3-Manifold Groups

Matthias Aschenbrenner

Stefan Friedl

Henry Wilton

University of California, Los Angeles, California, USA
E-mail address: matthias@math.ucla.edu

Fakultät für Mathematik, Universität Regensburg, Germany
E-mail address: sfriedl@gmail.com

Department of Pure Mathematics and Mathematical Statistics, Cambridge University, United Kingdom
E-mail address: h.wilton@maths.cam.ac.uk

Abstract. We summarize properties of 3-manifold groups, with a particular focus on the consequences of the recent results of Ian Agol, Jeremy Kahn, Vladimir Markovic and Dani Wise.

Contents

Introduction

The topic of this book is 3-*manifold groups,* that is, fundamental groups of compact 3-manifolds. This class of groups sits between the class of fundamental groups of surfaces, which are very well understood, and the class of fundamental groups of higher dimensional manifolds, which are very badly understood for the simple reason that given any finitely presented group π and integer $n \geq 4$, there exists a closed n-manifold with fundamental group π. (See [**CZi93**, Theorem 5.1.1] or [**SeT80**, Section 52] for a proof.) This fact poses a serious obstacle for understanding high-dimensional manifolds; for example, the unsolvability of the Isomorphism Problem for finitely presented groups [**Ady55, Rab58**] leads to a proof that closed manifolds of dimensions ≥ 4 cannot be classified in an algorithmically feasible way, see [**Mav58, Mav60, BHP68, Sht04**] and [**Sti93**, Section 9.4].

The study of the fundamental groups of 3-manifolds goes hand in hand with that of 3-manifolds themselves, since the latter are essentially determined by the former. More precisely, a closed, orientable, irreducible 3-manifold that is not a lens space is uniquely determined by its fundamental group. (See Theorem 2.1.3 below.) Despite the great interest and progress in 3-manifold topology during the last decades, survey papers focussing on the group-theoretic properties of fundamental groups of 3-manifolds seem to be few and far between. See [**Neh65, Sta71, Neh74, Hem76, Thu82a**], [**CZi93**, Section 5], and [**Kir97**] for some earlier surveys and lists of open questions.

This book grew out of an appendix originally planned for the monograph [**AF13**]. Its goal is to fill what we perceive as a gap in the literature, and to give an extensive overview of properties of fundamental groups of compact 3-manifolds with a particular emphasis on the impact of the Geometrization Conjecture of Thurston [**Thu82a**] and its proof by Perelman [**Per02, Per03a, Per03b**], the Tameness Theorem of Agol [**Ag04**] and Calegari–Gabai [**CaG06**], and the Virtually Compact Special Theorem of Agol [**Ag13**], Kahn–Markovic [**KM12a**] and Wise [**Wis12a**].

Our approach is to summarize many of the results in several flowcharts and to provide detailed references for each implication appearing in them. We will mostly consider fundamental groups of 3-manifolds which are either closed or have toroidal boundary, and we are interested in those properties of such 3-manifold groups π which can be formulated purely group-theoretically, i.e., without reference to the 3-manifold whose fundamental group is π. Typical examples are: *torsion-freeness*, *residual* properties such as being *residually finite* or *residually p*, *linearity* (over a field of characteristic zero), or *orderability.* We do not make any claims to originality—all results are either already in the literature, are simple consequences of established facts, or are well-known to the experts.

Organization of this book. As a guide for the reader, it may be useful to briefly go through some of the building blocks for our account of 3-manifold groups in the order they are presented in this book (which is roughly chronologically).

An important early result on 3-manifolds is the Sphere Theorem, proved by Papakyriakopoulos [**Pap57a**]. (See Section 1.3 below.) It implies that every orientable, irreducible 3-manifold with infinite fundamental group is an Eilenberg–Mac Lane space, and so its fundamental group is torsion-free. (See (A.2) and (C.3) in Sections 3.1 respectively 3.2.)

Haken [**Hak61a, Hak61b**] introduced the concept of a *sufficiently large* 3-manifold, later baptized *Haken manifold.* (See (A.10) in Section 3.1 for the definition.) He proved that Haken manifolds can be repeatedly cut along incompressible surfaces until the remaining pieces are 3-balls; this allows an analysis of Haken manifolds to proceed by induction. Soon thereafter, Waldhausen [**Wan68a, Wan68b**] produced many results on the fundamental groups of Haken 3-manifolds, e.g., the solution to the Word Problem.

A decade later, the Jaco–Shalen–Johannson (JSJ) decomposition [**JS79, Jon79a**] of an orientable, irreducible 3-manifold with incompressible boundary gave insight into the subgroup structure of the fundamental groups of Haken 3-manifolds. (See Section 1.6.) The JSJ-Decomposition Theorem also prefigured the Geometrization Conjecture. This conjecture was formulated and proved for Haken 3-manifolds by Thurston [**Thu82a**], and in the general case finally by Perelman [**Per02, Per03a, Per03b**]. (See Theorems 1.7.6 and 1.9.1.) After Perelman's epochal results, it became possible to prove that 3-manifold groups have many properties in common with linear groups: for example, they are residually finite [**Hem87**] (in fact, virtually residually p for all but finitely many prime numbers p [**AF13**], see (C.28) in Section 3.2 below) and satisfy the Tits Alternative (see items (C.26) and (L.2) in Sections 3.2 respectively 6).

The developments outlined in the paragraphs above (up to and including the proof of the Geometrization Conjecture and its fallout) are discussed in Chapters 1–3 of the present book. Flowchart 1 on p. 46 collects properties of 3-manifold groups that can be deduced using classical results of 3-manifold topology and Geometrization alone.

The Geometrization Conjecture also laid bare the special role played by hyperbolic 3-manifolds, which became a major focus of study in the last 30 years. During this period, our understanding of their fundamental groups has reached a level of completeness which seemed almost inconceivable only a short while ago. This is the subject of Chapters 4–6.

An important stepping stone in this process was the Subgroup Tameness Theorem (Theorem 4.1.2 below), which describes the finitely generated, geometrically infinite subgroups of fundamental groups of finite-volume hyperbolic 3-manifolds. This theorem is a consequence of the proof to Marden's Tameness Conjecture by Agol [**Ag04**] and Calegari–Gabai [**CaG06**], in combination with Canary's Covering Theorem [**Cay96**]. As a consequence, in order to understand the finitely generated subgroups of fundamental groups of hyperbolic 3-manifolds of this kind, one can mainly restrict attention to *geometrically finite* subgroups.

The results announced by Wise in [**Wis09**], with proofs provided in [**Wis12a**] (see also [**Wis12b**]), revolutionized the field. First and foremost, together with Agol's Virtual Fibering Theorem [**Ag08**], they imply that every Haken hyperbolic 3-manifold is virtually fibered (i.e., has a finite cover which is fibered over S^1). Wise in fact proved something stronger, namely that if N is a hyperbolic 3-manifold with an embedded geometrically finite surface, then $\pi_1(N)$ is virtually compact special. See Section 4.3 for the definition of a (*compact*) *special group.* (These groups arise as particular types of subgroups of *right-angled Artin groups* and carry a very combinatorial flavor.) As well as virtual fibering, Wise's theorem also implies that the fundamental group of a hyperbolic 3-manifold N as before is *subgroup separable* (i.e., each of its finitely generated subgroups is closed in the profinite topology) and *large* (i.e., has a finite-index subgroup which surjects onto a non-cyclic free group), and has some further, quite unexpected corollaries: for instance, $\pi_1(N)$ is linear over $\mathbb{Z}$.

Building on the aforementioned work of Wise and the proof of the Surface Subgroup Conjecture by Kahn–Markovic [**KM12a**], Agol [**Ag13**] was able to give a proof of Thurston's Virtually Haken Conjecture: every closed hyperbolic 3-manifold has a finite cover which is Haken. Indeed, he proved that *the fundamental group of any closed hyperbolic* 3*-manifold is virtually compact special.* (See Theorem 4.2.2 below.) Flowchart 2 on p. 66 contains the ingredients involved in the proof of this astounding fact, and the connections between various 'virtual' properties of 3-manifolds are summarized in Flowchart 3 on p. 77.

Complementing Agol's work, Przytycki–Wise [**PW12**] showed that fundamental groups of compact, orientable, irreducible 3-manifolds with empty or toroidal boundary which are not graph manifolds are virtually special. In particular such manifolds are also virtually fibered and their fundamental groups are linear over $\mathbb{Z}$. These and many other consequences of being virtually compact special are summarized in Flowchart 4 on p. 84, and we collect the consequences of these results for finitely generated infinite-index subgroups of 3-manifold groups in Flowchart 5 on page 103.

The combination of these results of Agol, Przytycki–Wise, and Wise, with a theorem of Liu [**Liu13**] also implies that *the fundamental group of a compact, orientable, aspherical* 3*-manifold N with empty or toroidal boundary is virtually special if and only if N is non-positively curved.* This very satisfying characterization of virtual speciality may be seen as a culmination of the work on 3-manifold groups in the last half-century.

We conclude the book with a discussion of some outstanding open problems in the theory of 3-manifold groups (in Chapter 7).

What this book is *not* about. As with any book, this one reflects the tastes and biases of the authors. We list some of the topics which we leave basically untouched:

(1) *Fundamental groups of non-compact* 3*-manifolds.* We note that Scott [**Sco73b**] showed that given a 3-manifold with finitely generated fundamental group, there exists a compact 3-manifold with the same fundamental group.
(2) *'Geometric' and 'large scale' properties of* 3*-manifold groups.* For some results in this direction see [**Ger94, KaL97, KaL98, BN08, Bn12, Sis11a**].
(3) *Automaticity, formal languages, Dehn functions and combings.* We refer to, for instance, [**Brd93, BrGi96, Sho92, ECHLPT92, Pin03**].
(4) *Recognition problems.* These are treated in [**Hen79, JO84, JLR02, JT95, Sel95, Mng02, JR03, Mae03, KoM12, SSh14, GMW12**]. We survey some of these results in a separate paper [**AFW13**].
(5) 3*-dimensional Poincaré duality groups.* We refer to, e.g., [**Tho95, Davb00, Hil11**] for further information. (But see also Section 7.1.1.)
(6) We rarely discuss specific properties of fundamental groups of *knot complements* (known as 'knot groups'), although they were some of the earliest and most popular examples of 3-manifold groups to be studied. We note that in general, irreducible 3-manifolds with non-empty boundary are not determined by their fundamental groups, but interestingly, prime knots in S^3 are in fact determined by their groups [**CGLS85, CGLS87, GLu89, Whn87**].
(7) *Fundamental groups of distinguished classes of* 3*-manifolds.* For example, arithmetic hyperbolic 3-manifolds exhibit a lot of special features [**MaR03, Lac11,**

Red07]. But they also tend to be quite rare [**Red91**], [**Bor81**, Theorem 8.2], [**Chi83**], [**BoP89**, Section 7], [**GrL12**, Appendix], [**Mai14**].

(8) The *representation theory of 3-manifolds* is a substantial field in its own right, which fortunately is served well by Shalen's survey paper [**Shn02**].

(9) The *history* of the study of 3-manifolds and their fundamental groups. We refer to [**Epp99, Gon99, McM11, Mil03, Mil04, Sti12, Vo96, Vo02, Vo13c, Vo14**] for some articles, dealing mostly with the early history of 3-manifold topology and the Poincaré Conjecture.

This book is not intended as a leisurely introduction to 3-manifolds. Even though most terms will be defined, we will assume that the reader is already somewhat acquainted with 3-manifold topology. We refer to [**Hem76, Hat, JS79, Ja80, Scs14**] for background material. Another gap we perceive is the lack of a post-Geometrization textbook on 3-manifolds. We hope that someone else will step forward to fill this gaping hole.

What is a 3-manifold? Throughout this book we tried to state the results in maximal generality. It is one of the curses of 3-manifold topology that at times authors make implicit assumptions on the 3-manifolds they are working with, for example that they are orientable, or compact, or closed, or that the boundary is toroidal. When we give a reference for a result, then to the best of our knowledge our assumptions match the ones given in the reference. For results concerning non-compact or non-orientable 3-manifolds, it is recommended to go back to the original reference.

A few sections in our book state in the beginning some assumptions on the 3-manifolds considered in that section, and that are in force throughout that section. The reader should be aware of those assumptions when studying a particular section, since we do not repeat them when stating definitions and theorems.

Conventions and notations. All topological spaces are assumed to be connected unless it says explicitly otherwise, but we do not put any other a priori restrictions on our spaces. All rings have an identity, and m, n range over the set $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ of natural numbers.

Notation	Definition
$\mathbb{Z}_n$	the cyclic group $\mathbb{Z}/n\mathbb{Z}$ with n elements ($n \geq 1$)
D^n	the closed n-ball $\{v \in \mathbb{R}^n : \lvert v\rvert \leq 1\}$
S^n	the n-dimensional sphere $\{v \in \mathbb{R}^{n+1} : \lvert v\rvert = 1\}$
I	the closed interval $[0, 1]$
T^2	the 2-dimensional torus $S^1 \times S^1$
K^2	the Klein bottle
$K^2 \widetilde{\times} I$	the unique oriented total space of (a necessarily twisted) I-bundle over K^2

Various notions of 'n-dimensional manifold' agree for $n = 3$; see Section 1.1. Unfortunately, there are different conventions for what a *topological n-dimensional submanifold* of a topological 3-manifold N should be. For us it is a subset S of N such that for any point in $p \in N$ there exists a homeomorphism $\varphi \colon U \to V$ from an open neighborhood of p in N to an open subset of

$$\mathbb{R}^3_{\geq 0} := \big\{(x, y, z) \in \mathbb{R}^3 : x, y \in \mathbb{R}, z \in \mathbb{R}_{\geq 0}\big\}$$

such that $\varphi(U \cap S) \subseteq (\{0\}^{3-n} \times \mathbb{R}^n) \cap \mathbb{R}^3_{\geq 0}$. For example, according to our definition Alexander's horned sphere [**Ale24b**], [**Rol90**, Section I] is not a topological 2-dimensional submanifold of S^3.

Let S be a submanifold of N. We denote by νS a tubular neighborhood of S in N. Given a surface Σ in N we refer to $N \setminus \nu\Sigma$ as N *cut along* Σ. Moreover, if N is orientable and Σ is an orientable surface in N, then at times we pick a product structure $\Sigma \times [-1, 1]$ for a neighborhood of Σ and we identify $\nu\Sigma$ with $\Sigma \times (-1, 1)$.

When we write 'a manifold with boundary' then we also include the case that the boundary is empty. If we want to ensure that the boundary is in fact non-empty, then we will write 'a manifold with non-empty boundary.' Finally, beginning with Convention 1.7, a hyperbolic 3-manifold is understood to be orientable and to have finite volume, unless we say explicitly otherwise.

A note about the bibliography. The bibliography to this book contains well over 1300 entries. We decided to refer to each paper or book by a combination of letters. We followed the usual approach, for a single-author paper we use the first two or three letters of the author's last name, for a multiple-author paper we used the first letters of each of the authors' last names. Unfortunately, with so many authors this approach breaks down at some point. For example, there are three single-author papers by three different Hamiltons. We tried to deal with each problem on an ad-hoc basis. We are aware that this produced some unusual choices for abbreviations. Nonetheless, we believe that using letter-based names for papers (rather than referring to each entry by a number, say) will make it easier to use this book. For example, for many readers it will be clear that [**Lac06**] refers to a paper by Marc Lackenby and that [**Wan68b**] refers to a paper by Waldhausen.

Acknowledgments. Aschenbrenner acknowledges support from the NSF through grant DMS-0969642. Friedl's work on this book was supported by the SFB 1085 'Higher Invariants' at the Universität Regensburg funded by the Deutsche Forschungsgemeinschaft (DFG). Wilton was partially supported by an EPSRC Career Acceleration Fellowship.

The authors thank Ian Agol, Jessica Banks, Igor Belegradek, Mladen Bestvina, Michel Boileau, Steve Boyer, Martin Bridson, Jack Button, Danny Calegari, Jim Davis, Daryl Cooper, Dave Futer, Cameron Gordon, Bernhard Hanke, Pierre de la Harpe, Matt Hedden, John Hempel, Jonathan Hillman, Neil Hoffman, Jim Howie, Takahiro Kitayama, Thomas Koberda, Tao Li, Viktor Kulikov, Marc Lackenby, Mayer A. Landau, Peter Linnell, Wolfgang Lück, Curtis McMullen, Matthias Nagel, Mark Powell, Piotr Przytycki, Alan Reid, Igor Rivin, Saul Schleimer, Kevin Schreve, Dan Silver, András Stipsicz, Stephan Tillmann, Stefano Vidussi, Liam Watson, Susan Williams and Raphael Zentner for helpful comments, discussions and suggestions. We are also grateful for the extensive feedback we got from many other people on earlier versions of this book. We especially thank the referee for a very long list of useful comments. Finally we thank Anton Geraschenko for bringing the authors together.

CHAPTER 1

Decomposition Theorems

When we investigate a class of complex objects in mathematics, we commonly try to first identify the 'simplest' members of this class, and then prove that all objects in the given class can be assembled from those basic ones in some comprehensible fashion. This general strategy is also applied during the first steps in the study of 3-manifolds.

In this realm, one class of basic objects consists of the *prime* 3-manifolds . These are the building blocks that one obtains when decomposing a closed, orientable 3-manifold by cutting along homotopically essential 2-spheres. In Section 1.2 of this chapter we introduce these notions, and state the Prime Decomposition Theorem. Section 1.3 contains two other fundamental facts about 3-manifolds, the Loop Theorem and the Sphere Theorem. These classical results are all of a topological nature, but they do have consequences for the structure of the fundamental groups of 3-manifolds. Some of them are observed in Section 1.4; for example, the Prime Decomposition Theorem applied to a (suitable) 3-manifold N leads to a decomposition of $\pi_1(N)$ as a free product.

Having decomposed a 3-manifold by cutting along 2-spheres, the natural next step is to cut along incompressible tori. This leads to the *JSJ-Decomposition* (named after Jaco, Shalen, and Johannson). Here, the building blocks are the *Seifert fibered manifolds* and the *atoroidal* 3-manifolds. We discuss Seifert fibered manifolds in Section 1.5. The nature of atoroidal 3-manifolds is elucidated by the Geometrization Conjecture of Thurston, which is now a theorem, thanks to the breakthrough work of Perelman. According to this theorem, stated in Section 1.7, we may replace the 'atoroidal' building blocks in the JSJ-Decomposition by 'hyperbolic 3-manifolds.' *Hyperbolic 3-manifolds* are one example of 3-manifolds carrying geometric structure. Thurston realized the importance of geometric structures in low-dimensional topology (see Section 1.8). In line with this philosophy, the Geometric Decomposition Theorem (stated in Section 1.9), asserts that all 3-manifolds may be decomposed into geometric pieces. In Section 1.10 we then look at this theorem in the instructive special case of *fibered* 3-manifolds.

In the final Section 1.11 we use the earlier results to classify those abelian, nilpotent, and solvable groups which appear as fundamental groups of 3-manifolds. We begin this chapter by briefly recalling (in Section 1.1) that, in the 3-dimensional world, we may freely pass between the topological, piecewise linear, and smooth categories.

1.1. Topological and smooth 3-manifolds

By the work of Radó [**Rad25**] any topological 2-manifold (not necessarily compact) admits a unique PL-structure and a unique smooth structure. (See also [**AS60**], [**Moi77**, p. 60], and [**Hat13**].) In general this does not hold for manifolds of dimension ≥ 4; see [**Mil56, Ker60, KeM63, KyS77, Fre82, Don83, Lev85, Mau13**] for details.

Fortunately the 3-dimensional case mirrors the 2-dimensional situation. More precisely, by Moise's Theorem [**Moi52**], [**Moi77**, p. 252 and 253] any topological 3-manifold (not necessarily compact) admits a unique piecewise-linear structure. (See also [**Bin59,**

Hama76, Shn84].) By work of Munkres [**Mun59**, p. 333], [**Mun60**, Theorems 6.2 and 6.3] and independently Whitehead [**Whd61**, Corollary 1.18] this implies that a topological 3-manifold admits a unique smooth structure. (See also [**Thu97**, Theorem 3.10.8 and 3.10.9].) We refer to [**Kui79, Kui99**] and [**Les12**, Section 7] for a more detailed discussion.

NOTATION. Let M and N be 3-manifolds. Throughout this book we write $M \cong N$ if M and N are homeomorphic, which by the above is the same as saying that they are diffeomorphic. If M and N are oriented, then $M \cong N$ means that there exists an orientation preserving homeomorphism. By a slight abuse of notation we will write $M = N$ if M and N are homeomorphic and if N is one of S^3, $S^1 \times D^2$ and $S^1 \times S^2$. At times, when we feel that there is no danger of confusion, we also write $M = N$ for $N = T^2 \times I$ and $N = K^2 \widetilde{\times} I$.

REMARKS.

