Z_2*Z_2$ and $Z$ have an infinite order. Thus we have a contradiction.

Conversely the 3-manifold which is obtained by gluing the twisted line bundle over $K_0$ and a solid torus is a prism manifold, if $K_0$ is incompressible in the 3-manifold.

**Lemma 3.2.** *Any incompressible Klein bottle $K$ in $N$ is ambient isotopic to $K_0$.*

**Proof.** If we regard $N$ as a line bundle over $K_0$, the restriction of the line bundle to $c_0$ is an annlus $A$, which is incompressible and is not parallel to $\mathrm{Bd}\, N$. Then we can deform $K$ so that $K\cap A$ consists of essential two-sided curves on $K$. We may assume that $A$ and $K\cap A$ are vertical with respect to the $S^1$-bundle $p$: $N\to B$, i.e. $p^{-1}p(A)=A$ and $p^{-1}p(K\cap A)=K\cap A$. Clearly $K\cap(N-\mathrm{Int}\, N(A))$ consists of annuli. Thus by an ambient isotopy of $N$ we can deform $K$ so that it is vertical with respect to $p$. Since the intersection of $K$ and the cross-section $B'$ of $p$: $N\to B$ is a centerline of $B'$, we can deform $A$ so that $K\cap B'=K_0\cap B'$. Then $K\cap\{N-\mathrm{Int}\, N(B')\}$ and $K_0\cap\{N-\mathrm{Int}\, N(B')\}$ are Möbius bands in the solid torus $N-\mathrm{Int}\, N(B')$. Therefore $K$ is ambient isotopic to $K_0$.

**Lemma 3.3.** *Suppose that $M_{\alpha\nu}$ is a prism manifold with $|\nu|\neq 2$. Then any self-homeomorphism $g$ of $M_{\alpha\nu}$ is isotopic to a self-homeomorphism $g_0$ such that $g_0(K_0)=K_0$.*

**Proof.** Deform $g(K_0)$ slightly so that it is in a general position with respect to $K_0$. Since $K_0$ is incompressible in $M_{\alpha\nu}$ and $M_{\alpha\nu}$ is irreducible, we can remove all inessential curves in $g(K_0)\cap K_0$. By using the uniqueness of a regular neighbourhood of $K_0$, we can deform $g(K_0)$ so that $g(K_0)\cap V$ consists of a finite number of annuli and at most one Möbius band.

Suppose that $g(K_0)\cap V$ contains no Möbius band. Let $c'$ be a simple closed curve in $g(K_0)\cap \mathrm{Bd}\, V$. As $g(K_0)$ is incompressible in $M_{\alpha\nu}$, $c'$ is not a meridian of $V$. Hence each component of $g(K_0)\cap V$ is parallel to Bd $V$. Thus we can remove all intersection curves in $g(K_0)\cap \mathrm{Bd}\, V$.

If there is a Möbius band in $g(K_0)\cap V$, a closed curve $c''$ in $g(K_0)\cap \mathrm{Bd}\, V$ is homologous to $\mathrm{Bd}\, B'$ in Bd $V$. Hence the intersection number of $c''$ with $x$ is $\pm\nu$. But $|\nu|\neq 2$. Hence $g(K_0)\cap V$ contains no Möbius band. Thus, using Lemma 3.2, we complete the proof.

Regarding $N$ as a line bundle over $K_0$, let $B_i$ be the restricted bundle over $c_i$, for $i=1, 2$. Then $B_i$ is a Möbius band and the intersection number of Bd $B$ with $x$ is $\pm\nu$. Assume that $|\nu|=2$. Then there exists a Möbius band $B_i'$ in $V$ such that $\mathrm{Bd}\, B_i'=\mathrm{Bd}\, B_i$, for $i=1, 2$, and $B_1'\cap B_2'=c_3$, where $c_3$ is a centerline of $V$. Let $K_i$ denote a Klein bottle $B_i\cap B_i'$, $i=1, 2$. Since $M_{\alpha\nu}$ is irreducible and $\Pi_1(M_{\alpha\nu})$ is finite, $K_i$ is incompressible in $M_{\alpha\nu}$.