(1) By work of Cerf [**Ce68**] and Hatcher [**Hat83**, p. 605], given any closed 3-manifold M the natural inclusion $\operatorname{Diff}(M) \to \operatorname{Homeo}(M)$ of the space of diffeomorphisms of M in the space of homeomorphisms of M is in fact a weak homotopy equivalence. (See also [**Lau85**].)
(2) Bing [**Bin52**] gave an example of a continuous involution on S^3 whose fixed point set is a wild S^2. In particular, this involution cannot be smoothed.
(3) Kwasik–Lee [**KwL88**, Corollary 2.2] showed that a topological action of a finite group on a closed 3-manifold N is smoothable if and only if it is simplicial in some triangulation of N, and that a topological action of a cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ on a closed 3-manifold N is smoothable if and only if for every subgroup H of $\mathbb{Z}_n$, the fixed point set N^H is tame [**KwL88**, Corollary 2.3]. (This was also proved earlier by Moise [**Moi80**].)

In the theory of 3-manifolds embedded surfaces play an important rôle. It is thus essential that we can replace topological 2-submanifolds by PL-submanifolds and by smooth submanifolds. This is indeed the case. More precisely, let N be a topological compact 3-manifold and let S be a topological 2-dimensional submanifold of N. Then there exists a smooth structure on N and a smoothly embedded surface S' in N that is isotopic to S. (The corresponding result for PL submanifolds also holds, with verbatim the same proof.) We denote by M the result of cutting N along S. Then there exist two copies S_1 and S_2 in ∂M and there exists a homeomorphism $f\colon S_1 \to S_2$ such that N is homeomorphic to the result of gluing S_1 to S_2 via f. By the above M admits a smooth structure. The topological submanifolds S_1 and S_2 of ∂M have two smooth submanifolds S_1' and S_2' of ∂M as deformation retracts. (Indeed, any connected topological submanifold of a surface contains a smooth submanifold such that inclusion induces an isomorphism on fundamental groups, the complementary regions are therefore annuli.) We replace f by a diffeomorphism $f'\colon S_1' \to S_2'$ that is isotopic to f under the above deformation retracts. The result of gluing S_1' to S_2' along f' is a smooth 3-manifold N' that contains the smooth submanifold $S_1' = S_2'$. It is straightforward to see that all these smoothings and maps can be done in such a way that there exists a homeomorphism $g\colon N' \to N$ such that $g(S')$ is isotopic to S.

1.2. The Prime Decomposition Theorem

Let N_1 and N_2 be oriented 3-manifolds and let $B_i \subseteq N_i$ $(i = 1, 2)$ be embedded closed 3-balls. For $i = 1, 2$ we endow ∂B_i with the orientation induced from the orientation as a boundary component of $N_i \setminus \operatorname{int} B_i$. We pick an orientation reversing homeomorphism $f\colon \partial B_1 \to \partial B_2$ and we refer to

$$N_1 \# N_2 := (N_1 \setminus \operatorname{int} B_1) \ \cup_f \ (N_2 \setminus \operatorname{int} B_2)$$

as the *connected sum of* N_1 *and* N_2. The homeomorphism type of $N_1 \# N_2$ does not depend on the choice of f [**Hem76**, Lemma 3.1].

Definition. An orientable 3-manifold N is called *prime* if it cannot be decomposed as a non-trivial connected sum of two manifolds, that is: if $N \cong N_1 \# N_2$, then $N_1 = S^3$ or $N_2 = S^3$. A 3-manifold N is called *irreducible* if every embedded 2-sphere in N bounds a 3-ball in N.

An orientable, irreducible 3-manifold is evidently prime. Conversely, if N is an orientable prime 3-manifold with no spherical boundary components, then either N is irreducible or $N = S^1 \times S^2$. (See [**Hem76**, Lemma 3.13] and also [**Cal14a**, Proposition 3.12].) It is a consequence of the generalized Schoenflies Theorem (see [**Broc60, CV77**]), that S^3 is irreducible.

The following theorem is due to Kneser [**Kn29**], Haken [**Hak61b**, p. 441f] and Milnor [**Mil62**, Theorem 1].

Theorem 1.2.1 (Prime Decomposition Theorem). *Let N be a compact, oriented 3-manifold with no spherical boundary components. There exist oriented prime 3-manifolds $N_1, \dots, N_m$ such that*

$$N \cong N_1 \# \cdots \# N_m.$$

In particular,

$$\pi_1(N) \cong \pi_1(N_1) * \cdots * \pi_1(N_m)$$

is the free product of fundamental groups of prime 3-manifolds. Moreover, if

$$N \cong N_1 \# \cdots \# N_m \quad \text{and} \quad N \cong N'_1 \# \cdots \# N'_n$$

with oriented prime 3-manifolds N_i and N'_i, then $m = n$ and (possibly after reordering) there exist orientation-preserving diffeomorphisms $N_i \to N'_i$.

The uniqueness statement of the theorem only concerns the homeomorphism types of the prime components. The decomposing spheres are not unique up to isotopy, but two different sets of decomposing spheres are related by 'slide homeomorphisms.' We refer to [**CdSR79**, Theorem 3], [**HL84**] and [**McC86**, Section 3] for details. See also [**Hak61b**, p. 441], [**Sco74**, Chapter III], [**Hem76**, Chapter 3], [**Hat**], [**Kin05**, Section 5.1.1], [**HAM08**], [**Scs14**, Section 5.5], and [**Cal14a**, Theorem 3.5]. More decomposition theorems in the bounded cases are in [**Grs69, Grs70, Swp70, Prz79**].

1.3. The Loop Theorem and the Sphere Theorem

The life of 3-manifold topology as a flourishing subject started with the proof of the Loop and Sphere Theorems by Papakyriakopoulos. We first state the Loop Theorem.

Theorem 1.3.1 (Loop Theorem). *Let N be a compact 3-manifold and $F \subseteq \partial N$ a subsurface. If the induced morphism $\pi_1(F) \to \pi_1(N)$ is not injective, then there exists a proper embedding $g\colon (D^2, \partial D^2) \to (N, F)$ such that $g(\partial D^2)$ represents a non-trivial element in $\operatorname{Ker}(\pi_1(F) \to \pi_1(N))$.*

A somewhat weaker version (usually called 'Dehn's Lemma') of this theorem was first stated by Dehn [**De10, De87**] in 1910, but Kneser [**Kn29**, p. 260] found a gap in the proof provided by Dehn. The Loop Theorem was finally proved by Papakyriakopoulos [**Pap57a, Pap57b**] building on work of Johansson [**Jos35**]. We refer to [**Hom57, SpW58, Sta60, Wan67b, Gon99, Bin83, Jon94, AiR04**], [**Cal14a**, Theorem 1.1], and [**Hem76**, Chapter 4] for alternative proofs, more details and several extensions.

Now we turn to the Sphere Theorem.

Theorem 1.3.2 (Sphere Theorem). *Every orientable* 3*-manifold* N *with* $\pi_2(N) \neq 0$ *contains an embedded* 2*-sphere that is homotopically non-trivial.*

This theorem was proved by Papakyriakopoulos [**Pap57a**] under a technical assumption which was removed by Whitehead [**Whd58a**]. (We refer to [**Whd58b, Bat71, Gon99, Bin83**], [**Cal14a**, Theorem 2.2] and [**Hem76**, Theorem 4.3] for alternative proofs, extensions and more information.) Gabai (see [**Gab83a**, p. 487] and [**Gab83b**, p. 79]) proved that for 3-manifolds the Thurston norm equals the singular Thurston norm. (See also [**Pes93**] for an alternative proof and Section 5.4.3 below for more on the Thurston norm.) This result can be viewed as a higher-genus analogue of the Loop Theorem and the Sphere Theorem.

1.4. Preliminary observations about 3-manifold groups

The main subject of this book are the properties of fundamental groups of compact 3-manifolds. In this section we argue that for most purposes it suffices to study the fundamental groups of compact, orientable, irreducible 3-manifolds whose boundary is either empty or toroidal. *Throughout this section* N *is a compact* 3*-manifold.*

We start out with the following basic observation.

Observation 1.4.1.

(1) *Let* $\widehat{N}$ *be the* 3*-manifold obtained from* N *by gluing* 3*-balls to all spherical components of* ∂N*. The inclusion* $N \to \widehat{N}$ *induces an isomorphism* $\pi_1(N) \xrightarrow{\cong} \pi_1(\widehat{N})$*.*
(2) *If* N *is non-orientable, then* N *has an orientable double cover.*

Most properties of groups of interest to us are preserved under going to free products of groups. (See, e.g., [**Nis40**] and [**Shn79**, Proposition 1.3] for linearity and [**Rom69, Bus71**] for being LERF.) Similarly most properties of groups that we consider are preserved under passing to a finite index supergroup. (See, e.g., (I.1)–(I.7) below.) Note though that this is not true for *all* interesting properties; for example, conjugacy separability does not pass to index-two extensions in general [**CMi77, Goa86**].

In light of Theorem 1.2.1 and Observation 1.4.1, we generally restrict ourselves to the study of orientable, irreducible 3-manifolds with no spherical boundary components.

A properly embedded surface $\Sigma \subseteq N$ with components $\Sigma_1, \dots, \Sigma_n$ is *incompressible* if for each $i = 1, \dots, n$ we have $\Sigma_i \neq S^2, D^2$ and the map $\pi_1(\Sigma_i) \to \pi_1(N)$ is injective. The following lemma is a well-known consequence of the Loop Theorem.

Lemma 1.4.2. *There exist compact* 3*-manifolds* $N_1, \dots, N_m$ *whose boundary components are incompressible and a free group* F *such that*

$$\pi_1(N) \cong \pi_1(N_1) * \cdots * \pi_1(N_m) * F.$$

Proof. By the above observation we can assume that N has no spherical boundary components. Let Σ be a component of ∂N such that $\pi_1(\Sigma) \to \pi_1(N)$ is not injective. By the Loop Theorem 1.3.1 there exists a properly embedded disk $D \subseteq N$ such that the curve $c = \partial D \subseteq \Sigma$ is essential. Here a curve $c \subseteq \Sigma$ is called *essential* if it does not bound an embedded disk in Σ.

Let N' be the result of capping off the spherical boundary components of $N \setminus \nu D$ by 3-balls. If N' is connected, then $\pi_1(N) \cong \pi_1(N') * \mathbb{Z}$; otherwise $\pi_1(N) \cong \pi_1(N_1) * \pi_1(N_2)$ where N_1, N_2 are the two components of N'. The lemma follows by induction on the lexicographically ordered pair $(-\chi(\partial N), b_0(\partial N))$ since we have either that $-\chi(\partial N') < -\chi(\partial N)$ (if Σ is not a torus), or that $\chi(\partial N') = \chi(\partial N)$ and $b_0(\partial N') < b_0(\partial N)$ (if Σ is a torus). □

We say that a group G is a *retract* of a group H if there exist morphisms $\varphi\colon G \to H$ and $\psi\colon H \to G$ such that $\psi \circ \varphi = \mathrm{id}_G$. In this case φ is injective and so we can view G as a subgroup of H via φ.

Lemma 1.4.3. *Suppose N has non-empty boundary. Then $\pi_1(N)$ is a retract of the fundamental group of a closed 3-manifold.*

Proof. Denote by M the double of N, i.e., the closed 3-manifold

$$M = N_1 \cup_{\partial N_1 = \partial N_2} N_2$$

where N_1, N_2 are homeomorphic copies of N. Let f be the natural map $N \to N_1 \xrightarrow{\subseteq} M$ and $g\colon M \to N$ be the map which restricts to a homeomorphism $N_i \to N$ on each of the two copies N_1, N_2 of N in M. Clearly $g \circ f = \mathrm{id}_N$ and hence $g_* \circ f_* = \mathrm{id}_{\pi_1(N)}$. □

Many properties of groups are preserved under retracts and taking free products; this way, many problems on 3-manifold groups can be reduced to the study of fundamental groups of closed 3-manifolds. Due to the important rôle played by 3-manifolds with toroidal boundary components we will be slightly less restrictive, and in the remainder we study fundamental groups of compact, orientable, irreducible 3-manifolds N whose boundary is either empty or toroidal.

1.5. Seifert fibered manifolds

Before we continue our systematic study of compact, oriented 3-manifolds we introduce Seifert fibered manifolds. These form a pretty well-understood class of 3-manifolds, and are an essential building block in the general theory. We refer to [**Sei33a, Or72, Hem76, Ja80, JD83, Sco83a, Brn93, LRa10**] for the proofs of the subsequent statements and for further information.

Definition. A *Seifert fibered manifold* is a compact 3-manifold N together with a decomposition of N into disjoint simple closed curves (called *Seifert fibers*) such that each Seifert fiber has a tubular neighborhood that forms a standard fibered torus. The *standard fibered torus* corresponding to a pair of coprime integers (a, b) with $a > 0$ is the surface bundle of the automorphism of a disk given by rotation by an angle of $2\pi b/a$, equipped with the natural fibering by circles. If $a = 1$, then the middle Seifert fiber is called *regular*, otherwise it is called *singular*.

A Seifert fibered manifold N has only a finite number of singular fibers, which necessarily lie in the interior of N. In particular a Seifert fibered structure defines a product structure on each boundary component, which implies that each boundary component is an S^1-bundle, which evidently means that each boundary component is either a torus or a Klein bottle.

On several occasions we make use of the following fact; see [**JS79**, Lemma II.4.2].

LEMMA 1.5.1. *Let N be a Seifert fibered manifold. All regular Seifert fibers of N are isotopic, and this isotopy class of curves defines a normal cyclic subgroup. Furthermore, if $\pi_1(N)$ is infinite, then this cyclic subgroup is infinite.*

The following theorem [**Ja80**, Theorems VI.17 and VI.18] (see also [**OVZ67**],[**Sco83a**, Theorem 3.8], [**JS79**, II.4.11]) says in particular that 'sufficiently complicated' Seifert fibered manifolds carry a unique Seifert fibered structure.

THEOREM 1.5.2. *Let N be an orientable 3-manifold that admits a Seifert fibered structure. If N has non-trivial boundary, then the Seifert fibered structure of N is unique up to isotopy if and only if N is not homeomorphic to one of the following:*

(1) $S^1 \times D^2$, $T^2 \times I$, $K^2 \widetilde{\times} I$.

If N is closed, then the Seifert fibered structure is unique up to homeomorphism if and only if N is not homeomorphic to one of the following:

(2) S^3, $S^1 \times S^2$ *or lens spaces (see Section 1.11 for the definition),*
(3) *prism 3-manifolds (i.e., the closed Seifert fibered 3-manifolds for which the base orbifold is S^2 with three cone points of order $(2, 2, n)$, $n \geq 2$),*
(4) *the double of $K^2 \widetilde{\times} I$.*

For an additively written abelian group G we denote by PG the set of equivalence classes of the equivalence relation $g \sim -g$ on $G \setminus \{0\}$, and for $g \in G \setminus \{0\}$ we denote by $[g]$ its equivalence class in PG. Given a Seifert fibered manifold N with a choice of a Seifert fibered structure and a boundary torus T, the regular fiber determines an element $f(N, T) \in PH_1(T)$ in a natural way. It follows from Theorem 1.5.2 that $f(N, T)$ is independent of the choice of the Seifert fibered structure, unless N is one of $S^1 \times D^2$, $T^2 \times I$ or $K^2 \widetilde{\times} I$. For later use we record:

LEMMA 1.5.3. *Let N_1 and N_2 be two Seifert fibered manifolds (where we allow $N_1 = N_2$) and let $N_1 \cup_{T_1 = T_2} N_2$ be the result of gluing N_1 and N_2 along boundary tori T_1 and T_2. Then the following are equivalent:*

(1) *there exists a Seifert fibered structure on $N_1 \cup_{T_1 = T_2} N_2$ that restricts (up to isotopy) to the Seifert fibered structures on N_1 and N_2.*
(2) $f(N_1, T_1) = f(N_2, T_2) \in PH_1(T_1) = PH_1(T_2)$.

In the subsequent discussion of the JSJ-decomposition we will need a precise understanding of which elements in $PH_1(\partial(K^2 \widetilde{\times} I))$ can be realized by Seifert fibered structures on $K^2 \widetilde{\times} I$. In order to state the result we need to carefully fix our notation.

In the following we think of the 2-torus T^2 as $S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$. Furthermore we make the corresponding identifications:

$$\pi_1(T^2) = \langle x, y \,|\, yxy^{-1} = x \rangle = \mathbb{Z}^2.$$

Also, we identify the Klein bottle K^2 with $\mathbb{R}^2/\sim$ where $(x, y) \sim (x + 1, y)$ and $(x, y) \sim (-x, y + \frac{1}{2})$ for all $x, y \in \mathbb{R}$. This gives us an identification

$$\pi_1(K^2) = \langle x, z \,|\, zxz^{-1} = x^{-1} \rangle.$$

With these identifications the obvious 2-fold covering $T^2 \to K^2$ induces the map $\pi_1(T^2) \to \pi_1(K^2)$ given by $x \mapsto x$ and $y \mapsto z^2$. Furthermore, the non-trivial deck transformation of $T^2 \to K^2$ induces the map $\pi_1(T^2) \to \pi_1(T^2)$ given by $x \mapsto -x$ and $y \mapsto y$. Identify K^2 with the 0-section of the twisted I-bundle $K^2 \widetilde{\times} I$. It is straightforward to see that we can identify $\partial(K^2 \widetilde{\times} I)$ with $T^2 = S^1 \times S^1$ in such a way that the above covering map $T^2 \to K^2$ corresponds to the map $\partial(K^2 \widetilde{\times} I) \to K^2 \widetilde{\times} I \to K$, where the last map is given by projecting onto the 0-section of the twisted I-bundle.

Proposition 1.5.4. *The only elements in*

$$PH_1\big(\partial(K^2 \widetilde{\times} I)\big) = PH_1(T^2) = P\mathbb{Z}^2$$

that can be realized by regular fibers of Seifert fibered structures on $K^2 \widetilde{\times} I$ are $[(1,0)]$ *and* $[(0,1)]$.

Proof. By Lemma 1.5.1 the regular fiber of a Seifert fibered structure of $K^2 \widetilde{\times} I$ defines an infinite cyclic normal subgroup. It is straightforward to see that the only such subgroups of $\pi_1(K^2 \widetilde{\times} I) = \pi_1(K^2) = \langle x, z \,|\, zxz^{-1} = x^{-1}\rangle$ are given by $\langle x^k\rangle$ or $\langle z^k\rangle$ with $0 \neq k \in \mathbb{Z}$. A Seifert fibered structure on $K^2 \widetilde{\times} I$ defines a product structure on $\partial(K^2 \widetilde{\times} I) = T^2$. It follows that a regular fiber is either $x^{\pm 1}$ or $z^{\pm 2} = y^{\pm 1}$ in $\pi_1(\partial(K^2 \widetilde{\times} I)) = \pi_1(T^2)$.

Conversely, the constructions of Seifert fibered structures on $K^2 \widetilde{\times} I$ in [**Ja80**, Section VI] or [**Scs14**, p. 90] show that $[(1,0)]$ and $[(0,1)]$ do indeed arise from Seifert fibered structures on $K^2 \widetilde{\times} I$. ◻

Before we continue with the formulation of the JSJ-Decomposition Theorem we need to consider the minor, but somewhat painful, special case of torus bundles and of twisted doubles of $K^2 \widetilde{\times} I$.

For this, let $A \in \mathrm{SL}(2,\mathbb{Z})$. By classical results there exists an orientation preserving homeomorphism $\varphi\colon T^2 \to T^2$ such that $\varphi_* = A$ in $\mathrm{SAut}(H_1(T;\mathbb{Z})) \cong \mathrm{SL}(2,\mathbb{Z})$, and φ is unique up to isotopy. (See, e.g., [**FaM12**, Theorem 2.5].) We write

$$\begin{array}{rcl} M(T^2,A) & = & (T^2 \times [0,1])/\!\sim \quad \text{where } (x,0) \sim (x,\varphi(x)), \text{ and} \\ D(K^2 \widetilde{\times} I, A) & = & (K^2 \widetilde{\times} I \times \{1\}) \cup_\psi (K^2 \widetilde{\times} I \times \{2\}) \\ & & \quad \text{where } \psi\colon \partial(K^2 \widetilde{\times} I) \times \{1\} \to \partial(K^2 \widetilde{\times} I) \times \{2\} \\ & & \quad \text{is given by } \psi(x,1) = (\varphi(x),2). \end{array}$$

We say that a 3-manifold is a *torus bundle* if it is homeomorphic to a manifold of the former type, and we say that it is a *twisted double of* $K^2 \widetilde{\times} I$ if it is homeomorphic to a manifold of the latter type.

We say that A is *Anosov* if it has two distinct real eigenvalues, and denote by J be the matrix corresponding to the non-trivial deck transformation of $T^2 \to K^2$, i.e., J is the diagonal matrix with entries -1 and 1.

Lemma 1.5.5.

(1) *The torus bundle $M(T^2,A)$ is Seifert fibered if and only if A is not Anosov.*
(2) *$D(T^2,A)$ is doubly covered by $M(T^2, JAJA^{-1})$.*
(3) *The following statements are equivalent:*
  (*a*) *$D(K^2 \widetilde{\times} I, A)$ is Seifert fibered;*
  (*b*) *$JAJA^{-1}$ is not Anosov;*
  (*c*) *one of the entries of A is zero;*

(*d*) *there exists a Seifert fibered structure on* $D(K^2 \widetilde{\times} I, A)$ *that restricts to a Seifert fibered structure on each of the two copies of* $K^2 \widetilde{\times} I$.