**Lemma 3.4.** *Let $M_{\alpha\nu}$ be a prism manifold with $|\nu|=2$. Then any incompressible Klein bottle $K$ in $M_{\alpha\nu}$ is ambient isotopic to $K_i$, $i=0$, 1 or 2.*

**Proof.** Suppose that $K\cap K_0$ contains either two one-sided curves or no one sided curve on $K_0$. Then $K\cap V$ consists of annuli. Hence the same argument as in the proof of Lemma 3.3 implies that $K$ is ambient isotopic to $K_0$.

Now we suppose that there is a Möbius band in $K\cap V$. We can deform $K$ so that $K\cap K_0$ contains $c_1$ or $c_2$. Since the Möbius band in $K\cap V$ intersects with $B_i'$ in a centerline of $B_i'$ and a finite nnmber of two-sided curves on $B_i'$, $K\cap K_i$ contains two one sided curves on $K_i$. Since $M_{\alpha\nu}-\mathrm{Int}\, N(K_i)$ is a solid torus and $K\cap\{M_{\alpha\nu}-\mathrm{Int}\, N(K_i)\}$ consists of annuli, we can deform $K$ so that $K$ coincides with $K_i$.

Now we construct a self-homeomorphism $f_i$ of $M_{\alpha\nu}$ such that $f_i(K_0)=K_i$, $i=1, 2$. Define a homeomorphism $f_i'$ of $K_0\cup K_i$ onto itself so that $f_i'(c_1)=c_i$ and $f_i'(c_2)=c_3$, for $i=1, 2$. Then $f_i'$ can be extended to an orientation preserving homemorphism $f_i$ of $N(K_0\cup K_i)$. If we consider $M_{\alpha\nu}$ as a Seifert fiber space $\{\pm(\alpha-3)/2; (o_1, 0); (2, 1), (2, 1), (2, 1)\}$ with the exceptional fibers $c_1$, $c_2$ and $c_3$, $M_{\alpha\nu}-\mathrm{Int}\, N(K_0\cup K_i)$ is a regular neighbourhood of a normal fiber of the Seifert fiber space and $f_i$ is a fiber preserving homeomorphism. Thus we can extend $f_i$ to a self-homeomorphism of $M_{\alpha\nu}$.

## 4. Homeotopy groups

In this section we calculate the homeotopy group $\mathscr{H}(M_{\alpha\nu})$ of $M_{\alpha\nu}$. First we have

**Lemma 4.1.** *$M_{\alpha\nu}$ does not admit an orientation reversing homeomorphism.*

**Proof.** Suppose that there exists an orientation reversing homeomorphism $r$ of $M_{\alpha\nu}$. We may assume that $r(K_0)=K_0$, $r(c_0)=c_0$, $r(q)=q$ and $r(N)=N$. Let $r_*$ denote the isomorphism from $H_1(\mathrm{Bd}\, V)$ onto itself induced by the restriction of $r$ to Bd $V$. Since $r(c_0)$ represents $c_0^{\varepsilon}$ and $r(c_1)$ represents $c_1^{\varepsilon}$ or $(c_1c_0)^{\varepsilon}$ in $\Pi_1(K_0, q)$, $\varepsilon=1$ or $-1$, $r_*(c)=\varepsilon c$ and $r_*(h)=-\varepsilon h$, where $c$ is the boundary of the cross-section $B_i'$ of $N$ and $h$ is a fiber on Bd $N$. Hence $r_*(x)=\varepsilon\alpha c-\varepsilon\nu h$. Since $r(x)$ is a meridian of $V$ and $\varepsilon\alpha c-\varepsilon\nu h\not\sim 0$ in $V$, we have a contradiction.

Every homeomorphism of $K_0$ isotopic to a fiber preserving homeomorphism and can be extended to an orientation preserving homeomorphism of $N$. Furthermore we can extend the homeomorphism to an homeomorphism of $M_{\alpha\nu}$. Thus there is a natural homomorphism $\Phi$: $\mathscr{H}(K_0)\to\mathscr{H}(M_{\alpha\nu})$. Note that $\mathscr{H}(K_0)\cong Z_2\times Z_2$. For $|\nu|\neq 2$, by Lemma 3.3, $\Phi$ is onto.

**Lemma 4.2.** *For $\alpha\neq 1$,* Ker $\Phi$ *is trival.*

**Proof.** Let $\mathscr{A}(G_{\alpha\nu})$ denote the group of outerautomorphisms of $G_{\alpha\nu}$. Then there exists a homomorphism $\Psi$: $\mathscr{H}(M_{\alpha\nu})\to\mathscr{A}(G_{\alpha\nu})$. We shall show that Ker $\Psi\Phi$ is trivial It is sufficient to prove that the conjugacy classes of $c_1$, $c_2$, $c_1^{-1}$ and $c_2^{-1}$ are mutually disjoint in $G_{\alpha\nu}$. The normal forms of $c_1$, $c_2$, $c_1^{-1}$ and $c_2^{-1}$ are $c_1$, $c_1c_0$, $c_1^{2\alpha-1}c_0^{|\nu|}$ and $c_1^{2\alpha-1}c_0^{|\nu|+1}$, respectively. Let $c_1^{\lambda}c_0^{\mu}$ be the normal form of an arbitrary element of $G_{\alpha\nu}$. Then

$$(c_1^{\lambda}c_0^{\mu})^{-1}c_1(c_1^{\lambda}c_0^{\mu})=c_1c_0^{2\mu} \quad \text{or} \quad c_1c_0^{2|\nu|-2\mu}$$
$$(c_1^{\lambda}c_0^{\mu})^{-1}c_1c_0(c_1^{\lambda}c_0^{\mu})=c_1c_0^{2\mu+1} \quad \text{or} \quad c_1c_0^{2\mu-1}\,.$$

Hence, $\alpha=1$ and $\nu$ is odd if and only if there exist integers $\lambda$ and $\mu$ such that $(c_1^{\lambda}c_0^{\mu})^{-1}c_1(c_1^{\lambda}c_0^{\mu})=c_1^{2\alpha-1}c_0^{|\nu|+1}$ and $(c_1^{\lambda}c_0^{\mu})^{-1}c_1c_0(c_1^{\lambda}c_0^{\mu})=c_1^{2\alpha-1}c_0^{|\nu|}$. Similarly, $\alpha=1$ and $\nu$ is even if and only if there exist integers $\lambda'$ and $\mu'$ such that $(c_1^{\lambda'}c_0^{\mu'})^{-1}c_1(c_1^{\lambda'}c_0^{\mu'})=c_1^{2\alpha-1}c_0^{|\nu|}$, $(c_1^{\lambda'}c_0^{\mu'})^{-1}c_1c_0(c_1^{\lambda'}c_0^{\mu'})=c_1^{2\alpha-1}c_0^{|\nu|+1}$. Thus the conjugacy classes of $c_1$, $c_2$, $c_1^{-1}$ and $c_2^{-1}$ are mutually disjoint except for $\alpha=1$.

For $\alpha=1$, $M_{\alpha\nu}$ is an $S^1$-bundle over a projective plane. The following proposition will be proved easily.

**Proposition 4.3.** *Let $p$: $Q\to T$ be an $S^1$-bundle over a surface $T$ and $\{i_t;\ 0\leqq t\leqq 1\}$ an isotopy of $T$. Suppose that there exists a homeomorphism $F$ of $Q$ onto itself* *such that $pF=i_0p$. Then there exists an isotopy $\{I_t;\ 0\leqq t\leqq 1\}$ of $Q$ such that $pI_t=i_tp$ and $I_0=F$.*

Since there is an isotopy of a projective plane $P^2$ which takes a one-sided curve on $P^2$ onto the same curve with opposite orientation, it follows from Proposition 4.3 that for $\alpha=1$ there exists an isotopy of $M_{\alpha\nu}$ which takes $\{c_1, c_2\}$ to $\{c_1^{-1}, c_2^{-1}\}$. Hence we have