PROOF. The first statement follows from [**Sco83a**, Theorem 5.3]. By taking the 2-fold orientation covers on both copies of $K^2 \widetilde{\times} I$ we obtain a 2-fold cover of $D(K^2 \widetilde{\times} I, A)$ that is given by gluing two copies of $T^2 \times I$ along the boundary. It is straightforward to see that this manifold equals $M(T^2, JAJA^{-1})$.

We turn to the proof of the equivalence of (a)–(d). It follows from the first statement that if $D(K^2 \widetilde{\times} I, A)$ is Seifert fibered, then $JAJA^{-1}$ is not Anosov. An elementary, albeit slightly painful, exercise involving matrices in $\mathrm{SL}(2, \mathbb{Z})$ shows that $JAJA^{-1}$ is Anosov, unless one of the entries of A is zero. It is a consequence of Lemma 1.5.3 and Proposition 1.5.4 that if one of the entries of A is zero, then $D(K^2 \widetilde{\times} I, A)$ has a Seifert fibered structure restricting to a Seifert fibered structure on each of the two copies of $K^2 \widetilde{\times} I$. □

We conclude this section on Seifert fibered manifolds with some remarks:

(1) By [**Hat**, Proposition 1.12], [**Ja80**, Lemma VI.7], [**Scs14**, Theorem 3.7.17] the non-prime manifold $\mathbb{R}P^3 \# \mathbb{R}P^3$ is Seifert fibered. Furthermore, any orientable Seifert fibered manifold that is not prime is homeomorphic to $\mathbb{R}P^3 \# \mathbb{R}P^3$.
(2) A compact, orientable 3-manifold admits a Seifert fibered structure if and only if it admits a foliation by circles [**Eps72**, p. 81].
(3) For the most part the non-uniqueness of the Seifert fibered structures will be of no importance to us. Sometimes, later in the text, we will therefore slightly abuse language and say that a 3-manifold is Seifert fibered if it admits the structure of a Seifert fibered manifold.
(4) It is often useful to think of a Seifert fibered manifold as a circle bundle over a 2-dimensional orbifold.

1.6. The JSJ-Decomposition Theorem

In the previous sections we have seen that a compact, oriented 3-manifold with no spherical boundary components admits a decomposition along spheres into prime 3-manifolds $\neq S^3$ such that the set of resulting components are unique up to diffeomorphism. The goal of this section is to formulate the JSJ-Decomposition Theorem, which gives a canonical decomposition along incompressible tori.

Following [**JS79**, p. 185] we say that a 3-manifold N is *atoroidal* if any map $T \to N$ from a torus to N which induces a monomorphism $\pi_1(T) \to \pi_1(N)$ can be homotoped into the boundary of N. (In the literature at times slightly different definitions of 'atoroidal' are being used. For example, in [**Ja80**, p. 210] a 3-manifold is called atoroidal if any incompressible torus is boundary parallel. It follows from Theorem 1.7.7 and standard results on Seifert fibered manifolds [**Brn93**] that these two notions agree except for 'small Seifert fibered manifolds.') There exist orientable, irreducible 3-manifolds that cannot be cut into atoroidal components in a unique way (e.g., the 3-torus). Nonetheless, as we will see, any compact, orientable, irreducible 3-manifold with empty or toroidal boundary admits a canonical decomposition along tori into atoroidal and Seifert fibered manifolds.

The following theorem was first announced by Waldhausen [**Wan69**] and proved independently by Jaco–Shalen [**JS79**, p. 157] and Johannson [**Jon75, Jon79a**]. The initials of Jaco, Shalen and Johannson gave the theorem its catchy name.

THEOREM 1.6.1 (JSJ-Decomposition Theorem). *Let N be a compact, orientable, irreducible 3-manifold with empty or toroidal boundary. There exists a (possibly empty) collection of disjointly embedded incompressible tori $T_1, \ldots, T_m$ such that each component of N cut along $T_1 \cup \cdots \cup T_m$ is atoroidal or Seifert fibered. Any such collection of tori with a minimal number of components is unique up to isotopy.*

In the case of knot complements and graph manifolds, the JSJ-Decomposition Theorem was foreshadowed by the work of Schubert [**Sct49, Sct53, Sct54**] (see also [**Lic97, Sulb00, BZH14**]) and Waldhausen [**Wan67c**]. The JSJ-Decomposition Theorem was extended to the non-orientable case by Bonahon–Siebenmann [**BoS87**].

DEFINITION. Let $T_1, \ldots, T_m$ be a collection of tori as in Theorem 1.6.1 of minimal cardinality m. The isotopy class of these tori is well-defined, and we refer to $T_1, \ldots, T_m$ as the *JSJ-tori* of N. We refer to the components of N cut along $T_1 \cup \cdots \cup T_m$ as the *JSJ-components of N*.

The goal of this book is to understand fundamental groups of compact 3-manifolds. We thus digress for a moment to explain the group theoretic significance of the JSJ-Decomposition Theorem. Let M be a JSJ-component of N. After picking base points for N and M and a path connecting them, the inclusion $M \subseteq N$ induces a map on the level of fundamental groups. This map is injective since the tori we cut along are incompressible. (We refer to [**LyS77**, Chapter IV.4] for details.) We can thus view $\pi_1(M)$ as a subgroup of $\pi_1(N)$, which is well defined up to the above choices, i.e., well defined up to conjugation. This leads to a description of $\pi_1(N)$ as the fundamental group of a graph of groups with vertex groups the fundamental groups of the JSJ-components and with edge groups the fundamental groups of the JSJ-tori. Graphs of groups are explained in [**Ser77, Ser80, Bas93**].

Our next goal is to give a criterion for showing that a collection of tori in N are in fact the JSJ-tori. In order to state it we recall from Section 1.5 that given a Seifert fibered manifold M with a choice of a Seifert fibered structure and a boundary torus T we refer to the image of the regular fiber in $PH_1(T)$ by $f(M,T)$. Also recall that by Theorem 1.5.2 the only orientable Seifert fibered 3-manifolds M with non-empty incompressible boundary for which $f(M,T)$ depends on the choice of the Seifert fibered structure are $T^2 \times I$ and $K^2 \widetilde{\times} I$.

PROPOSITION 1.6.2. *Let N be a compact, orientable, irreducible 3-manifold with empty or toroidal boundary. Let $T_1, \ldots, T_m$ be disjointly embedded incompressible tori in the 3-manifold N. The T_i are the JSJ-tori of N if and only if*

(1) *the components $M_1, \ldots, M_n$ of N cut along $T_1 \cup \cdots \cup T_m$ are Seifert fibered or atoroidal;*
(2) *for any choice of Seifert fibered structures for all the Seifert fibered components among $M_1, \ldots, M_n$ the following two conditions hold:*
    (a) *if a torus T_i cobounds two Seifert fibered components M_r and M_s with $r \neq s$, then $f(M_r, T_i) \neq f(M_s, T_i) \in PH_1(T_i)$; and*
    (b) *if a torus T_i abuts two boundary tori T_i' and T_i'' of the same Seifert fibered component M_r, then $f(M_r, T_i') \neq f(M_r, T_i'') \in PH_1(T_i) = PH_1(T_i') = PH_1(T_i'')$;*

    and
(3) *if one of the M_i equals $T^2 \times I$, then $m = n = 1$, i.e., N is a torus bundle.*

PROOF. The 'only if' direction of the proposition follows easily from the definitions and Lemma 1.5.3. Suppose conversely that $T_1, \dots, T_m$ are tori that satisfy (1), (2) and (3). It follows from [**CyM04**, Theorem 2.9.3] that the T_i are the JSJ tori once we have verified the following: if some T_i cobounds components M_r and M_s (where it is possible that $r = s$), then the result of gluing M_r to M_s along T_i is not Seifert fibered.

So suppose that the result of gluing M_r to M_s along T_i, is in fact Seifert fibered. It follows from [**Ja80**, Theorem VI.34] and our conditions (2) and (3) that this is only possible if the Seifert fibered manifolds M_r, M_s are copies of $K^2 \widetilde{\times} I$, i.e., $N = D(K^2 \widetilde{\times} I, A)$ for some $A \in \mathrm{SL}(2, \mathbb{Z})$. By Lemma 1.5.5 there exist Seifert fibered structures on M_r and M_s defining the same element in $PH_1(T_i)$. This contradicts (2). □

We conclude this section with the following theorem, which is an immediate consequence of Proposition 1.6.2 and Lemma 1.5.3. This theorem often allows us to reduce proofs to the closed case:

THEOREM 1.6.3. *Suppose the compact, orientable, irreducible* 3*-manifold N has non-empty* (*toroidal*) *boundary, with boundary tori $S_1, \dots, S_m$ and JSJ-tori $T_1, \dots, T_n$. Let $M = N \cup_{\partial N} N$ be the double of N along its boundary. Then the two copies of T_i together with the S_j which bound non-Seifert fibered components are the JSJ-tori for M.*

More generally, if we glue two compact, orientable, irreducible 3-manifolds N_1, N_2 with non-trivial toroidal boundary along boundary components T_1 and T_2, then Proposition 1.6.2 makes it possible to determine the JSJ-decomposition of $N_1 \cup_{T_1 = T_2} N_2$ in terms of the JSJ-decompositions of N_1 and N_2. This conclusion is not true if the boundary torus is compressible. For example, gluing a solid torus $S^1 \times D^2$ to a knot exterior $N = S^3 \setminus \nu K$ can dramatically change the topology and geometry of N. Detailed surveys for what can happen in this context are [**Gon98, Boy02**].

1.7. The Geometrization Theorem

Now we turn to the study of atoroidal 3-manifolds. We say that a closed 3-manifold is *spherical* if it admits a complete metric of constant curvature +1. The universal cover of a spherical 3-manifold is isometric to the 3-sphere with the usual metric (see [**Lan95**, Theorem IX.3.13]). It follows that spherical 3-manifolds are closed and that their fundamental groups are finite; in particular spherical 3-manifolds are atoroidal. Moreover, a 3-manifold N is spherical if and only if it is homeomorphic to the quotient of S^3 by a finite group, which acts freely and isometrically. In particular, we can then view $\pi_1(N)$ as a finite subgroup of SO(4) which acts freely on S^3. By Hopf [**Hop26**, §2] such a group is isomorphic to precisely one of the following types of groups:

(1) the trivial group,
(2) $Q_{4n} := \langle x, y \,|\, x^2 = (xy)^2 = y^n \rangle$ where $n \geq 2$,
(3) $P_{48} := \langle x, y \,|\, x^2 = (xy)^3 = y^4, x^4 = 1 \rangle$,
(4) $P_{120} := \langle x, y \,|\, x^2 = (xy)^3 = y^5, x^4 = 1 \rangle$,
(5) the dihedral group $D_{2^m(2n+1)} := \langle x, y \,|\, x^{2^m} = 1, y^{2n+1} = 1, xyx^{-1} = y^{-1} \rangle$, where $m \geq 2$ and $n \geq 1$,
(6) the group
$$P'_{8 \cdot 3^m} := \langle x, y, z \,|\, x^2 = (xy)^2 = y^2, zxz^{-1} = y, zyz^{-1} = xy, z^{3^m} = 1 \rangle,$$
where $m \geq 1$,
(7) the direct product of any of the above groups with a cyclic group of relatively prime order.

Here Q_{4n} is an extension of the dihedral group D_{2n} by $\mathbb{Z}_2$, P_{48} is an extension of the octahedral group by $\mathbb{Z}_2$, and P_{120} is an extension of the icosahedral group by $\mathbb{Z}_2$. (See also [**SeT30, SeT33**],[**Mil57**, Theorem 2], [**DuV64**], [**Or72**, Chapter 6.1, Theorem 1], [**CoS03**].)

Spherical 3-manifolds are Seifert fibered; see, e.g., [**SeT33**, §7, Hauptsatz], [**Or72**, Chapter 6.1, Theorem 5], [**Sco83a**, §4], or [**Bon02**, Theorem 2.8]. By [**EvM72**, Theorem 3.1] the fundamental group of a spherical 3-manifold N is solvable unless $\pi_1(N)$ is isomorphic to the binary dodecahedra1 group P_{120} or the direct sum of P_{120} with a cyclic group of order relatively prime to 120. Evidently a solvable group with trivial abelianization is already trivial. It follows that the Poincaré homology sphere, i.e., the unique spherical 3-manifold with fundamental group P_{120}, is the spherical 3-manifold that is a homology sphere. (We refer to [**Vo13b**] for a detailed discussion of the Poincaré homology sphere.) We also note that by [**GMW12**, Lemma 7.1] the fundamental group of any spherical 3-manifold contains a normal cyclic subgroup whose index divides 240. In particular any spherical 3-manifold admits a finite regular cover whose index divides 240 which is a lens space. Finally we refer to [**Mil57, Lee73, Tho78, Tho79, Dava83, Tho86, Rub01**] and also to [**Hab14**, Section 4] for some pre-Geometrization results on the classification of finite 3-manifold groups.

A spherical 3-manifold admits a complete metric of constant curvature +1. We say that a compact 3-manifold is *hyperbolic* if its interior admits a complete metric of constant curvature −1. The universal cover of a hyperbolic 3-manifold is isometric to the hyperbolic space $\mathbb{H}^3$, see [**Lan95**, Theorem IX.3.12]. The following theorem is due to Mostow [**Mos68**, Theorem 12.1] in the closed case and due to Prasad [**Pra73**, Theorem B] and Marden [**Man74**] independently in the case of non-empty boundary. (See [**Thu79**, Section 6], [**Muk80**], [**Grv81b**], [**BeP92**, Chapter C], [**Rat06**, Chapter 11], and [**BBI13**, Corollary 1] for alternative proofs and [**Frg04**] for an extension to hyperbolic 3-manifolds with geodesic boundary.)

THEOREM 1.7.1 (Rigidity Theorem). *Let M, N be finite-volume hyperbolic 3-manifolds. Every isomorphism $\pi_1(M) \to \pi_1(N)$ is induced by a unique isometry $M \to N$.*

REMARKS.

(1) This theorem implies in particular that the geometry of finite-volume hyperbolic 3-manifolds is determined by their topology. This is not the case if we drop the finite-volume condition. More precisely, the Ending Lamination Theorem states that hyperbolic 3-manifolds with finitely generated fundamental groups are determined by their topology and by their 'ending laminations.' This theorem was conjectured by Thurston [**Thu82a**] and proved by Brock–Canary–Minsky [**BCM12, Miy10**], building on earlier work of Masur–Minsky [**MMy99, MMy00**]. We also refer to [**Miy94, Miy03, Miy06, Ji12**] for more background information and to [**Bow05, Bow06, Bow11**], [**Ree04**], and [**Som10**] for alternative approaches.

(2) Applying the Rigidity Theorem to a 3-manifold equipped with two different finite volume hyperbolic structures implies that the two hyperbolic structures are the same up to an isometry which is *homotopic* to the identity. This does not imply that the set of hyperbolic metrics on a finite volume 3-manifold is path connected. Path connectedness was later shown by Gabai–Meyerhoff–N. Thurston [**GMT03**, Theorem 0.1] building on earlier work of Gabai [**Gab94a, Gab97**].

(3) Gabai [**Gab01**, Theorem 1.1] showed that if N is a closed hyperbolic 3-manifold, then the inclusion of the isometry group $\operatorname{Isom}(N)$ into the diffeomorphism group $\operatorname{Diff}(N)$ is a homotopy equivalence.

A hyperbolic 3-manifold has finite volume if and only if it is either closed or has toroidal boundary and it is not homeomorphic to $T^2 \times I$. (See [**Thu79**, Theorem 5.11.1] or [**Bon02**, Theorem 2.9], [**BeP92**, Proposition D.3.18], and [**Thu82a**, p. 359].) In this book we are mainly interested in orientable 3-manifolds with empty or toroidal boundary, so we henceforth restrict ourselves to hyperbolic 3-manifolds with finite volume and work with the following understanding.

Convention. Unless we say explicitly otherwise, in the remainder of the book, a hyperbolic 3-manifold is always understood to be *orientable* and to have *finite volume.*

With this proviso, hyperbolic 3-manifolds are atoroidal; in fact, the following slightly stronger statement holds. (See [**Man74**, Proposition 6.4], [**Thu79**, Proposition 5.4.4], and also [**Sco83a**, Corollary 4.6].)

Theorem 1.7.2. *Let N be a hyperbolic 3-manifold, and let Γ be an abelian non-cyclic subgroup of $\pi = \pi_1(N)$. Then there exists a boundary torus S of N and $h \in \pi$ such that $\Gamma \subseteq h\,\pi_1(S)\,h^{-1}$.*

The Elliptization and Hyperbolization Theorems (Theorems 1.7.3 and 1.7.5 below) together imply that every atoroidal 3-manifold which is not homeomorphic to $S^1 \times D^2$, $T^2 \times I$, or $K^2 \widetilde{\times} I$, is either spherical or hyperbolic. Both theorems had been conjectured by Thurston [**Thu82a, Thu82b**]. The latter was also foreshadowed by Riley [**Ril75a, Ril75b, Ril82, Ril13**]. The Hyperbolization Theorem was proved by Thurston for Haken manifolds. (See [**Wala83, Thu86c, Mor84, Sul81, McM96, Ot96, Ot01**] and [**BB11**, Theorem 1.3] for the fibered case and [**Thu86b, Thu86d, Mor84, McM92, Ot98, Kap01**] for the non-fibered case.) A full proof of both theorems was first given by Perelman in his seminal papers [**Per02, Per03a, Per03b**], building on earlier work of R. Hamilton [**Hamc82, Hamc95, Hamc99**]. We refer to [**MTi07, MTi14**] for full details and to [**CZ06a, CZ06b, KlL08, BBBMP10**] for further information on the proof. Expository accounts can be found in [**Mil03, Anb04, Ben06, Bei07, Lot07, Ecr08, McM11**].

Theorem 1.7.3 (Elliptization Theorem). *Every closed 3-manifold with finite fundamental group is spherical.*

As we mentioned before, by [**Lan95**, Theorem IX.3.13] any simply connected spherical 3-manifold is isometric to S^3. Hence the Elliptization Theorem implies the Poincaré Conjecture, formulated in 1904 [**Poi04, Poi96, Poi10**]:

Corollary 1.7.4 (Poincaré Conjecture). *Each closed, simply connected 3-manifold is homeomorphic to S^3.*

The Poincaré Conjecture had a long and eventful history, and over the 20th century it survived many attempts of proving it, see, e.g., [**Whd34, Whd35a, Kos58, Sta66, Vo96, Sz08, Vo14**] for details.

Now we turn to atoroidal 3-manifolds with infinite fundamental groups.

Theorem 1.7.5 (Hyperbolization Theorem). *Let N be a compact, orientable, irreducible 3-manifold with empty or toroidal boundary. Suppose that N is atoroidal and* *not homeomorphic to* $S^1 \times D^2$*,* $T^2 \times I$*, or* $K^2 \widetilde{\times} I$*. If* $\pi_1(N)$ *is infinite, then* N *is hyperbolic.*

REMARK. The first examples of finite volume hyperbolic 3-manifolds were given by Gieseking [**Gie12, Ada87**], Löbell [**Lö31, Ves87**], Seifert–Weber [**SeW33**] and Best [**Bet71**]. But until the work of Thurston there were only few 3-manifolds which were known to be hyperbolic. The picture has changed quite dramatically. Indeed, several results in Section 7.4 show that a 'generic' 3-manifold is hyperbolic.

Combining the JSJ Decomposition Theorem with the Elliptization Theorem and the Hyperbolization Theorem (and the aforementioned facts that spherical 3-manifolds as well as $S^1 \times D^2$, $T^2 \times I$, and $K^2 \widetilde{\times} I$ are Seifert fibered, and that hyperbolic 3-manifolds are atoroidal) we obtain the following:

THEOREM 1.7.6 (Geometrization Theorem). *Let* N *be a compact, orientable, irreducible* 3*-manifold with empty or toroidal boundary. There exists a* (*possibly empty*) *collection of disjointly embedded incompressible tori* $T_1, \dots, T_m$ *in* N *such that each component of* N *cut along* $T_1 \cup \dots \cup T_m$ *is hyperbolic or Seifert fibered. Furthermore any such collection of tori with a minimal number of components is unique up to isotopy.*

REMARKS.

(1) The Geometrization Conjecture has also been formulated for non-orientable 3-manifolds; we refer to [**Bon02**, Conjecture 4.1] for details. To the best of our knowledge this has not been fully proved yet. Note however that by (E.3), a non-orientable 3-manifold has infinite fundamental group, i.e., it cannot be spherical. It follows from [**DL09**, Theorem H] that a closed atoroidal 3-manifold with infinite fundamental group is hyperbolic.
(2) In 1982 Thurston [**Thu82a**] announced a proof of the Orbifold Geometrization Theorem for irreducible orbifolds with a non-empty singular set. A detailed sketch of the proof appears in [**CHK00**], and full details of the proof appear in [**BP01, BLP01, BLP05**] and [**BMP03**, Theorem 3.27].
(3) Stallings [**Sta66**, Section 3] and Jaco [**Ja69**, Theorem 6.3] gave algebraic reformulations of the Poincaré Conjecture, and Myers [**Mye00**, Theorem 1.3] gave an algebraic reformulation of the Geometrization Conjecture.

We conclude this section on the Geometrization Theorem with a discussion of the characteristic submanifold.