**Lemma 4.4.** *For $\alpha=1$,* Ker $\Phi\cong Z_2$.

Assume that $|\nu|=2$. Let $c_3$ be a centerline of $V$. Then $c_3$ represents the conjugacy class of $c_1^{2\alpha'}c_0^{\nu'}$ or $(c_1^{2\alpha'}c_0^{\nu'})^{-1}$ in $G_{\alpha\nu}$, where $\alpha'$ and $\nu'$ are integers with $2\alpha'-\alpha\nu'=1$. Furthermore, the conjugacy class of $c_1^{2\alpha'}c_0^{\nu'}$ is $\{c_1{}^{\alpha'}c_0^{\nu'}, c_1^{2\alpha'}c_0^{-\nu'}\}$. Hence the conjugacy classes of $c_i$ and $c_i^{-1}$, $i=1, 2, 3$, are mutually disjoint. Thus there is an epimorphism $\Psi'$ from $\mathscr{A}(G_{\alpha\nu})$ onto the symmetric group $S_3$ of degree 3 which is defined by

$$\Psi'(\phi)=\begin{pmatrix}1 & 2 & 3\\ \gamma_1 & \gamma_2 & \gamma_3\end{pmatrix} \quad \text{if } \phi(c_i) \text{ is } c_{\gamma_i} \text{ or } c_{\gamma_i}^{-1} \text{ in } G_{\alpha\nu}\,.$$

Define a homomorphism $\Phi'$ from $S_3$ into $\mathscr{H}(M_{\alpha\nu})$ by $\Phi'(2, 3)=[f_1]$, $\Phi'(1, 2, 3)=[f_2]$, where $[f_i]$ denotes the isotopy class of $f_i$, $i=1, 2$. Then the exact sequence

$$1\rightarrow \mathrm{Ker}\,\Psi'\rightarrow\mathscr{H}(M_{\alpha\nu})\rightarrow S_3\rightarrow 1$$

splits by $\Phi'$. Clearly Ker $\Psi'$ is a proper subgroup of $\Phi\{\mathscr{H}(K_0)\}$. Hence Ker $\Psi'\cong Z_2$.

We summarize our results in the following theorem.

**Theorem 4.5.**
$$\mathscr{H}(M_{\alpha\nu})\cong\begin{cases} Z_2 & \alpha=1\,, \quad |\nu|\neq 2\\ S_3 & \alpha=1\,, \quad |\nu|=2\\ Z_2\times Z_2 & \alpha\neq 1\,, \quad |\nu|\neq 2\\ S_3\times Z_2 & \alpha\neq 1\,, \quad |\nu|=2\,.\end{cases}$$

For the case $\alpha=1$ and $|\nu|=2$, $\mathscr{H}(M_{\alpha\nu})$ has been determined by T. M. Price [5].

Let $L(2k, p)$ be a lens space of type $(2k, p)$ and let $q$ be the integer with $pq=\pm 1 \bmod 2k$ and $0<q<k$. It follows from [1] that $k-q=1$, i.e. $2k=4n$ and $p=2n\pm 1$, if and only if $L(2k, p)$ contains an incompressible Klein bottle $K$. As Bd $N(K)$ is compressible in $L(4n, 2n\pm 1)$, $L(4n, 2n\pm 1)-\mathrm{Int}\,N(K)$ is a solid torus. Hence $L$(4n, $2n\pm 1$) is a prism manifold. Since $|\nu|=1$ if and only if $G_{\alpha\nu}\cong Z_{4\alpha}$, we have

**Corollary 4.6.**
$$\mathscr{H}(L(4n, 2n\pm 1))\cong\begin{cases} Z_2 & \text{if } n=1\\ Z_2\times Z_2 & \text{if } n\neq 1\,.\end{cases}$$

**Note added in proof.** The same result has been obtained independently by J. H. Rubinstein. His paper will appear in Trans. Amer. Math. Soc.

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