DEFINITION. Let N be a compact, orientable, irreducible 3-manifold with empty or toroidal boundary. The *characteristic submanifold of* N is the union of the following:

(1) all JSJ-components of N that are Seifert fibered;
(2) all boundary tori of N which cobound an atoroidal JSJ-component;
(3) all JSJ-tori of N.

Now we can formulate the following variation on the 'Characteristic Pair Theorem' of Jaco–Shalen [**JS79**, p. 138]. (This theorem shows that our definition of characteristic submanifold is equivalent to the original definition in [**JS79**].)

THEOREM 1.7.7. *Let* N *be a compact, orientable, irreducible* 3*-manifold with empty or toroidal boundary. Let* M *be a Seifert fibered manifold not homeomorphic to* S^3*,* $S^1 \times D^2$*, or* $S^1 \times S^2$*. Then every* π_1*-injective map* $M \to N$ *is homotopic to a map* $g\colon M \to N$ *such that* $g(M)$ *lies in a component of the characteristic submanifold of* N*.*

Proof. In the case where N admits a non-trivial JSJ-decomposition or the boundary of N is non-empty, the theorem is an immediate consequence of the 'Characteristic Pair Theorem' of Jaco–Shalen [**JS79**, p. 138]. Now suppose that the JSJ-decomposition is trivial. If N is Seifert fibered, then there is nothing to prove. By Theorem 1.7.6 it remains to consider the case that N is closed and hyperbolic. By (C.3) we know that $\pi_1(N)$ is torsion-free, thus it can not contain the fundamental group of a spherical 3-manifold. On the other hand it follows easily from the consideration of centralizers in Theorems 2.5.1 and 2.5.2 below that $\pi_1(N)$ can not contain a subgroup that is the fundamental group of a non-spherical Seifert fibered manifold different from $S^1 \times D^2$ and $S^1 \times S^2$. □

1.8. Geometric 3-manifolds

The decomposition in Theorem 1.7.6 may be considered as being somewhat *ad hoc* ('Seifert fibered vs. hyperbolic'). The geometric point of view introduced by Thurston gives rise to an elegant reformulation of Theorem 1.7.6. He introduced the notion of a *geometry of a 3-manifold* and of a *geometric 3-manifold*. We give a quick summary of the definitions and the most relevant results, and refer to the expository papers by Scott [**Sco83a**] and Bonahon [**Bon02**] as well as to Thurston's book [**Thu97**] for proofs and further references.

A *3-dimensional geometry* is a smooth, simply connected 3-manifold X equipped with a smooth, transitive action of a Lie group G by diffeomorphisms on X, with compact point stabilizers. The Lie group G is called the *group of isometries* of X. A *geometric structure* on a 3-manifold N (modeled on X) is a diffeomorphism from the interior of N to X/π, where π is a discrete subgroup of G acting freely on X. The geometry X is said to *model* N, and N is said to *admit an X-structure*, or just to be an *X-manifold*. There is also one technical condition, which rules out redundant examples of geometries: the group of isometries is required to be maximal among Lie groups acting transitively on X with compact point stabilizers.

Thurston showed that, up to a certain equivalence, there exist precisely eight 3-dimensional geometries that model compact 3-manifolds. These are: the *3-sphere* with the standard spherical metric; *Euclidean 3-space*; *hyperbolic 3-space*; $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$; the *universal cover* $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ *of* $\mathrm{SL}(2,\mathbb{R})$; and two further geometries called Nil and Sol. We refer to [**Thu97, Sco83a, KLs14**] for details. The spherical and the hyperbolic manifolds in this sense are precisely the type of manifolds we introduced in the previous section. A 3-manifold is called *geometric* if it is an X-manifold for some geometry X.

By [**Bon02**, Proposition 2.1] a geometric 3-manifold that is either spherical, Nil or $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ is in fact orientable. No other geometries have non-orientable examples.

The following theorem summarizes the relationship between Seifert fibered manifolds and geometric 3-manifolds.

Theorem 1.8.1. *Let N be a compact, orientable 3-manifold with empty or toroidal boundary, which is not homeomorphic to $S^1 \times D^2$, $T^2 \times I$, or $K^2 \widetilde{\times} I$. Then N is Seifert fibered if and only if N admits a geometric structure modeled on one of the following geometries: the 3-sphere, Euclidean 3-space, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, or* Nil.

We refer to [**Bon02**, Theorem 4.1] and [**Bon02**, Theorems 2.5, 2.7, 2.8] for the proof and for references; see also [**Sco83a**, Theorem 5.3] and [**FoM10**, Lecture 31].

(In [**Bon02**] the geometries $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ and Nil are referred to as $\mathbb{H}^2\widetilde{\times}\mathbb{E}^1$ and $\mathbb{E}^2\widetilde{\times}\mathbb{E}^1$, respectively.)

For closed orientable Sol-manifolds the combination of [**Sco83a**, Theorem 5.3 (i)] and Lemma 1.5.5 gives the following characterization. As in Section 1.5 we denote by J the 2×2 diagonal matrix with entries -1 and 1.

THEOREM 1.8.2. *A closed orientable* 3*-manifold is a* Sol*-manifold if and only if it is homeomorphic to one of the following two types of manifolds:*

(1) *a mapping torus* $M(T^2, A)$ *where* A *is Anosov,*
(2) *a twisted double* $D(K^2 \widetilde{\times} I, A)$ *where* $JAJA^{-1}$ *is Anosov.*

Before we continue our discussion of geometric 3-manifolds we introduce a definition. Given a property $\mathcal{P}$ of groups we say that a group π is *virtually* $\mathcal{P}$ if π admits a (not necessarily normal) subgroup of finite index that satisfies $\mathcal{P}$. Completely analogously, given a property $\mathcal{P}$ of manifolds, we say that a connected manifold virtually has property $\mathcal{P}$ if it has a finite cover which has the property $\mathcal{P}$.

In Table 1.1 we summarize some of the key properties of geometric 3-manifolds. In the table, a *surface group* is the fundamental group of a closed, orientable surface of genus at least one. Given a geometric 3-manifold N, the first column lists the geometric type, the second describes the fundamental group of N and the third describes the topology of N (or a finite-sheeted cover). The table is very similar in content to the flowcharts given in [**Thu97**, p. 281 and 283].

A geometric 3-manifold is Seifert fibered if its geometry is neither Sol nor hyperbolic, by Theorem 1.8.1. One can think of a Seifert fibered manifold as an S^1-bundle over an orbifold. We denote by χ the orbifold Euler characteristic of the base orbifold and by e the Euler number. We refer to [**Sco83a**, pp. 427, 436] for precise definitions.

Now we give the references for Table 1.1. We refer to [**Sco83a**, p. 478] for the last two columns, and the first three rows to [**Sco83a**, p. 449, 457, 448] and [**HzW35**]. Details regarding Nil-geometry can be found in [**Sco83a**, p. 467], and regarding Sol-geometry in [**Bon02**, Theorem 2.11] and [**Sco83a**, Theorem 5.3]. Finally we refer to [**Sco83a**, pp. 459, 462, 448] for details regarding the last three geometries. The fact that the fundamental group of a hyperbolic 3-manifold does not contain a non-trivial abelian normal subgroup will be shown in Theorem 2.5.5.

By Lemma 1.5.1, if N is a non-spherical Seifert fibered manifold, then the subgroup of $\pi_1(N)$ generated by a regular fiber is infinite cyclic and normal in $\pi_1(N)$. It follows from the above table that the geometry of a geometric manifold can be read off from its fundamental group. In particular, if a 3-manifold admits a geometric structure, then the type of that structure is unique (see also [**Sco83a**, Theorem 5.2] and [**Bon02**, Section 2.5]). Some of these geometries are very rare: there are only finitely many 3-manifolds with Euclidean or $S^2 \times \mathbb{R}$ geometry [**Sco83a**, p. 459]. Finally, the geometry of a geometric 3-manifold with non-empty boundary is either $\mathbb{H}^2 \times \mathbb{R}$ or hyperbolic.

1.9. The Geometric Decomposition Theorem

We are now in a position to state the 'geometric version' of Theorem 1.7.6 (see [**Mor05**, Conjecture 2.2.1] and [**FoM10**, p. 5])) and to deduce it from Theorems 1.7.6, 1.8.1 and 1.8.2.

THEOREM 1.9.1 (Geometric Decomposition Theorem). *Let* N *be a compact, orientable, irreducible* 3*-manifold with empty or toroidal boundary such that* $N \neq S^1 \times D^2$,

Geometry	Fundamental group	Topology	χ	e
Spherical	π is finite	N is finitely covered by S^3	>0	$\neq 0$
$S^2\times\mathbb{R}$	$\pi=\mathbb{Z}$ or π is the infinite dihedral group	N or a double cover equals $S^1\times S^2$	>0	$=0$
Euclidean	π is virtually $\mathbb{Z}^3$	N is finitely covered by $S^1\times S^1\times S^1$	0	0
Nil	π is virtually nilpotent but not virtually $\mathbb{Z}^3$	N finitely covered by a torus bundle with nilpotent monodromy	0	$\neq 0$
Sol	π is solvable but not virtually nilpotent	N or a double cover is a torus bundle with Anosov monodromy		
$\mathbb{H}^2\times\mathbb{R}$	π is virtually a product $\mathbb{Z}\times F$ with F a non-cyclic free group or a surface group	N is finitely covered by $S^1\times\Sigma$ where Σ is a compact surface with $\chi(\Sigma)<0$	<0	0
$\widetilde{\mathrm{SL}(2,\mathbb{R})}$	π is a non-split extension of a surface group by $\mathbb{Z}$	N is finitely covered by a non-trivial S^1-bundle over a closed surface Σ with $\chi(\Sigma)<0$	<0	$\neq 0$
hyperbolic	π infinite and π does not contain a non-trivial abelian normal subgroup	N is atoroidal		

TABLE 1.1. Geometries of 3-manifolds.

$N\neq T^2\times I$ and $N\neq K^2\widetilde{\times} I$. There is a (possibly empty) collection of disjointly embedded incompressible surfaces $S_1,\dots,S_m$ which are either tori or Klein bottles, such that each component of N cut along $S_1\cup\dots\cup S_m$ is geometric. Any such collection with a minimal number of components is unique up to isotopy.

PROOF. If N is already geometric, then there is nothing to prove. For the remainder of the proof we therefore assume that N is not geometric. In light of Theorem 1.10.1 we can thus in particular assume that N is not a torus bundle. It follows from Lemma 1.5.5 and Theorems 1.8.1 and 1.8.2 that N is not a twisted double of $K^2\widetilde{\times} I$.

By Theorem 1.7.6 there exists a minimal collection $\mathcal{T}$ of disjointly embedded incompressible tori such that each component of N cut along $\mathcal{T}$ is either hyperbolic or Seifert fibered. We denote the components of N cut along $\mathcal{T}$ by $M_1,\dots,M_n$. It follows from our assumption that $N\neq S^1\times D^2$ and from the fact that the tori are incompressible that none of the M_i's equals $S^1\times D^2$. Furthermore, our assumption that N is not a torus bundle together with the minimality of a JSJ decomposition implies that none of the M_i's equals $T^2\times I$.

We replace all components of $\mathcal{T}$ that bound a copy of $K^2\widetilde{\times} I$ in N by the Klein bottle which is the core of the twisted I-bundle. We denote by $\mathcal{S}$ the resulting collection of Klein bottles and tori. By the above none of the components of N cut along $\mathcal{S}$ equals one of $S^1 \times D^2$, $T^2 \times I$, $K^2 \widetilde{\times} I$. It is straightforward to verify (using Theorem 1.8.1) that cutting N along $\mathcal{S}$ decomposes N into geometric pieces.

Now suppose that we are given another collection $\mathcal{S}'$ of tori and Klein bottles such that each component of N cut along $\mathcal{S}'$ is geometric. We replace each Klein bottle S in $\mathcal{S}'$ by the torus $\partial \nu S$ and we denote by $\mathcal{T}'$ the resulting collection of tori. It follows from Theorem 1.8.1 that the components of N cut along $\mathcal{T}'$ are either Seifert fibered or hyperbolic. By the minimality of a JSJ-decomposition we have $\#\mathcal{S}' = \#\mathcal{T}' \geq \#\mathcal{T} = \#\mathcal{S}$. The uniqueness part of Theorem 1.6.1 says that if $\#\mathcal{T}' = \#\mathcal{T}$, then $\mathcal{T}'$ and $\mathcal{T}$ are isotopic. It follows easily that $\mathcal{S}'$ and $\mathcal{S}$ are also isotopic. □

In the following we refer to the union of tori and Klein bottles from Theorem 1.9.1 as the *geometric decomposition surface of* N. The proof of Theorem 1.9.1 also shows how to obtain the geometric decomposition surface from the JSJ-tori given by Theorem 1.7.6 and vice versa. More precisely, the combination of Lemma 1.5.5 and Theorem 1.8.2 together with the proof of Theorem 1.9.1 gives us the following proposition.

PROPOSITION 1.9.2. *Let N be a compact, orientable, irreducible* 3*-manifold with empty or toroidal boundary with $N \neq S^1 \times D^2$, $N \neq T^2 \times I$ and $N \neq K^2 \widetilde{\times} I$.*

(1) *If N is a* Sol*-manifold, then N is geometric, i.e., the geometric decomposition surface is empty. On the other hand N has one JSJ-torus, namely, if N is a torus bundle, then the fiber is the JSJ-torus, and if N is a twisted double of $K^2 \widetilde{\times} I$, then the JSJ-torus is given by the boundary of $K^2 \widetilde{\times} I$.*
(2) *Suppose that N is not a* Sol*-manifold, and denote by $T_1, \dots, T_m$ the JSJ-tori of N. We assume that they are ordered such that the tori $T_1, \dots, T_n$ do not bound copies of $K^2 \widetilde{\times} I$ and that for $i = n+1, \dots, m$, each T_i cobounds a copy of $K^2 \widetilde{\times} I$. Then the geometric decomposition surface of N is given by*

$$T_1 \cup \cdots \cup T_n \cup K_{n+1} \cup \cdots \cup K_m.$$

Conversely, if $T_1 \cup \cdots \cup T_n \cup K_{n+1} \cup \cdots \cup K_m$ is the geometric decomposition such that $T_1, \dots, T_n$ are tori and $K_{n+1}, \dots, K_m$ are Klein bottles, then the JSJ-tori are given by

$$T_1 \cup \cdots \cup T_n \cup \partial \nu K_{n+1} \cup \cdots \cup \partial \nu K_m.$$

The following theorem says that the geometric decomposition surface behaves well under passing to finite covers:

THEOREM 1.9.3. *Let N be a compact, orientable, irreducible* 3*-manifold with empty or toroidal boundary such that $N \neq S^1 \times D^2$, $N \neq T^2 \times I$ and $N \neq K^2 \widetilde{\times} I$. Let $p\colon N' \to N$ be a finite cover and let S be the geometric decomposition surface of N. Then N' is irreducible and $p^{-1}(S)$ is the geometric decomposition surface of N'. Furthermore N' is hyperbolic if and only if N is hyperbolic.*

The preimages of the JSJ-tori of N under a finite covering map $p\colon N' \to N$ are in general not the JSJ-tori of N'. For example, if N is a Sol-manifold, then N' is also a Sol-manifold. A Sol-manifold has precisely one JSJ-torus, but the preimage of the JSJ-torus of N under the map p might have many components. The precise behavior of the JSJ-tori under finite covers can easily be obtained from Proposition 1.9.2 together with Theorem 1.9.3.

Proof. The fact that N' is again irreducible follows from the Equivariant Sphere Theorem (see [**MSY82**, p. 647] and see also [**Duw85, Ed86, JR89**]). (The assumption that N is orientable is necessary, see, e.g., [**Row72**, Theorem 5].)

If N is geometric, then N' is clearly also geometric, and there is nothing to prove. We henceforth assume that N is not geometric. It follows from Lemma 1.5.5, and from Theorems 1.8.1 and 1.8.2 that N is not a twisted double of $K^2 \widetilde{\times} I$.

Let $\mathcal{S}$ be the collection of tori and Klein bottles that form the geometric decomposition surface of N and denote by $\mathcal{T}$ the corresponding collection of JSJ tori. Set $\mathcal{S}' := p^{-1}(\mathcal{S})$ and denote by $\mathcal{T}'$ the collection of tori in N' that is given by replacing each Klein bottle K in $\mathcal{S}'$ by $\partial \nu K$. In light of Proposition 1.9.2 it suffices to show that $\mathcal{T}'$ are the JSJ tori of N', or equivalently, that $\mathcal{T}'$ satisfies the conditions of Propositions 1.6.2.

So let T' be a surface in $\mathcal{T}'$. A moments thought shows that the uniqueness of Seifert fibered structures on manifolds with non-empty boundary that are not homeomorphic to $S^1 \times D^2$, $T^2 \times I$ and $K^2 \widetilde{\times} I$ (see Theorem 1.5.2) and the fact that $\mathcal{T}$ satisfies the conditions of Propositions 1.6.2 takes care of all cases, except the following: T' corresponds to a toroidal component of S' such that $S := p(S')$ is a Klein bottle.

We turn our attention to this special case. By definition S cobounds a copy X of $K^2 \widetilde{\times} I$ in N. By the above N is not a twisted double of $K^2 \widetilde{\times} I$. It follows that S also bounds a component M of N cut along $\mathcal{S}$ that is not homeomorphic to $K^2 \widetilde{\times} I$. We denote by M' a component of $p^{-1}(M)$ that bounds S'. Furthermore we pick an identification of $K^2 \widetilde{\times} I$ with the twisted Klein bottle cobounded by S.

We identify $\partial M'$ with the standard 2-torus T' that we described in Section 1.9. Moreover, we denote by $J' \colon T' \to T'$ the deck transformation described in Section 1.9. The component of $p^{-1}(X \cup_{\partial X = \partial M} M)$ that contains S' is given by

$$-M' \cup_{\partial M' = \partial X' = T' \xrightarrow{J'} T' = \partial X' = \partial M'} M'.$$

If M is hyperbolic, then so is M', and T' satisfies the condition of Proposition 1.6.2. Now suppose that M is Seifert fibered. By the above M is not one of $S^1 \times D^2$, $T^2 \times I$, $K^2 \widetilde{\times} I$. By Theorem 1.5.2 the Seifert fibered structure on M is unique up to isotopy.

We write $T = p(T')$ and we denote by $J \colon T \to T$ the deck transformation map. By the minimality of $\mathcal{T}$ the Seifert fibered structure of M does not extend to a Seifert fibered structure of X. By Proposition 1.5.4 this implies that $f(M,T) \not\in \{(1,0),(0,1)\}$. But this also implies that $J'(f(M',T')) \neq \pm f(M',T')$, i.e., the Seifert fibered structures in the above gluing via J' do not match up. □

Here is another way of succinctly stating the behavior of the JSJ-decomposition under finite covers: if $p \colon N' \to N$ is a finite cover as in Theorem 1.9.3 such that N is not a Sol-manifold, then the preimage of the characteristic manifold of N is the characteristic submanifold of N'. This result was generalized in [**MeS86**, p. 290] and [**JR89**] to the study of characteristic submanifolds of compact, orientable, irreducible 3-manifolds with a finite group action.

1.10. The Geometrization Theorem for fibered 3-manifolds

In this section we discuss the Geometrization Theorem for the special case of fibered 3-manifolds, i.e., for 3-manifolds that are the total space of a surface bundle over S^1.

Given a compact surface Σ we denote by $M(\Sigma)$ the mapping class group of Σ, that is, the group of isotopy classes of orientation preserving self-diffeomorphisms of Σ. For

$\varphi \in M(\Sigma)$ we denote by

$$M(\Sigma, \varphi) = \big(\Sigma^2 \times [0,1]\big)/(x,0) \sim \big(x, \varphi(x)\big)$$

the corresponding *mapping torus* with *monodromy* φ.

The geometry of a mapping torus can be studied in terms of the monodromy. We start out with the 'baby case' that Σ is the torus $T^2 = S^1 \times S^1$. Here recall that for the torus T^2 we can identify $M(T^2)$ with $\mathrm{SAut}(H_1(T^2;\mathbb{Z}))$; see, e.g., [**FaM12**, Theorem 2.5] for details. Given $\varphi \in \mathrm{SAut}(H_1(T^2;\mathbb{Z}))$ it follows from an elementary linear algebraic argument that one of the following occurs:

(1) $\varphi^n = \mathrm{id}$ for some $n \in \{1, 2, 4, 6\}$, or
(2) φ has two distinct real eigenvalues, or
(3) φ is non-diagonalizable but has eigenvalue ± 1.

Accordingly we say that φ is *periodic*, or *nilpotent*, or *Anosov*. We then have the following geometrization theorem for torus bundles.

THEOREM 1.10.1. *Let $\varphi \in \mathrm{SAut}(H_1(T^2;\mathbb{Z}))$ and $N = M(T^2, \varphi)$. Then*

(1) *φ is periodic $\Rightarrow$ N Euclidean;*
(2) *φ is Anosov $\Rightarrow$ N is a* Sol*-manifold; and*
(3) *φ is nilpotent $\Rightarrow$ N is a* Nil*-manifold.*

The Nielsen–Thurston Classification Theorem says that if Σ is a compact, orientable surface with negative Euler characteristic, then there exists also trichotomy for elements in $M(\Sigma)$. More precisely, any class $\varphi \in M(\Sigma)$ is either

(1) periodic, i.e., φ is represented by f with $f^n = \mathrm{id}_\Sigma$ for some $n \geq 1$, or
(2) pseudo-Anosov, i.e., there exists $f\colon \Sigma \to \Sigma$ which represents φ and a pair of transverse measured foliations and a $\lambda > 1$ such that f stretches one measured foliation by λ and the other one by λ^{-1}, or
(3) reducible, i.e., there exists $f\colon \Sigma \to \Sigma$ which represents φ and a non-empty embedded 1-manifold Γ in Σ consisting of essential curves with a f-invariant tubular neighborhood $\nu\Gamma$ such that on each f-orbit of $\Sigma \setminus \nu\Gamma$ the restriction of f is either finite order or pseudo-Anosov.

We refer to [**Nie44, BlC88, Thu88**], [**FaM12**, Chapter 13], [**FLP79a, FLP79b**], and [**CSW11**, Theorem 2.15] for details, to [**Iva92**, Theorem 1] for an extension, and to [**Gin81, Milb82, HnTh85**] for the connection between the work of Nielsen and Thurston. If Σ is a closed surface of genus $g \geq 2$ and if $\varphi \in M(\Sigma)$ is periodic then it follows from classical work of Wiman [**Wim95**] from 1895 (see also [**FaM12**, Corollary 7.6] and [**HiK14**]).

The following theorem, due to Thurston, determines the geometric type of a mapping torus in terms of the monodromy. (See [**Thu86c**] and [**Ot96, Ot01**].)

THEOREM 1.10.2. *Let Σ be a compact, orientable surface with negative Euler characteristic, let $\varphi \in M(\Sigma)$, and let $N = M(\Sigma, \varphi)$.*

(1) *If φ is periodic, then N admits an $\mathbb{H}^2 \times \mathbb{R}$ structure.*
(2) *If φ is pseudo-Anosov, then N is hyperbolic.*
(3) *If φ is reducible, then N admits a non-trivial JSJ-decomposition. More precisely, if Γ is a 1-manifold in Σ as in the definition of a reducible element in the mapping class group, then the JSJ-tori of N are given by the φ-mapping tori of the 1-manifold Γ, where φ is also as in the definition of a reducible element in the mapping class group.*

In the third case the geometric decomposition of $M(\Sigma)$ can be obtained by applying the theorem again to the mapping torus of $\Sigma \setminus \nu\Gamma$ and by iterating this process.

1.11. 3-manifolds with (virtually) solvable fundamental group

We finish this chapter by classifying the abelian, nilpotent and solvable groups which appear as fundamental groups of 3-manifolds.

THEOREM 1.11.1. *Let N be an orientable, non-spherical 3-manifold such that no boundary component is a 2-sphere. Then the following are equivalent:*

(1) $\pi_1(N)$ *is solvable;*
(2) $\pi_1(N)$ *is virtually solvable;*
(3) N *is one of the following six types of manifolds:*
  (a) $N = \mathbb{R}P^3 \# \mathbb{R}P^3$;
  (b) $N = S^1 \times D^2$;
  (c) $N = S^1 \times S^2$;
  (d) N *has a finite solvable cover which is a torus bundle;*
  (e) $N = T^2 \times I$;
  (f) $N = K^2 \widetilde{\times} I$.

Before we prove the theorem, we state a useful lemma.

LEMMA 1.11.2. *Let π be a group. If π decomposes non-trivially as an amalgamated free product $\pi \cong A *_C B$, then π has a non-cyclic free subgroup unless $[A : C] \leq 2$ and $[B : C] \leq 2$. Similarly, if π decomposes non-trivially as an HNN-extension $\pi \cong A*_C$, then π contains a non-cyclic free subgroup unless one of the inclusions of C into A is an isomorphism.*

The proof of the lemma is a standard application of Bass–Serre theory [**Ser77, Ser80**]. Now we are ready to prove the theorem.

PROOF OF THEOREM 1.11.1. The implication (1) $\Rightarrow$ (2) is obvious. The fundamental group of $\mathbb{R}P^3 \# \mathbb{R}P^3$ is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$, the infinite dihedral group, so is solvable. It is clear that if N is one of the remaining types (b)–(f) of 3-manifolds, then $\pi_1(N)$ is also solvable. This shows (3) $\Rightarrow$ (1).

Finally, assume that (2) holds. We will show that (3) holds. Let A and B be two non-trivial groups. By Lemma 1.11.2, $A * B$ contains a non-cyclic free group (in particular it is not virtually solvable) unless $A = B = \mathbb{Z}_2$. By the Elliptization Theorem, any 3-manifold M with $\pi_1(M) \cong \mathbb{Z}_2$ is diffeomorphic to $\mathbb{R}P^3$. It follows that if N has solvable fundamental group, then either $N \cong \mathbb{R}P^3 \# \mathbb{R}P^3$ or N is prime.

Since $S^1 \times S^2$ is the only orientable prime 3-manifold which is not irreducible we can henceforth assume that N is irreducible. Now let N be an irreducible 3-manifold such that no boundary component is a 2-sphere and such that $\pi = \pi_1(N)$ is infinite and solvable. The proof of Lemma 1.4.2 in combination with Lemma 1.11.2 and the Poincaré Conjecture show that our assumption that $\pi_1(N)$ is virtually solvable implies that one of the following occurs: either $N = S^1 \times D^2$ or the boundary of N is incompressible. We can thus assume that the boundary of N is incompressible. Since $\pi_1(N)$ is virtually solvable this implies that N is either closed or that its boundary is toroidal. From now on we also assume that $N \neq T^2 \times I$ and that $N \neq K^2 \widetilde{\times} I$. It follows from Theorem 1.9.1 that $\pi_1(N)$ is the fundamental group of a graph of groups where the vertex groups are fundamental groups of geometric 3-manifolds. By Lemma 1.11.2, $\pi_1(N)$ contains a non-cyclic free group unless the 3-manifold is already geometric. By the discussion preceding this theorem, if N is geometric, then N is either a Euclidean manifold, a Sol-manifold or a Nil-manifold, and N is finitely covered by a torus bundle. It follows from the discussion of these geometries in [**Sco83a**] that the finite cover is in fact a finite solvable cover. (Alternatively we could have applied [**EvM72**, Theorems 4.5 and 4.8], [**EvM72**, Corollary 4.10], and [**Tho79**, Section 5] for a proof of the theorem without using the full Geometrization Theorem, only requiring the Elliptization Theorem.) □

Remark. The proof of the above theorem shows that every compact 3-manifold with nilpotent fundamental group is either spherical, Euclidean, or a Nil-manifold. Using the discussion of these geometries in [**Sco83a**] one can then determine the list of nilpotent groups which can appear as fundamental groups of compact 3-manifolds. This list was already determined pre-Geometrization by Thomas [**Tho68**, Theorem N] for the closed case and by Evans–Moser [**EvM72**, Theorem 7.1] in general.

We conclude this section with a discussion of 3-manifolds with abelian fundamental groups. In order to do this we first recall the definition of lens spaces. Given coprime natural numbers p and q we denote by $L(p,q)$ the corresponding *lens space*, defined as

$$L(p,q) := S^3/\mathbb{Z}_p = \{(z_1,z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}/\mathbb{Z}_p,$$

where $k \in \mathbb{Z}_p$ acts on S^3 by $(z_1,z_2) \mapsto (z_1 e^{2\pi ik/p}, z_2 e^{2\pi ikq/p})$. Note that $\pi_1(L(p,q)) \cong \mathbb{Z}_p$ and that the Hopf fibration on S^3 descends to a decomposition of $L(p,q)$ into circles endowing $L(p,q)$ with the structure of a Seifert fibered 3-manifold. We refer to [**Vo13a**] for a summary of the long history of lens spaces in 3-manifold topology.

abelian group π	compact 3-manifolds with fundamental group π
$\mathbb{Z}$	$S^1 \times S^2$, $S^1 \times D^2$
$\mathbb{Z}^3$	$S^1 \times S^1 \times S^1$
$\mathbb{Z}_p$	the lens spaces $L(p,q)$, $q \in \{1,\dots,p-1\}$ with $(p,q)=1$
$\mathbb{Z} \oplus \mathbb{Z}$	$T^2 \times I$
$\mathbb{Z} \oplus \mathbb{Z}_2$	$S^1 \times \mathbb{R}P^2$

Table 1.2. Abelian fundamental groups of 3-manifolds.

In Table 1.2 we give the complete list of all compact, orientable 3-manifolds without spherical boundary components for which the fundamental group is abelian. The table can be obtained in a straightforward fashion from Lemma 1.4.2, the Prime Decomposition Theorem and the Geometrization Theorem. The fact that the groups in the table are indeed the only abelian groups that appear as fundamental groups of compact 3-manifolds is in fact a classical pre-Geometrization result. The list of abelian fundamental groups of closed 3-manifolds was first determined by Reidemeister [**Rer36**, p. 28], and in general by Epstein [**Eps61a**, Theorem 3.3] [**Eps61b**, Theorem 9.1] and Specker [**Sp49**, Satz IX']. (See also [**Hem76**, Theorems 9.12 and 9.13].)

CHAPTER 2

The Classification of 3-Manifolds by their Fundamental Groups

We have already seen that the structure of the fundamental group of a 3-manifold N can determine many topological properties of N. The Poincaré Conjecture is a profound example of this. In this chapter we systematically investigate this phenomenon.

In Sections 2.1 and 2.2 we ask whether isomorphism (of fundamental groups) implies homeomorphism (of the corresponding 3-manifolds). In the closed and irreducible case (Section 2.1), the answer is affirmative, except in the case of *lens spaces*. The picture is somewhat more complicated in the case of non-empty boundary (Section 2.2), where the *peripheral structure* of a 3-manifold needs to be taken into account. In Section 2.3 we briefly consider the relationship between two submanifolds in the interior of a given 3-manifold having conjugate fundamental groups. In Section 2.4 we ask about the kinds of topological properties of a 3-manifold which can be inferred from a knowledge of its fundamental group alone, using only the results in Chapter 1, and not the more recent results discussed in Chapter 4 below. Finally, Section 2.5 contains a description of the centralizers of elements of 3-manifold groups, and some applications.

2.1. Closed 3-manifolds and fundamental groups

It is well known that closed, compact surfaces are determined by their fundamental groups; more generally, compact surfaces with possibly non-empty boundary are determined by their fundamental groups together with the number of boundary components. (See, e.g., [**Brh21, FW99, Gra71, Gra84, SeT34, SeT80, Msy81**].) In 3-manifold theory a similar, but more subtle, picture emerges.

There are several ways for constructing pairs of compact, orientable, non-diffeomorphic 3-manifolds with isomorphic fundamental groups.

(A) Consider lens spaces $L(p_1, q_1)$ and $L(p_2, q_2)$. They are diffeomorphic if and only if $p_1 = p_2$ and $q_1 q_2^{\pm 1} \equiv \pm 1 \bmod p_i$, but they are homotopy equivalent if and only if $p_1 = p_2$ and $q_1 q_2^{\pm 1} \equiv \pm t^2 \bmod p_i$ for some t, and their fundamental groups are isomorphic if and only if $p_1 = p_2$.
(B) Let M, N be compact, oriented 3-manifolds. Denote by $\overline{N}$ the manifold N with opposite orientation. Then $\pi_1(M \# N) \cong \pi_1(M \# \overline{N})$ but if neither M nor N has an orientation reversing diffeomorphism, then $M \# N$ and $M \# \overline{N}$ are not diffeomorphic.
(C) Let M_1, M_2 and N_1, N_2 be compact, oriented 3-manifolds with $\pi_1(M_i) \cong \pi_1(N_i)$ and such that M_1 and N_1 are not diffeomorphic. Then $\pi_1(M_1 \# M_2) \cong \pi_1(N_1 \# N_2)$ but in general $M_1 \# M_2$ is not diffeomorphic to $N_1 \# N_2$.

Reidemeister [**Rer35**, p. 109] and Whitehead [**Whd41a**] classified lens spaces in the PL-category. (See also [**Tur02**, Corollary 10.3].) The classification of lens spaces up to homeomorphism (i.e., the first statement above), then follows from Moise's proof [**Moi52**] of the 'Hauptvermutung' in dimension 3. Alternatively, this follows from Reidemeister's argument together with [**Chp74**]. We refer to [**Mil66**] and [**Hat**, Section 2.1] for more modern accounts and to [**Fo52**, p. 455], [**Bry60**, p. 181], [**Tur76**], [**Bon83**], [**HoR85**, Theorem 5.3], [**PY03**] and [**Gre13**, Section 1.4] for different approaches. The fact that lens spaces with the same fundamental group are not necessarily homeomorphic was first conjectured by Tietze [**Tie08**, p. 117], [**Vo13a**] and was first proved by Alexander [**Ale19, Ale24a**]; see also [**Sti93**, p. 258–260]. The other two statements follow from the uniqueness of the prime decomposition. In the subsequent discussion we will see that (A), (B), and (C) form in fact a complete list of methods for finding examples of pairs of closed, orientable, non-diffeomorphic 3-manifolds with isomorphic fundamental groups.

Recall that Theorem 1.2.1 implies that the fundamental group of a compact, orientable 3-manifold is isomorphic to a free product of fundamental groups of prime 3-manifolds. The Kneser Conjecture implies that the converse holds.

THEOREM 2.1.1 (Kneser Conjecture). *Let N be a compact, oriented 3-manifold with incompressible boundary. If $\pi_1(N) \cong \Gamma_1 * \Gamma_2$, then there exist compact, oriented 3-manifolds N_1 and N_2 with $\pi_1(N_i) \cong \Gamma_i$ for $i = 1, 2$ and $N \cong N_1 \# N_2$.*

This theorem was first stated by Kneser [**Kn29**], with a (rather obscure) proof relying on Dehn's Lemma. The statement was named 'Kneser's Conjecture' by Papakyriakopoulos. The Kneser Conjecture was first proved completely by Stallings [**Sta59a, Sta59b**] in the closed case, and by Heil [**Hei72**, p. 244] in the bounded case. (We refer to [**Eps61c**], [**Hem76**, Section 7], and [**Cal14a**, Theorem 3.9] for details, and to [**Gon99**] for the history.)

The following theorem is a consequence of the Geometrization Theorem, the Rigidity Theorem 1.7.1, work of Waldhausen [**Wan68a**, Corollary 6.5] and Scott [**Sco83b**, Theorem 3.1] and classical work on spherical 3-manifolds (see [**Or72**, p. 113]). We refer to [**Boe72**], [**Jon75**, Corollary 8], [**Swp78**, Theorem 5], and [**Rak81**] for earlier work.

THEOREM 2.1.2. *Let N and N' be closed, orientable, prime 3-manifolds and let $\varphi\colon \pi_1(N) \to \pi_1(N')$ be an isomorphism.*

1. *If N and N' are not lens spaces, then N and N' are homeomorphic.*
2. *If N and N' are not spherical, then there exists a homeomorphism which induces φ.*

REMARK. The Borel Conjecture states that every homotopy equivalence $N \to N'$ between closed and aspherical topological manifolds (of the same dimension) is homotopic to a homeomorphism. In dimensions ≥ 5 this conjecture is known to hold for large classes of fundamental groups, e.g., if the fundamental group is word-hyperbolic [**BaL12**, Theorem A]. The high-dimensional results also extend to dimension 4 if the fundamental groups are good in the sense of Freedman [**Fre84**]. The Borel Conjecture holds for all 3-manifolds: if N and N' are orientable, this is immediate from Theorem 2.1.2; the case where N or N' is non-orientable was proved by Heil [**Hei69a**].

Summarizing, Theorems 1.2.1, 2.1.1 and 2.1.2 show that fundamental groups determine closed 3-manifolds up to orientation of the prime factors and up to the indeterminacy arising from lens spaces. More precisely, we have the following theorem.

THEOREM 2.1.3. *Let N and N' be closed, oriented 3-manifolds with isomorphic fundamental groups. Then there exist natural numbers $p_1, \dots, p_m$, $q_1, \dots, q_m$ and $q'_1, \dots, q'_m$ and oriented manifolds $N_1, \dots, N_n$ and $N'_1, \dots, N'_n$ such that*

(1) *we have homeomorphisms*

$$N \cong L(p_1, q_1) \# \cdots \# L(p_m, q_m) \# N_1 \# \cdots \# N_n \text{ and}$$
$$N' \cong L(p_1, q_1') \# \cdots \# L(p_m, q_m') \# N_1' \# \cdots \# N_n';$$

(2) N_i *and* N_i' *are homeomorphic (but possibly with opposite orientations); and*
(3) *for* $i = 1, \ldots, m$ *we have* $q_i' \not\equiv \pm q_i^{\pm 1} \bmod p_i$.

2.2. Peripheral structures and 3-manifolds with boundary

According to Theorem 2.1.2, orientable, prime 3-manifolds with infinite fundamental groups are determined by their fundamental groups, provided they are closed. The same conclusion does not hold if we allow boundary. For example, if $J \subseteq S^3$ is the trefoil knot with an arbitrary orientation, then $S^3 \setminus \nu(J \# J)$ and $S^3 \setminus \nu(J \# {-J})$ (i.e., the exteriors of the granny knot and the square knot) have isomorphic fundamental groups, but the spaces are not homeomorphic. This can be seen by studying the linking form (see [**Sei33b**, p. 826]) or the Blanchfield form [**Bla57**], which in turn can be studied using Levine–Tristram signatures (see [**Kea73, Lev69, Tri69**]).

Before we can discuss to what degree the fundamental group determines the homeomorphism of a 3-manifold with boundary we need to introduce a few more definitions.

Definition. Let N be a 3-manifold.

(1) Suppose N has incompressible boundary. The fundamental group of N together with the set of conjugacy classes of its subgroups determined by the boundary components is called the *peripheral structure of* N.
(2) We say that a properly embedded surface S in N is *boundary parallel* if there exists an isotopy, through properly embedded surfaces, from S to a subsurface of ∂N.
(3) An *essential annulus* in N is a properly embedded annulus which is incompressible and not boundary parallel.

The following definition is from [**Jon79a**, p. 227].

Definition. Let N and N' be two compact 3-manifolds and let $W \subseteq N$ be a solid torus such that $\overline{\partial W \setminus \partial M}$ consists of essential annuli in N. We say N' is obtained from N by a *Dehn flip along* W if the following conditions hold:

(1) there exists a solid torus W' in N' such that $\overline{\partial W' \setminus \partial N'}$ consists of essential annuli in N',
(2) there is a homeomorphism $h\colon \overline{W \setminus N} \to \overline{W' \setminus N'}$ with
$$h(\overline{\partial W \setminus \partial N}) = \overline{\partial W' \setminus \partial N'},$$
and
(3) there is a homotopy equivalence $f\colon W \to W'$ with
$$f(\overline{\partial W \setminus \partial N}) = \overline{\partial W' \setminus \partial N'}.$$

Now we have the following theorem.

Theorem 2.2.1. *Let N and N' be compact, orientable, irreducible 3-manifolds with non-spherical, non-trivial incompressible boundary.*

(1) *If* $\pi_1(N) \cong \pi_1(N')$ *are isomorphic, then N can be turned into N' using finitely many Dehn flips.*

(2) *Up to homeomorphism there exist only finitely many compact, orientable, irreducible* 3*-manifolds with non-spherical, non-empty incompressible boundary such that the fundamental group is isomorphic to* $\pi_1(N)$.
(3) *If there exists an isomorphism* $\pi_1(N) \to \pi_1(N')$ *which sends the peripheral structure of* N *isomorphically to the peripheral structure of* N'*, then* N *and* N' *are homeomorphic.*

The first two statements of the theorem were proved by Johannson [**Jon79a**, Theorem 29.1 and Corollary 29.3]. (See also [**Swp80a**] for a proof of the second statement.) The third statement was proved by Waldhausen. We refer to [**Wan68a**, Corollary 7.5] and [**JS76**] for details. If the manifolds N and N' have no Seifert fibered JSJ-components, then any isomorphism of fundamental groups is in fact induced by a homeomorphism (this follows, e.g., from [**Jon79c**, Theorem 1.3]).

We conclude this section with a short discussion of knots. A knot is a connected 1-submanifold of S^3. A knot is called *prime* if it is not the connected sum of two non-trivial knots. Somewhat surprisingly, in light of the above discussion, prime knots are in fact determined by their fundamental groups. More precisely, if J_1 and J_2 are two prime knots with $\pi_1(S^3 \setminus \nu J_1) \cong \pi_1(S^3 \setminus \nu J_2)$, then there exists a homeomorphism $f\colon S^3 \to S^3$ with $f(J_1) = J_2$. This was first proved by Gordon–Luecke [**GLu89**, Corollary 2.1] (see also [**Gra92**]) extending earlier work of Culler–Gordon–Luecke–Shalen [**CGLS85, CGLS87**] and Whitten [**Whn86, Whn87**]. See [**Tie08, Fo52, Neh61a, Sim76b, FeW78, Sim80, Swp80b, Swp86**] for earlier discussions and work on this result. Non-prime knots are determined by their 'quandles', see [**Joy82**] and [**Mae82**], by their '2-generalized knot groups', see [**LiN08, NN08, Tuf09**], and by certain quotients of the knot groups induced by the longitudes, see [**Zim91**, Theorem 1].

2.3. Submanifolds and subgroups

Let M be a connected submanifold of a 3-manifold N. If M has incompressible boundary, then the inclusion-induced map $\pi_1(M) \to \pi_1(N)$ is injective, and $\pi_1(M)$ can be viewed as a subgroup of $\pi_1(N)$, which is well defined up to conjugacy. In the previous two sections we have seen that 3-manifolds are, for the most part, determined by their fundamental groups. The following theorem, due to Jaco–Shalen [**JS79**, Corollary V.2.3], says that submanifolds of 3-manifolds are, under mild assumptions, completely determined by the subgroups they define.

THEOREM 2.3.1. *Let* N *be a compact, irreducible* 3*-manifold and let* M, M' *be two compact connected submanifolds in the interior of* N *whose boundaries are incompressible. Then* $\pi_1(M)$ *and* $\pi_1(M')$ *are conjugate if and only if there exists a homeomorphism* $f\colon N \to N$ *which is isotopic (relative to* ∂N*) to the identity, with* $f(M) = f(M')$.

2.4. Properties of 3-manifolds and their fundamental groups

In the previous section we saw that orientable, closed irreducible 3-manifolds with infinite fundamental groups are determined by their fundamental groups. We also saw that the fundamental group of a given compact, orientable 3-manifold determines the fundamental groups of the prime factors of this manifold. It is interesting to ask which topological properties of 3-manifolds can be 'read off' from the fundamental group.

Given a 3-manifold N we denote by $\mathrm{Diff}(N)$ the group of self-diffeomorphisms of N which restrict to the identity on the boundary. Furthermore we denote by $\mathrm{Diff}_0(N)$ the identity component of $\mathrm{Diff}(N)$. The quotient $\mathrm{Diff}(N)/\mathrm{Diff}_0(N)$ is denoted by $\mathcal{M}(N)$. Also, given a group π, we denote by $\mathrm{Out}(\pi)$ the group of outer automorphisms of π (i.e., the quotient of the group of automorphisms of π by its normal subgroup of inner automorphisms of π). If $N \neq S^1 \times D^2$ is a compact, irreducible 3-manifold which is not an I-bundle over a surface, then the natural morphism

$$\Phi\colon \mathcal{M}(N) \to \big\{\varphi \in \mathrm{Out}(\pi) : \varphi \text{ preserves the peripheral structure}\big\}$$

is injective. This follows from the Rigidity Theorem 1.7.1, work of Waldhausen [**Wan68a**, Corollary 7.5], Scott [**Sco85b**, Theorem 1.1] (see also [**Som06b**, Theorem 0.2]), Boileau–Otal [**BO86**], [**BO91**, Théorème 3], and the Geometrization Theorem.

(1) A Seifert fibered manifold is called *small* if it is not Haken. Small Seifert fibered manifolds have been classified, see [**Brn93**, Section 2.3.3] and [**Ja80**, Theorem VI.15]. Inspection of trivial cases and [**McC91**, p. 21] show that if N is a small Seifert manifold, then $\mathrm{Out}(\pi_1(N))$ is finite. On the other hand, if N is Seifert fibered but not small, then the map Φ above is in fact an isomorphism.

(2) Let N be a hyperbolic 3-manifold. As a consequence of the Rigidity Theorem, $\mathrm{Out}(\pi_1(N))$ is finite and naturally isomorphic to the isometry group of N. (See [**BeP92**, Theorem C.5.6] and also [**Jon79b**], [**Jon79a**, p. 213] for details.)

(3) Kojima [**Koj88**] (see also [**BeL05**, Theorem 1.1]) showed that every finite group appears as the full isometry group of a closed hyperbolic 3-manifold. Also, Cooper and Long [**CoL00**] showed that for each finite group G there is a rational homology sphere with a free G-action.

    On the other hand Milnor [**Mil57**, Corollary 1] gave restrictions on finite groups which can act freely on an integral homology sphere. See also [**MeZ04, MeZ06, Reni01, ReZ02, Zim02a, Zim02b, Zim04, GMZ11, GZ13**] for further information and extensions. Also, by [**Kaw90**, Theorem 10.5.3], if $N = S^3 \setminus \nu K$ is the exterior of a knot $K \subseteq S^3$, then $\mathcal{M}(N)$ is either cyclic or a finite dihedral group.

(4) For the fundamental group π of any compact 3-manifold, $\mathrm{Out}(\pi)$ is residually finite, as shown by Antolín–Minasyan–Sisto [**AMS13**, Corollary 1.6], extending [**AKT06, AKT09**].

(5) If N is a closed irreducible 3-manifold which is not Seifert fibered, then any finite subgroup of $\mathrm{Out}(\pi_1(N))$ can be represented by a finite group of diffeomorphisms of N. This follows from the above discussion of the hyperbolic case, from Zimmermann [**Zim82**, Satz 0.1], and from the Geometrization Theorem. (See also [**HeT87**].) The case of Seifert fibered 3-manifolds is somewhat more complicated and is treated by Zieschang and Zimmermann [**ZZ82, Zim79**] and Raymond [**Ray80**, p. 90]; see also [**RaS77, HeT78, HeT83**].

(6) Suppose N is hyperbolic or Haken (see (A.10) for the definition). It follows from work of Gabai [**Gab01**, Theorem 1.2] and Hatcher [**Hat76, Hat83, Hat99**] and Ivanov [**Iva76, Iva80**] (extending [**Lau74**]) that $\mathrm{Diff}(N)$ is weakly homotopy equivalent to the space of homotopy equivalences of N fixing the boundary.

(7) If N is a Seifert fibered manifold, then N admits a fixed-point free S^1-action, and $\mathrm{Diff}(N)$ thus contains torsion elements of arbitrarily large order. On the other hand Kojima [**Koj84**, Theorem 4.1] showed that if N is a closed irreducible 3-manifold which is not Seifert fibered, then there is a bound on the order of finite subgroups of $\mathrm{Diff}(N)$.

For 3-manifolds which are spherical or not prime the map Φ is in general neither injective nor surjective. See [**Gab94b, McC90, McC95**] for more information.

Now we give a few more situations in which topological information can be 'directly' obtained from the fundamental group.

(1) Let N be a compact 3-manifold and $\phi \in H^1(N;\mathbb{Z}) = \operatorname{Hom}(\pi_1(N),\mathbb{Z})$ be non-trivial. Work of Stallings (see [**Sta62**, Theorem 2] and (L.9)), together with the resolution of the Poincaré Conjecture, shows that $\phi\colon \pi_1(N) \to \mathbb{Z}$ is a fibered morphism (i.e., can be realized by a surface bundle $N \to S^1$) if and only if its kernel is finitely generated. (If $\pi_1(N)$ is a one-relator group the latter condition can be verified easily using Brown's criterion, see [**Brob87**, §4] and [**BR88, Mas06a, Dun01, FTi15, FST15**].) We refer to [**Neh63b**, p. 381] for an alternative proof for knots, to [**Siv87**, p. 86] and [**Bie07**, p. 953] for a homological reformulation of Stallings' criterion, and to [**Zu97**, Theorem 5.2] for a group-theoretic way to detect semibundles. Finally we refer to (M.2) for various strenghtenings of Stallings' result.

(2) Let N be a closed, orientable, irreducible 3-manifold such that $\pi_1(N)$ is an amalgamated product $A_1 *_B A_2$ where B is the fundamental group of a closed surface $S \neq S^2$. Feustel [**Feu72a**, Theorem 1] and Scott [**Sco72**, Theorem 2.3] showed that this splitting of $\pi_1(N)$ can be realized geometrically, i.e., there exists an embedding of S into N such that $N \setminus S$ consists of two components N_1, N_2 such that the triples $(B, B \to A_1, B \to A_2)$ and $(\pi_1(S), \pi_1(S) \to \pi_1(N_1), \pi_1(S) \to \pi_1(N_2))$ are isomorphic in the obvious sense. See [**Feu73**] and Scott [**Sco74**, Theorem 3.6] for more details. Moreover, we refer to Feustel–Gregorac [**FeG73**, Theorem 1] and [**Sco80**, Corollary 1.2 (a)] (see also [**TY99**]) for a similar result corresponding to HNN-extensions where the splitting is given by closed surfaces or annuli.

    More generally, if $\pi_1(N)$ admits a non-trivial decomposition as a graph of groups (e.g., as an amalgamated product or an HNN-extension), then this decomposition gives rise to a decomposition along incompressible surfaces of N with the same underlying graph. (Some care is needed here: in the general case the edge and vertex groups of the new decomposition may be different from those of the original one.) We refer to Culler–Shalen [**CuS83**, Proposition 2.3.1] for details and for [**Hat82, HO89, Rat90, LiR91, SZ01, ChT07, HoSh07, Gar11, DG12**] for extensions of this result.

(3) If N is a geometric 3-manifold, then the geometry of N is determined by the properties of $\pi_1(N)$, by the discussion of Section 1.9.

(4) The Thurston norm $H^1(N;\mathbb{R}) \to \mathbb{R}^{\geq 0}$ measures the minimal complexity of surfaces dual to cohomology classes. We refer to [**Thu86a**] and Section 5.4.3 for a precise definition and for details.
   (a) It follows from [**FeG73**, Theorem 1] that if N is a closed 3-manifold with $b_1(N) = 1$, then the Thurston norm can be recovered in terms of splittings of fundamental groups along surface groups. By work of Gabai [**Gab87**, Corollary 8.3] this also gives a group-theoretic way to recover the genus of a knot in S^3.
   (b) It is shown in [**FSW13**] that the genus of a knot J in S^3 is determined by the possible HNN-splittings of $\pi_1(S^3 \setminus \nu J)$.

(c) Let N be a 3-manifold such that the boundary consists of a single torus and with $H_2(N;\mathbb{Z}) = 0$, i.e., N is the exterior of a knot K in a rational homology sphere. Calegari [**Cal09a**, Proof of Proposition 4.4] gave a group-theoretic interpretation of the Thurston seminorm of N in terms of the 'stable commutator length' of a longitude of K.

However, there does not seem to be a good group-theoretical equivalent to the Thurston norm for general 3-manifolds. Nevertheless, the Thurston norm and the hyperbolic volume can be recovered from the fundamental group alone using the Gromov norm; see [**Grv82**], [**Gab83a**, Corollary 6.18], [**Gab83b**, p. 79], and [**Thu79**, Theorem 6.2], for background and details. For most 3-manifolds, the Thurston norm can be obtained from the fundamental group using twisted Alexander polynomials; see [**FKm06, FV12, DFJ12**].

(5) Scott and Swarup algebraically characterized the JSJ-decomposition of a compact, orientable 3-manifold with incompressible boundary; see [**SSw01**, Theorem 2.1] and also [**SSw03**].

In many cases, however, it is difficult to obtain topological information about N by just applying group-theoretical methods to $\pi_1(N)$. For example, it is obvious that given a closed 3-manifold N, the minimal number $r(N)$ of generators of $\pi_1(N)$ is a lower bound on the Heegaard genus $g(N)$ of N. It has been a long standing question of Waldhausen to determine for which 3-manifolds the equality $r(N) = g(N)$ holds. (See [**Hak70**, p. 149] and [**Wan78b**].) The case $r(N) = 0$ is equivalent to the Poincaré conjecture. It has been known for a while that $r(N) \neq g(N)$ for graph manifolds [**BoZ83, BoZ84, Zie88, Mon89, Wei03, ScW07, Won11**], and evidence for the inequality for some hyperbolic 3-manifolds was given in [**AN12**, Theorem 2]. In contrast to this, work of Souto [**Sou08**, Theorem 1.1] and Namazi–Souto [**NaS09**, Theorem 1.4], and also Biringer [**Bir09**, Theorem 1.1] together with work of Bachmann-Schleimer [**BcS05**, Corollary 3.4], yields that $r(N) = g(N)$ for 'sufficiently complicated' hyperbolic 3-manifolds. (See also [**Mas06a**] and [**BiS14**] for more examples.) Recently Li [**Lia13**] showed that there also exist hyperbolic 3-manifolds with $r(N) < g(N)$. See [**Shn07**] for some background.

2.5. Centralizers

Let π be a group. The *centralizer* of a subset X of π is defined to be the subgroup

$$C_\pi(X) := \{g \in \pi : gx = xg \text{ for all } x \in X\}$$

of π. For $x \in G$ we also write $C_\pi(x) := C_\pi(\{x\})$. Determining the centralizers is often one of the key steps in understanding a group. In the world of 3-manifold groups, thanks to the Geometrization Theorem, an almost complete picture emerges. In this section we only consider 3-manifolds to which Theorem 1.7.6 applies, i.e., 3-manifolds that are compact, orientable, and irreducible, with empty or toroidal boundary. But many of the results of this section also generalize fairly easily to fundamental groups of compact 3-manifolds in general, using the arguments of Sections 1.2 and 1.4.

The following theorem reduces the determination of centralizers to the case of Seifert fibered manifolds.

THEOREM 2.5.1. *Let N be a compact, orientable, and irreducible 3-manifold with empty or toroidal boundary. Let $1 \neq g \in \pi := \pi_1(N)$ be such that $C_\pi(g)$ is non-cyclic. One of the following holds:*

(1) *there is a JSJ-torus T of N and $h \in \pi$ with $g \in h\,\pi_1(T)\,h^{-1}$ and*
$$C_\pi(g) = h\,\pi_1(T)\,h^{-1};$$
(2) *there is a boundary component S of N and $h \in \pi$ with $g \in h\,\pi_1(S)\,h^{-1}$ and*
$$C_\pi(g) = h\,\pi_1(S)\,h^{-1};$$
(3) *there is a Seifert fibered component M of N and $h \in \pi$ with $g \in h\pi_1(M)h^{-1}$ and*
$$C_\pi(g) = h\,C_{\pi_1(M)}\,(h^{-1}gh)h^{-1}.$$

Remark. One could formulate the theorem more succinctly: if $g \neq 1$ and $C_\pi(g)$ is non-cyclic, then there exists a component C of the characteristic submanifold of N and $h \in \pi$ such that $g \in h\,\pi_1(C)\,h^{-1}$ and
$$C_\pi(g) = h\,C_{\pi_1(C)}\,(h^{-1}gh)h^{-1}.$$

We will provide a short proof of Theorem 2.5.1 which makes use of the deep results of Jaco–Shalen and Johannson and of the Geometrization Theorem for non-Haken manifolds. Alternatively the theorem can be proved using the Geometrization Theorem much more explicitly—we refer to [**Fri11**] for details.

Proof. We first consider the case that N is hyperbolic. In Section 5 we will see that we can view π as a discrete, torsion-free subgroup of $\mathrm{PSL}(2,\mathbb{C})$. Let $g \in \pi$, $g \neq 1$.

The centralizer of any non-trivial matrix in $\mathrm{PSL}(2,\mathbb{C})$ is abelian; this can be seen easily using the Jordan normal form of such a matrix. Since π is a discrete, torsion-free subgroup of $\mathrm{PSL}(2,\mathbb{C})$ it is straightforward to see that the centralizer of g in $\pi_1(N)$ is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}^2$. If $C_\pi(g)$ is not infinite cyclic, then it is a free abelian group of rank two. It follows from Theorem 1.7.2 that either (1) or (2) holds.

If N is Seifert fibered, then the theorem holds trivially. By Theorem 1.7.6 it remains to consider the case where N admits a non-trivial JSJ-decomposition. In that case N is Haken (see (A.10) for the definition), and the theorem follows from [**JS79**, Theorem VI.1.6]. (See also [**JS78**, Theorem 4.1], [**Jon79a**, Proposition 32.9], and [**Sim76a**, Theorem 1].) □

Now we turn to the study of centralizers in Seifert fibered manifolds. Let M be a Seifert fibered manifold with a given Seifert fiber structure. Then there exists a projection map $p\colon M \to B$ onto the base orbifold B of M. We denote by $B' \to B$ the orientation cover; this is either the identity or a 2-fold cover. We denote by $\pi_1^{\mathrm{orb}}(B')$ the orbifold fundamental group of B' (see [**BMP03**]), and following [**JS79**, p. 23] we refer to $(p_*)^{-1}(\pi_1^{\mathrm{orb}}(B'))$ as the *canonical subgroup* of $\pi_1(M)$. (If B is orientable, then the canonical subgroup of $\pi_1(M)$ is just $\pi_1(M)$ itself.) If f is a regular Seifert fiber of the Seifert fibration, then we refer to the subgroup of $\pi_1(M)$ generated by f as the *Seifert fiber subgroup* of $\pi_1(M)$. Recall that if M is non-spherical, then the Seifert fiber subgroup of $\pi_1(M)$ is infinite cyclic and normal. (Recall that in general, the Seifert fiber subgroup is only unique up to conjugacy.)

Remark. The definition of the canonical subgroup and of the Seifert fiber subgroup depend on the Seifert fiber structure. As we saw in Theorem 1.5.2, 'most' Seifert fibered 3-manifolds admit a unique Seifert fibered structure up to homeomorphism.

The following theorem, together with Theorem 2.5.1, now classifies centralizers of compact, orientable, irreducible 3-manifolds with empty or toroidal boundary.

THEOREM 2.5.2. *Suppose N is a non-spherical Seifert fibered manifold with a given Seifert fiber structure. Let $g \in \pi = \pi_1(N)$, $g \neq 1$. Then*

(1) *if g lies in the Seifert fiber subgroup of π, then $C_\pi(g)$ equals the canonical subgroup of π;*
(2) *if g does not lie in the Seifert fiber subgroup of π, then the intersection of $C_\pi(g)$ with the canonical subgroup of π is abelian—in particular, $C_\pi(g)$ admits an abelian subgroup of index at most two;*
(3) *if g does not lie in the canonical subgroup of π, then $C_\pi(g)$ is infinite cyclic.*

PROOF. The first statement is [**JS79**, Proposition II.4.5]. The second and the third statement follow from [**JS79**, Proposition II.4.7]. □

It follows immediately from the theorem that if N is a non-spherical Seifert fibered manifold and $g \in \pi = \pi_1(N)$ does not lie in the Seifert fiber group of a Seifert fiber structure, then $C_\pi(g)$ is isomorphic to one of $\mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}$, or the fundamental group of a Klein bottle. (See also [**JS78**, p. 82].)

Let π be a group and $g \in \pi$. We say $h \in \pi$ is a *root* of g if $h^n = g$ for some n. The roots of g necessarily lie in $C_\pi(g)$. We denote by $\operatorname{roots}_\pi(g)$ the set of all roots of g in π. As in [**JS79**, p. 32] we say that g has *trivial root structure* if $\operatorname{roots}_\pi(g)$ lies in a cyclic subgroup of π. We say that g has *nearly trivial root structure* if $\operatorname{roots}_\pi(g)$ lies in a subgroup of π which has an abelian subgroup of index at most 2.

The theorem below follows immediately from Theorem 2.5.1 and from [**JS79**, Proposition II.4.13].

THEOREM 2.5.3. *Let N be a compact, orientable, irreducible 3-manifold with empty or toroidal boundary. Let $g \in \pi = \pi_1(N)$.*

(1) *If g does not have trivial root structure, then there exists a Seifert fibered JSJ-component M of N and $h \in \pi$ such that g lies in $h\pi_1(M)h^{-1}$ and*
$$\operatorname{roots}_\pi(g) = h \operatorname{roots}_{\pi_1(M)}(h^{-1}gh)\, h^{-1}.$$
(2) *If N is Seifert fibered and g does not have nearly trivial root structure, then hgh^{-1} lies in a Seifert fiber group of N.*
(3) *If N is Seifert fibered and g lies in the Seifert fiber group, then all roots of hgh^{-1} are conjugate to an element represented by a power of a singular Seifert fiber of N.*

REMARKS.

(1) By [**JS79**, Addendum II.4.14] we get the following strengthening of conclusion (2) provided the Seifert fibered manifold N does not contain any embedded Klein bottles: either g has trivial root structure or is conjugate to an element in a Seifert fiber group of N.
(2) Let N be a 3-manifold. Kropholler [**Kr90a**, Proposition 1] showed, without using the Geometrization Theorem, that if g is an element of infinite order of $\pi_1(N)$ and $k, l \in \mathbb{Z}$ are such that g^k is conjugate to g^l, then $k = \pm l$. (See also [**Ja75**] and [**Shn01**].) This fact also follows immediately from Theorem 2.5.3.

We also say that an element g of a group π is *divisible by n* if there exists an $h \in \pi$ with $g = h^n$. Theorem 2.5.3 has the following corollary.

Corollary 2.5.4. *Let N be as in Theorem 2.5.3. No element of $\pi_1(N) \setminus \{1\}$ is divisible by infinitely many n.*

Remark. For Haken 3-manifolds this result had been proved in [**EJ73**, Corollary 3.3], [**Shn75**, p. 327] and [**Ja75**, p. 328]; see also [**Wan69**] and [**Swp73, Feu76a, Feu76b, Feu76c**].

As we saw in Lemma 1.5.1, the fundamental group of a non-spherical Seifert fibered manifold has a normal infinite cyclic subgroup, namely its Seifert fiber subgroup. The following consequence of Theorem 2.5.1 shows that the converse holds.

Theorem 2.5.5. *Let N be a compact, orientable, irreducible 3-manifold with empty or toroidal boundary. If $\pi_1(N)$ has a normal infinite cyclic subgroup, then N is Seifert fibered.*

Proof. Suppose $\pi = \pi_1(N)$ admits a normal infinite cyclic subgroup G. Recall that $\operatorname{Aut} G$ is canonically isomorphic to $\mathbb{Z}_2$. The conjugation action of π on G defines a morphism $\varphi\colon \pi \to \operatorname{Aut} G = \mathbb{Z}_2$. We write $\pi' = \operatorname{Ker}(\varphi)$. Clearly $\pi' = C_\pi(G)$. It follows immediately from Theorem 2.5.1 that either N is Seifert fibered, $\pi' = \mathbb{Z}$, or $\pi' = \mathbb{Z}^2$. But the latter case also implies that N is one of $S^1 \times D^2$, $T^2 \times I$, or $K^2 \widetilde{\times} I$. In particular, N is again Seifert fibered. □

Remarks.

(1) This theorem was proved before the Geometrization Theorem:
   (a) Casson–Jungreis [**CJ94**] and Gabai [**Gab92**], extending earlier work of Tukia [**Tuk88a, Tuk88b**], showed that every word-hyperbolic group with boundary is homeomorphic to S^1 acts properly discontinuously and cocompactly on $\mathbb{H}^2$ with finite kernel. See Section 4.4 below for the definition of 'word-hyperbolic group' and [**BrH99**] for the definition of the boundary of such a group.
   (b) Mess [**Mes88**] showed that this result on word-hyperbolic groups implies Theorem 2.5.5.

   We also refer to [**Neh60, Neh63a, Mur65, BZ66, Wan67a, Wan68a, BdM70, GoH75**] and [**Ja80**, Theorem VI.24] for partial results, [**Bow04**, Corollary 0.5], [**KKl04**, Theorem 1.4], and [**Mac13**, Theorem 1.4] for alternative proofs, and [**Mai01**][**Mai03**, Theorem 1.3], [**Whn92**, Theorem 1], [**HeW94**], and [**Win94**] for extensions to orbifolds, to the non-orientable case and to the non-compact case. We refer to [**Pre14**] and to [**Cal14b**] for a survey on the development of Theorem 2.5.5.

(2) By [**JS79**, Lemma II.4.8] a more precise conclusion holds if the boundary of N is non-empty: every normal infinite cyclic subgroup of $\pi_1(N)$ is the Seifert fiber group for some Seifert fibration of N.

Given an element g of a group π, the set of conjugacy classes of g is in a canonical bijection with the set $\pi/C_\pi(g)$. We thus obtain the following corollary to Theorem 2.5.1.

Theorem 2.5.6. *Let N be a compact, orientable, irreducible 3-manifold with empty or toroidal boundary. If N is not a Seifert fibered manifold, then each element of $\pi_1(N)$ has infinitely many conjugacy classes.*

This result was proved in slightly greater generality and using different methods by de la Harpe–Préaux [**dlHP07**, p. 563]. We refer to [**dlHP07**] for an application of this result to the von Neumann algebra $W^*_\lambda(\pi_1(N))$.

The following was shown by Hempel [**Hem87**, p. 390], generalizing work of Noga [**No67**], without using the Geometrization Theorem.

THEOREM 2.5.7. *Let N be a compact, orientable, irreducible 3-manifold with toroidal boundary, and let S be a JSJ-torus or a boundary component. Then $\pi_1(S)$ is a maximal abelian subgroup of $\pi_1(N)$.*

PROOF. The result is well known to hold for Seifert fibered manifolds. The general case follows immediately from Theorem 2.5.1. □

A subgroup A of a group π is called *malnormal* if $A \cap gAg^{-1} = 1$ for all $g \in \pi \setminus A$.

THEOREM 2.5.8. *Let N be a compact, orientable, irreducible 3-manifold with empty or toroidal boundary.*

(1) *Let S be a boundary component of N. If the JSJ-component of N which contains S is hyperbolic, then $\pi_1(S)$ is a malnormal subgroup of $\pi_1(N)$.*
(2) *Let T be a JSJ-torus of N. If both of the JSJ-components of N abutting T are hyperbolic, then $\pi_1(T)$ is a malnormal subgroup of $\pi_1(N)$.*

The first statement was proved by de la Harpe–Weber [**dlHW11**, Theorem 3] and can be viewed as a strengthening of the previous theorem. See [**Fri11**, Theorem 4.3] for an alternative proof. The second statement can be proved using the same techniques.

The following theorem was first proved by Epstein [**Eps61d, Eps62**]:

THEOREM 2.5.9. *Let N be a compact, orientable, irreducible 3-manifold with empty or toroidal boundary. Suppose $\pi_1(N) \cong A \times B$ where A is infinite and B is non-trivial. Then $N = S^1 \times \Sigma$ where Σ is a surface.*

PROOF. Any element of A has a non-cyclic centralizer in $\pi_1(N)$. It follows easily from Theorem 2.5.1 that N is a Seifert fibered manifold. The case of a Seifert fibered manifold then follows from an elementary argument. □

REMARK. An infinite group π is said to be *presentable by a product* if there is a morphism $\varphi\colon \Gamma_1 \times \Gamma_2 \to \pi$ onto a finite-index subgroup of π such that for $i = 1, 2$ the groups $\varphi(\Gamma_i)$ are infinite. Kotschick–Neofytidis [**KoN13**, Theorem 8] and Kotschick–de la Harpe [**KdlH14**, Theorem 6.4], building on work of Kotschick–Löh [**KoL09, KoL13**] showed that a 3-manifold is Seifert fibered if and only if its fundamental group is presentable by a product.

Given a group π we define an *ascending sequence of centralizers of length m* in π to be a sequence of subgroups of the form:

$$C_\pi(g_1) \subsetneq C_\pi(g_2) \subsetneq \cdots \subsetneq C_\pi(g_m).$$

Let $m(\pi)$ denote the supremum (in $\mathbb{N} \cup \{\infty\}$) of the lengths of ascending sequences of centralizers in π. If $m(\pi)$ is finite, then π satisfies the maximal condition on centralizers ('Max-c'); see [**Kr90a**] for details. It follows from Theorems 2.5.1 and 2.5.2 that for any compact, orientable, aspherical 3-manifold with empty or toroidal boundary the inequality $m(\pi_1(N)) \leq 3$ holds. Furthermore, by [**Kr90a**, Lemma 5] we have $m(\pi_1(N)) \leq 16$ if N is spherical. It follows from [**Kr90a**, Lemma 4.2], combined with the basic facts of Sections 1.2 and 1.4 and some elementary arguments, that $m(\pi_1(M)) \leq 17$ for any compact 3-manifold M. We refer to [**Kr90a**] for an alternative proof of this fact which does not require the Geometrization Theorem. See also [**Hil06**] for a different approach.

We finish this section by illustrating how the results discussed so far can often be used to quickly determine all 3-manifolds whose fundamental groups have a given group-theoretic property. More precisely, we will describe all fundamental groups of compact 3-manifolds that are CA and CSA. A group is said to be *CA* (short for *centralizer abelian*) if the centralizer of any non-identity element is abelian. Equivalently, a group is CA if and only if the intersection of any two distinct maximal abelian subgroups is trivial, if and only if 'commuting' is an equivalence relation on the set of non-identity elements. For this reason, CA groups are also sometimes called 'commutative transitive groups' (or CT groups, for short).

LEMMA 2.5.10. *Let π be a CA group and $g \in \pi$, $g \neq 1$, such that $C_\pi(g)$ is infinite cyclic. Then $C_\pi(g)$ is self-normalizing.*

PROOF. Let x generate C, and let $y \in \pi$ such that $yC = Cy$. Then $yxy^{-1} = x^{\pm 1}$ and hence $y^2xy^{-2} = x$. Thus x commutes with y^2, and since y^2 commutes with y, we see that x commutes with y. Hence y commutes with g, thus $y \in C_\pi(g) = C$. □

The class of CSA groups was introduced by Myasnikov–Remeslennikov [**MyR96**] as a natural (in the sense of first-order logic, universally axiomatizable) generalization of torsion-free word-hyperbolic groups. (See Section 4.4 below.) A group is said to be *CSA* (short for *conjugately separated abelian*) if all of its maximal abelian subgroups are malnormal. Alternatively, a group is CSA if and only if the centralizer of every non-identity element is abelian and self-normalizing. (As a consequence, every subgroup of a CSA group is again CSA.) It is easy to see that a group that is CSA is also CA. There are CA groups which are not CSA, e.g., the infinite dihedral group, see [**MyR96**, Remark 5]. But for fundamental groups of 3-manifolds we have the following result:

COROLLARY 2.5.11. *Let N be a compact, orientable, irreducible* 3*-manifold with empty or toroidal boundary. Suppose $\pi = \pi_1(N)$ is not abelian. The following are equivalent:*

(1) *Every JSJ-component of N is hyperbolic.*
(2) *π is CA.*
(3) *π is CSA.*

PROOF. We only need to show (1) ⇒ (3) and (2) ⇒ (1). Suppose all JSJ-components of N are hyperbolic. By Theorem 2.5.1, the centralizer $C_\pi(g)$ of each $g \neq 1$ in π is abelian (so π is CA). It remains to show that each such $C_\pi(g)$ is self-normalizing. This follows from the preceding lemma if $C_\pi(g)$ is cyclic, and by Theorems 2.5.1 and 2.5.8 otherwise. This shows (1) ⇒ (3). The implication (2) ⇒ (1) follows easily from Theorems 2.5.1 and 2.5.2, (1), and the fact that subgroups of CA groups are CA. □

CHAPTER 3

3-manifold groups after Geometrization

In Section 1.4 we argued that for our purposes it suffices to study the fundamental groups of compact, orientable, 3-manifolds with empty or toroidal boundary. The following theorem about such manifolds is an immediate consequence of the Prime Decomposition Theorem, the Geometric Decomposition Theorem 1.2.1 and 1.9.1, and Table 1.1.

Theorem 3.0.1. *Let N be a compact, orientable 3-manifold with empty or toroidal boundary. Then N admits a decomposition*

$$N \cong S_1 \# \dots \# S_k \;\#\; T_1 \# \dots \# T_l \;\#\; N_1 \# \dots \# N_m \qquad (k, l, m \in \mathbb{N})$$

as a connected sum of orientable prime 3-manifolds, where:

(1) *$S_1, \dots, S_k$ are spherical;*
(2) *for any $i = 1, \dots, l$ the manifold T_i is either one of $S^1 \times S^2$, $S^1 \times D^2$, $T^2 \times I$, $K^2 \widetilde{\times} I$, or it has a finite solvable cover which is a torus bundle; and*
(3) *$N_1, \dots, N_m$ are irreducible 3-manifolds which are either hyperbolic, or finitely covered by an S^1-bundle over a surface Σ with $\chi(\Sigma) < 0$, or have a non-trivial geometric decomposition.*

The decomposition above can also be stated in terms of fundamental groups:

(1) $S_1, \dots, S_k$ are the prime components of N with finite fundamental groups,
(2) $T_1, \dots, T_l$ are the prime components of N with infinite solvable fundamental groups,
(3) $N_1, \dots, N_m$ are the prime components of N with fundamental groups which are neither finite nor solvable.

The first two types of manifolds are well understood, so from now on we mostly restrict our attention to fundamental groups of compact, orientable, irreducible 3-manifolds N with empty or toroidal boundary, which are neither finite nor solvable. (These assumptions on N imply that its boundary is incompressible: the only irreducible 3-manifold with compressible, toroidal boundary is $S^1 \times D^2$ [**Nemd99**, p. 221].)

In this chapter we summarize the properties of $\pi_1(N)$ that can be deduced from either classical techniques or the Geometrization Theorem. This can be thought of as the state of the art immediately before the intervention of Agol, Calegari–Gabai, Kahn–Markovic and Wise discussed in Chapter 4 below.

The principal instrument of our summary is Flowchart 1 on p. 46. The flowchart describes a set of implications about 3-manifold groups. Section 3.1 contains the definitions of the properties in question, as well as the conventions adopted in the flowchart. Section 3.2 contains the justifications for the implications in the flowchart. Finally, Section 3.3 contains some further implications not listed in the flowchart.

3.1. Definitions and conventions

We first give some of the definitions which we use in Flowchart 1, roughly in the order that they appear there. Below we let N be a 3-manifold and π be a group.

(A.1) A space X is *aspherical* if X is connected and if $\pi_i(X) = 0$ for $i \geq 2$.

(A.2) A space X is an *Eilenberg–Mac Lane space* for π, written as $X = K(\pi, 1)$, if $\pi_1(X) \cong \pi$ and if X is aspherical.

(A.3) The deficiency of a finite presentation $\langle g_1, \ldots, g_m \,|\, r_1, \ldots, r_n \rangle$ of a group is defined to be $m - n$. The *deficiency of a finitely presented group* is defined to be the supremum of the deficiencies of all its finite presentations. (Note that some authors use the negative of this quantity.)

(A.4) A group is called *coherent* if each of its finitely generated subgroups is finitely presented.

(A.5) The L^2-Betti numbers $b_i^{(2)}(X, \alpha)$, for a given topological space X and a group morphism $\alpha\colon \pi_1(X) \to \Gamma$, were introduced by Atiyah [**At76**]; see [**Ecn00**] and [**Lü02**] for the definition. Set $b_i^{(2)}(X) := b_i^{(2)}(X, \mathrm{id})$ for $\Gamma = \pi_1(X)$, $\alpha = \mathrm{id}$.

(A.6) A *cofinal* (*normal*) *filtration* of π is a decreasing sequence $\{\pi_n\}$ of finite-index (normal) subgroups of π such that $\bigcap_n \pi_n = \{1\}$. A *cofinal* (*regular*) *tower* of N is a sequence $\{\tilde{N}_n\}$ of connected covers of N such that $\{\pi_1(\tilde{N}_n)\}$ is a cofinal (normal) filtration of $\pi_1(N)$. Let R be an integral domain. If the limit

$$\lim_{n\to\infty} \frac{b_1(\tilde{N}_n; R)}{[N : \tilde{N}_n]}$$

exists for any cofinal regular tower $\{\tilde{N}_n\}$ of N, and if all the limits agree, then we denote this unique limit by

$$\lim_{\tilde{N}} \frac{b_1(\tilde{N}; R)}{[N : \tilde{N}]}.$$

(A.7) We denote by $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$.

(A.8) The *Frattini subgroup* of π is the intersection $\Phi(\pi)$ of all maximal subgroups of π. If π does not admit a maximal subgroup, then we define $\Phi(\pi) = \pi$. (By an elementary argument, $\Phi(\pi)$ also agrees with the intersection of all maximal normal subgroups of π.)

(A.9) Let R be a commutative ring. We say that π is *linear over R* if there exists an embedding $\pi \to \mathrm{GL}(n, R)$ for some n. In this case, π also admits an embedding into $\mathrm{SL}(n+1, R)$.

(A.10) Unless we say specifically otherwise we will mean by a surface in N a compact, orientable surface, properly embedded in N. If N is orientable, then a surface in our sense will always be two-sided. A surface Σ in N is called
- (a) *separating* if $N \setminus \Sigma$ is disconnected;
- (b) a (*non-*) *fiber surface* if it is incompressible, connected, and (not) the fiber of a surface bundle map $N \to S^1$;
- (c) *separable* if Σ is connected and $\pi_1(\Sigma)$ is separable in $\pi_1(N)$. (See (A.22) for the definition of a separable subgroup.)

A 3-manifold is *Haken* (or *sufficiently large*) if it is compact, orientable, irreducible, and has an embedded incompressible surface.

(A.11) A group is *large* if it contains a finite-index subgroup which admits a surjective morphism onto a non-cyclic free group.

(A.12) We refer to

$$\operatorname{coker}\big\{H_1(\partial N;\mathbb{Z}) \to H_1(N;\mathbb{Z})\big\}$$

as the *non-peripheral homology of* N. A standard transfer argument shows that if N has non-peripheral homology of rank at least n, then so does any finite cover of N. We say that N is *homologically large* if given any n, N has a finite regular cover which has non-peripheral homology of rank at least n.

(A.13) Let R be an integral domain with fraction field F. We write $vb_1(\pi;R)=\infty$ if for any n there is a finite-index (not necessarily normal) subgroup π' of π with

$$\operatorname{rank}_R(H_1(\pi';R)) := \dim_F(H_1(\pi';F)) \geq n.$$

In that case we say that π has *infinite virtual first R-Betti number.* We write $vb_1(N;R)=\infty$ if $vb_1(\pi_1(N);R)=\infty$, and sometimes $vb_1(N)=vb_1(N;\mathbb{Z})$.

Suppose $\pi=\pi_1(N)$, where N is irreducible, non-spherical, compact, with empty or toroidal boundary, and $vb_1(\pi;R)=\infty$. Then for any n there exists also a finite-index *normal* subgroup π' of π with $\operatorname{rank}_R(H_1(\pi';R))\geq n$. Indeed, if $\operatorname{char}(R)=0$, then this follows from elementary group-theoretic arguments, and if $\operatorname{char}(R)\neq 0$, from [**Lac09**, Theorem 5.1], since the Euler characteristic of $N=K(\pi,1)$ is zero. (Here we used that our assumptions on N imply that $N=K(\pi,1)$—see (C.1) and (C.2).)

(A.14) A group is called *indicable* if it admits a surjective morphism onto $\mathbb{Z}$, and *locally indicable* if each of its finitely generated subgroups is indicable.

(A.15) A *left ordering* on π is a total ordering '$\leq$' on π which is left-invariant, i.e., if $g,h,k\in\pi$ with $g\leq h$, then $kg\leq kh$. Similarly one defines the notion of *right ordering* on π. A *bi-ordering* on π is an ordering on π which is both a left ordering and a right ordering on π. One calls π *left orderable* if it has a left ordering, and similarly with 'right' and 'bi' in place of 'left.'

(A.16) Following Bowditch [**Bow00**] we say that a group π is *diffuse* if every finite non-empty subset A of π has an extremal point, i.e., an element $a\in A$ such that for any $g\in\pi\setminus\{1\}$ one of ga or $g^{-1}a$ is not in A.

(A.17) See [**Pie84**] or [**CSC14**, Section 4] for the definition of an amenable group.

(A.18) Given a property $\mathcal{P}$ of groups we say that a group is *virtually* $\mathcal{P}$ if it has a finite-index subgroup (not necessarily normal) which satisfies $\mathcal{P}$.

(A.19) Given a class $\mathcal{P}$ of groups we say that π is *residually* $\mathcal{P}$ if given any $g\in\pi$, $g\neq 1$, there exists a surjective group morphism α onto a group from $\mathcal{P}$ such that $\alpha(g)\neq 1$. A case of particular importance is when $\mathcal{P}$ is the class of finite groups, in which case π is said to be *residually finite.* Another important case is when $\mathcal{P}$ is the class of finite p-groups for p a prime (that is, the class of groups of p-power order), and then π is said to be *residually p.*

(A.20) Given a class $\mathcal{P}$ of groups we say that π is *fully residually* $\mathcal{P}$ if given any $g_1,\dots,g_n\in\pi\setminus\{1\}$, there exists a surjective group morphism α onto a group from $\mathcal{P}$ such that $\alpha(g_i)\neq 1$ for all $i=1,\dots,n$.

(A.21) The *profinite topology* on π is the coarsest topology with respect to which every morphism from π to a finite group, equipped with the discrete topology, is continuous. Note that π is residually finite if and only if the profinite topology on π is Hausdorff. Similarly, the *pro-p topology* on π is the coarsest topology with respect to which every morphism from π to a finite p-group, equipped with the discrete topology, is continuous.

(A.22) A subset S of π is *separable* if S is closed in the profinite topology on π; equivalently, for any $g \in \pi \setminus S$, there exists a morphism α from π to a finite group with $\alpha(g) \notin \alpha(S)$. The group π is called
  (1) *locally extended residually finite* (*LERF,* or *subgroup separable*) if each finitely generated subgroup of π is separable,
  (2) *AERF* (or *abelian subgroup separable*) if any finitely generated abelian subgroup of π is separable.

(A.23) We say that π is *double-coset separable* if for all finitely generated subgroups A, B of π and $g \in \pi$, the subset $AgB \subseteq \pi$ is separable. Note that AgB is separable if and only if $(g^{-1}Ag)B$ is, and so to prove double-coset separability it suffices to show that products of finitely generated subgroups are separable.

(A.24) Let Γ be a subgroup of π. We say that π *induces the full profinite topology on* Γ if the restriction of the profinite topology on π to Γ is the full profinite topology on Γ; equivalently, for any finite-index subgroup $\Gamma' \subseteq \Gamma$ there exists a finite-index subgroup π' of π such that $\pi' \cap \Gamma \subseteq \Gamma'$.

(A.25) Suppose N is compact, orientable, irreducible, with empty or toroidal boundary. We say that N is *efficient* if the graph of groups corresponding to the JSJ-decomposition is efficient, i.e., if
  (a) $\pi_1(N)$ induces the full profinite topology on the fundamental groups of the JSJ-tori and of the JSJ-components; and
  (b) the fundamental groups of the JSJ-tori and the JSJ-components, viewed as subgroups of $\pi_1(N)$, are separable.

  We refer to [**WZ10**] for more about this notion.

(A.26) Suppose π is finitely presentable. We say that the *word problem for π is solvable* if given any finite presentation for π there exists an algorithm which can determine whether or not a given word in the generators is trivial. Similarly, the *conjugacy problem for π is solvable* if given any finite presentation for π there exists an algorithm to determine whether or not any two given words in the generators represent conjugate elements of π. We refer to [**CZi93**, Section D.1.1.9] and to [**AFW13**] for details. (By [**Mila92**, Lemma 2.2] the word problem is solvable for one finite presentation if and only if it is solvable for every finite presentation; similarly for the conjugacy problem.)

(A.27) A group is called *Hopfian* if it is not isomorphic to a proper quotient of itself.

(A.28) The *Whitehead group* $\mathrm{Wh}(\pi)$ is defined as the quotient of $K_1(\mathbb{Z}[\pi])$ by $\pm\pi$; here $K_1(\mathbb{Z}[\pi])$ is the abelianization of the direct limit $\varinjlim_{n\to\infty} \mathrm{GL}(n, \mathbb{Z}[\pi])$ of the general linear groups over $\mathbb{Z}[\pi]$. We refer to [**Mil66**] for details.

(A.29) If every countable group embeds in a quotient of π, then π is called *SQ-universal.*

Before we move on to Flowchart 1, we state a few conventions which we apply there.

(B.1) In Flowchart 1, N is a compact, orientable, irreducible 3-manifold whose boundary consists of a (possibly empty) collection of tori. Furthermore we assume throughout Flowchart 1 that $\pi := \pi_1(N)$ is neither solvable nor finite. Without these extra assumptions some of the implications do not hold. For example, not every Seifert fibered manifold N has a finite cover N' with $b_1(N') \geq 2$, but this is the case if π additionally is neither solvable nor finite.

(B.2) Arrow 6 splits into three arrows, this means that precisely one of the three possible conclusion holds.

(B.3) Red arrows indicate that the conclusion holds *virtually*. For example, Arrow 11 says that if N is a Seifert fibered manifold such that $\pi_1(N)$ is neither finite nor solvable, then N contains virtually an incompressible torus.

(B.4) If a property $\mathcal{P}$ of groups is written in green, then the following conclusion always holds: If N is a compact, orientable, irreducible 3-manifold with empty or toroidal boundary such that the fundamental group of a finite (not necessarily regular) cover of N is $\mathcal{P}$, then $\pi_1(N)$ also is $\mathcal{P}$. We will justify on page 58 that the green properties in Flowchart 1 do indeed have the above property.

(B.5) The Conventions (B.3) and (B.4) imply that a concatenation of red and black arrows which leads to a green property means that the initial group also has the green property.

(B.6) An arrow with a condition next to it indicates that this conclusion only holds if this extra condition is satisfied.

Finally, one last disclaimer: the flowchart is meant as a guide to the precise statements in the text and in the literature; it should not be used as a reference in its own right.

3.2. Justifications

Now we give the justifications for the implications of Flowchart 1. In the subsequent discussion we strive for maximal generality; in particular, unless we say otherwise, we will only assume (as we do throughout this book) that N is a connected 3-manifold. We will give the required references and arguments for the general case, so each justification can be read independently of all the other steps.

In many cases, after the justification we will give further information and background material to put the statements in context.

(C.1) Suppose N is irreducible, orientable, with infinite fundamental group. It follows from the irreducibility of N and the Sphere Theorem 1.3.2 that $\pi_2(N) = 0$. Since $\pi_1(N)$ is infinite, it follows from the Hurewicz theorem that $\pi_i(N) = 0$ for any $i > 2$, i.e., N is aspherical. (For the exterior of an alternating knot in S^3 this result was first proved by Aumann [**Aum56**].[1])

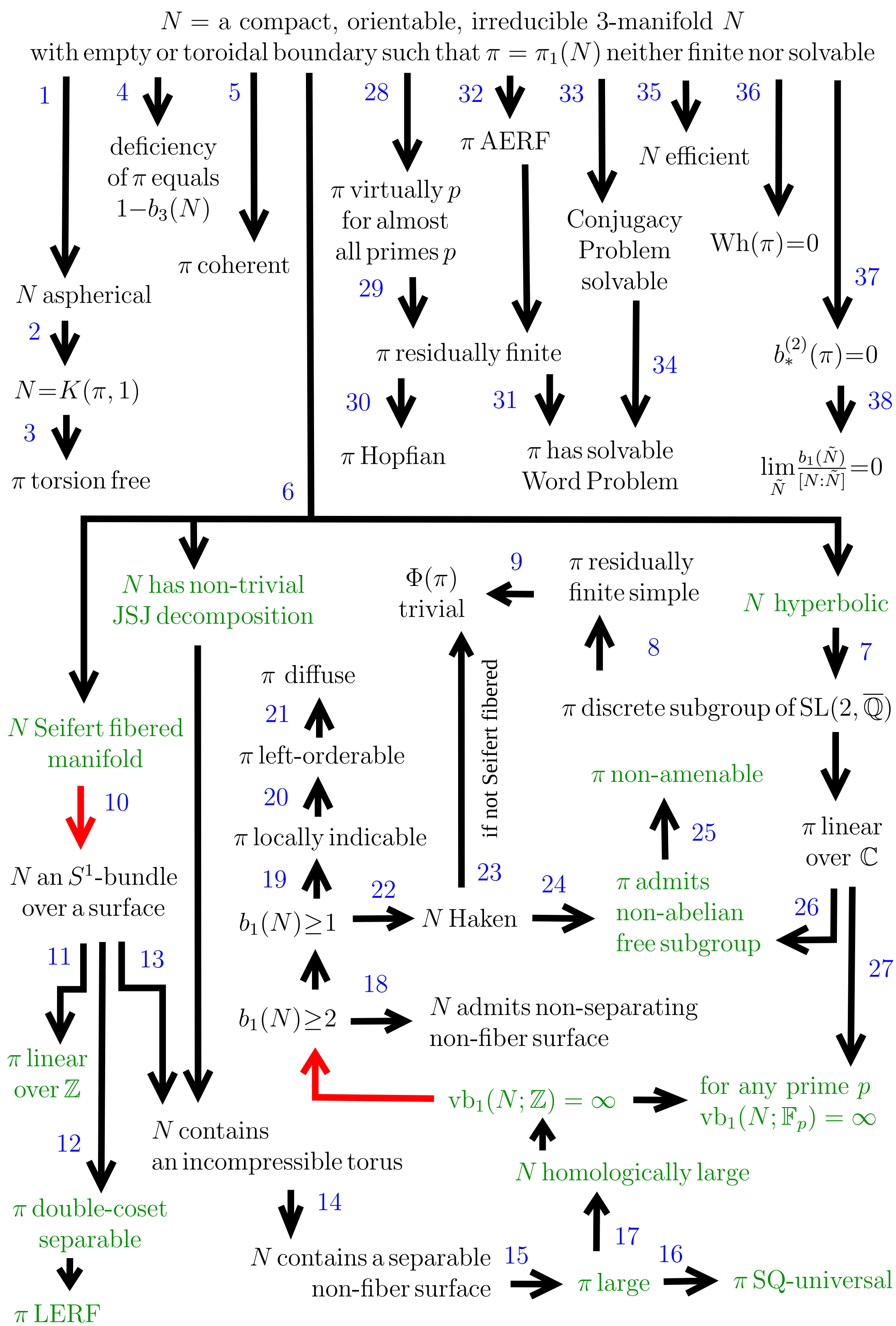
If N is non-orientable, then the above conclusion does not hold since for example $S^1 \times \mathbb{R}P^2$ is not aspherical. Following [**Hem76**, p. 88] we say that a 3-manifold is $\mathbb{R}P^2$-*irreducible* if it is irreducible and if it contains no 2-sided projective planes. If N is a non-orientable $\mathbb{R}P^2$-irreducible 3-manifold, then its 2-fold orientable cover $\tilde{N}$ is by [**Hem76**, Lemma 10.4] also $\mathbb{R}P^2$-irreducible. Hence by the above $\tilde{N}$ (and thus also N) are aspherical.

(C.2) It follows immediately from the definitions that if X is an aspherical space, then X is an Eilenberg–Mac Lane space for its fundamental group. In particular by (C.1), if N is orientable, irreducible, with infinite fundamental group, then N is an Eilenberg–Mac Lane space for its fundamental group.

If N is compact, orientable, irreducible, with infinite fundamental group and non-empty boundary, then it has a deformation retract to the 2-skeleton, so $\pi_1(N)$ has a 2-dimensional Eilenberg–Mac Lane, space by the above.

An argument as in [**FJR11**, p. 458] shows that if N is compact, orientable, irreducible, with non-trivial toroidal boundary, and if P is a presentation of $\pi_1(N)$ of deficiency 1, then the 2-complex X corresponding to P is also an Eilenberg–Mac Lane space for $\pi_1(N)$. (This argument relies on the fact

[1]Robert Aumann won a Nobel Memorial Prize in Economic Sciences in 2005.



Flowchart 1. Consequences of the Geometrization Theorem.

that $\pi_1(N)$ is locally indicable, see (C.19).) In fact by (C.36) the complex X is simple homotopy equivalent to N.

If N is an Eilenberg–Mac Lane space for $\pi_1(N)$, then the cohomology ring of N is an invariant of $\pi_1(N)$. Postnikov [**Pos48**], Sullivan [**Sula75**] and Turaev [**Tur83, Tur84**] classified the graded rings that appear as cohomology rings of closed 3-manifolds.

(C.3) Suppose N is orientable, irreducible, with infinite fundamental group. By (C.1) and (C.2) we have $N = K(\pi_1(N), 1)$. Since the Eilenberg–Mac Lane space is finite-dimensional it follows by standard arguments that $\pi_1(N)$ is torsion-free. (Indeed, if X is an aspherical space and if $g \in \pi_1(X)$ is an element of finite order k, then consider the covering space of X corresponding to the subgroup of $\pi_1(X)$ generated by g. Then $\tilde{X}$ is an Eilenberg–Mac Lane space for $\mathbb{Z}_k$, hence $H_*(\mathbb{Z}_k; \mathbb{Z}) = H_*(\tilde{X}; \mathbb{Z})$; but the only finite cyclic group with finite homology is the trivial group. See [**Hat02**, Proposition 2.45].)

(C.4) Suppose N is compact, irreducible, with empty or toroidal boundary and with infinite fundamental group. By (C.1) and (C.2), N is an Eilenberg–Mac Lane space. By work of Epstein it follows that the deficiency of $\pi_1(N)$ equals $1 - b_3(N)$; see [**Eps61a**, Lemmas 2.2 and 2.3], [**Eps61a**, Theorem 2.5].

The fact that the fundamental group of any closed 3-manifold admits a balanced presentation, i.e., a presentation of deficiency 0, is an immediate consequence of the existence of a Heegaard splitting, a surface which splits the 3-manifold into two handlebodies. (See [**Hee1898, Hee16**], [**Hem76**, Theorem 2.5], [**CZi93**, Theorem 5.1.2], and [**Sav12**, Theorem 1.1].) The question which groups with a balanced presentation are fundamental groups of a compact 3-manifold is studied in [**Neh68, Neh70, OsS74, OsS77a, OsS77b, Osb78, Sts75, Hog00**].

(C.5) Scott [**Sco73b**] proved the Core Theorem: if Y is any 3-manifold such that $\pi_1(Y)$ is finitely generated, then Y has a compact submanifold M such that the natural morphism $\pi_1(M) \to \pi_1(Y)$ is an isomorphism. By applying the Core Theorem to all covers of the given 3-manifold N corresponding to finitely generated subgroups of $\pi_1(N)$ we see that $\pi_1(N)$ is coherent. (See also [**Sco73a, Sco74, Sta77, RS90**] and [**Cal14a**, Theorem 3.6].)

(C.6) The Geometrization Theorem 1.7.6 implies that any compact, orientable, irreducible 3-manifold with empty or toroidal boundary
 (a) is Seifert fibered, or
 (b) is hyperbolic, or
 (c) admits an incompressible torus.

(C.7) Suppose N is hyperbolic. Then the fundamental group of N admits a faithful discrete representation $\pi_1(N) \to \text{Isom}^+(\mathbb{H}^3)$, where $\text{Isom}^+(\mathbb{H}^3)$ denotes the group of orientation preserving isometries of 3-dimensional hyperbolic space. (Here recall that we assume throughout that hyperbolic 3-manifolds are orientable.) By [**Bon09**, Theorem 9.8] we can identify $\text{Isom}^+(\mathbb{H}^3)$ with $\text{PSL}(2, \mathbb{C})$, which thus gives rise to a faithful discrete representation $\pi_1(N) \to \text{PSL}(2, \mathbb{C})$. As a consequence of the Rigidity Theorem 1.7.1, this representation is unique up to conjugation and complex conjugation. Another consequence of rigidity is that there exists in fact a faithful discrete representation $\rho\colon \pi_1(N) \to \text{PSL}(2, \overline{\mathbb{Q}})$ over the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ [**MaR03**, Corollary 3.2.4]. Thurston showed that the representation ρ lifts to a faithful discrete representation $\tilde{\rho}\colon \pi_1(N) \to$

$\mathrm{SL}(2,\overline{\mathbb{Q}})$. (See [**Cu86**, Corollary 2.2] and [**Shn02**, Section 1.6].) The set of lifts of ρ to a representation $\pi_1(N) \to \mathrm{SL}(2,\overline{\mathbb{Q}})$ is in a natural one-to-one correspondence with the set of Spin-structures on N, with the number of Spin-structures on N given by $|H^1(N;\mathbb{Z}_2)|$; see [**MFP14**, Section 2] for details.

Let T be a boundary component of N and $\tilde{\rho}\colon \pi_1(N) \to \mathrm{SL}(2,\overline{\mathbb{Q}})$ be a lift as above. Then $\tilde{\rho}(\pi_1(T))$ is a discrete subgroup of $\mathrm{SL}(2,\overline{\mathbb{Q}})$ isomorphic to $\mathbb{Z}^2$, so up to conjugation, we have

$$\tilde{\rho}(\pi_1(T)) \subseteq \left\{ \begin{pmatrix} \varepsilon & a \\ 0 & \varepsilon \end{pmatrix} \,:\, \varepsilon \in \{-1,1\},\ a \in \overline{\mathbb{Q}} \right\}.$$

By [**Cal06**, Corollary 2.4], if $a \in \pi_1(T)$ is represented by a curve on T which cobounds a surface in N, then $\mathrm{tr}(\tilde{\rho}(a)) = -2$.

Button [**But12**] studied the question of which non-hyperbolic 3-manifolds admit (non-discrete) embeddings of their fundamental groups into $\mathrm{SL}(2,\mathbb{C})$. For example, he showed that if N is given by gluing two figure-8 knot complements along their boundary, then there are some gluings for which such an embedding exists, and there are some for which it does not.

(C.8) Long–Reid [**LoR98**, Theorem 1.2] showed that if a subgroup π of $\mathrm{SL}(2,\overline{\mathbb{Q}})$ is isomorphic to the fundamental group of a compact, orientable, non-spherical 3-manifold, then π is residually finite simple. (The assumption that π is a non-spherical 3-manifold group is necessary since not all subgroups of $\mathrm{SL}(2,\overline{\mathbb{Q}})$ are residually simple.) Reading the proof of [**LoR98**, Theorem 1.2] shows that under the same hypothesis as above a slightly stronger conclusion holds: π is *fully residually simple.*

We refer to [**But11b**] for more results on 3-manifold groups (virtually) surjecting onto finite simple groups.

(C.9) Residually simple groups clearly have trivial Frattini subgroup. The combination of (C.7) and (C.8) thus shows that the Frattini subgroup of the fundamental group of a hyperbolic 3-manifold is trivial. Platonov [**Pla66**] showed that the Frattini subgroup of any finitely generated linear group is nilpotent. In particular the combination of (C.8) with Platonov's result gives the weaker statement that the Frattini subgroup of the fundamental group of a hyperbolic 3-manifold is nilpotent, which in this context means that it is abelian.

(C.10) Below we argue that if N is Seifert fibered, then it is finitely covered by an S^1-bundle N' over an orientable, connected surface. In the diagram we restrict ourselves to 3-manifolds such that the fundamental group is neither finite nor solvable. Theorem 1.11.1 implies that $\pi_1(N')$ is solvable if and only if $\pi_1(N)$ is.

So let N be a Seifert fibered 3-manifold. We first consider the case that N is closed. Denote by B the base orbifold of N. If B is a 'good' orbifold in the sense of [**Sco83a**, p. 425], then B is finitely covered by an orientable connected surface F. This cover $F \to B$ gives rise to a map $Y \to N$ of Seifert fibered manifolds. Since the base orbifold of the Seifert fibered manifold Y is a surface it follows that Y is in fact an S^1-bundle over F.

The 'bad' orbifolds are classified in [**Sco83a**, p. 425], and in the case of base orbifolds the only two classes of bad orbifolds which can arise are $S^2(p)$ and $S^2(p,q)$ (see [**Sco83a**, p. 430]). The former arises from the lens space $L(p,1)$ and the latter from the lens space $L(p,q)$. But lens spaces are covered by S^3 which is an S^1-bundle over the sphere.

Now consider the case that N has boundary. We consider the double $M = N \cup_{\partial N} N$, which is again a Seifert fibered manifold. By the above there exists a finite-sheeted covering map $p\colon Y \to M$ where Y is an S^1-bundle over a surface and p preserves the Seifert fibers. It follows that $p^{-1}(N) \subseteq Y$ is a sub-Seifert fibered manifold. In particular any component of $p^{-1}(N)$ is also an S^1-bundle over a surface.

Let N be any compact, orientable, irreducible 3-manifold with empty or toroidal boundary. A useful generalization of the above statement says that N has a finite cover all of whose Seifert fibered JSJ-components are in fact S^1-bundles over a surface. See [**AF13**, Section 4.3] and [**Hem87, Hamb01**].

(C.11) The fundamental group of an S^1-bundle over a surface is linear over $\mathbb{Z}$. This fact is well-known, we will provide a proof that was suggested to us by Boyer.

Before we give this proof, we recall that the fundamental group of a compact surface F is linear over $\mathbb{Z}$. Indeed, this is obvious if F is a sphere or a torus. If F has boundary, then $\pi_1(F)$ is free and hence embeds into $\mathrm{SL}(2,\mathbb{Z})$. Newman [**New85**, Lemma 1] showed that if F is closed, then there exists an embedding $\pi_1(F) \to \mathrm{SL}(8,\mathbb{Z})$. In the closed case we may also argue as follows: Scott [**Sco78**, Section 3] showed that $\pi_1(F)$ is a subgroup of a right angled Coxeter group on 5 generators, hence by [**Bou81**, Chapitre V, §4, Section 4] can be embedded into $\mathrm{SL}(5,\mathbb{Z})$. Or we may use that if F is closed then $\pi_1(F)$ embeds into a RAAG (see, e.g., [**DSS89**, p. 576], [**CrW04**, Theorem 3], or [**Rov07**]), hence is linear over $\mathbb{Z}$ by (H.31).

We first consider the case that N is a trivial S^1-bundle, i.e., $N \cong S^1 \times F$. Then $\pi_1(N) = \mathbb{Z} \times \pi_1(F)$ is the direct product of $\mathbb{Z}$ with the fundamental group of a compact surface, hence is $\mathbb{Z}$-linear by the above.

Now assume that N is a non-trivial S^1-bundle over a surface F. If N has boundary, then F also has boundary and we obtain $H^2(F;\mathbb{Z}) = 0$, so the Euler class of the S^1-bundle $N \to F$ is trivial, so N would be a trivial S^1-bundle. Hence F is a closed surface. If $F = S^2$, then the long exact sequence in homotopy theory shows that $\pi_1(N)$ is cyclic, hence linear. If $F \neq S^2$, then it follows again from the long exact sequence in homotopy theory that the subgroup $\langle t\rangle$ of $\pi_1(N)$ generated by a fiber is normal and infinite cyclic, and that we have a short exact sequence

$$1 \to \langle t\rangle \to \pi_1(N) \to \pi_1(F) \to 1.$$

Let $e \in H^2(F;\mathbb{Z}) \cong \mathbb{Z}$ be the Euler class of F. By the discussion on [**Sco83a**, p. 435], a presentation for $\pi := \pi_1(N)$ is given by

$$\pi = \left\langle a_1, b_1, \dots, a_n, b_n, t : \prod_{i=1}^{n} [a_i, b_i] = t^e,\ t \text{ central} \right\rangle.$$

Let π_e be the subgroup of π generated by the a_i, b_i, and t^e. It is straightforward to check that the assignment

$$\rho(a_1) := \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho(b_1) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho(t^e) := \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $\rho(a_i) := \rho(b_i) := 1$ for $i = 2, \ldots, n$ yields a morphism $\rho\colon \pi_e \to \mathrm{SL}(3, \mathbb{Z})$ such that $\rho(t^e)$ has infinite order. Now let σ be the composition

$$\pi_e \to \pi = \pi_1(N) \to \pi_1(F) \to \mathrm{SL}(n, \mathbb{Z}),$$

where the last arrow is a faithful representation of $\pi_1(F)$ in $\mathrm{SL}(n, \mathbb{Z})$, which exists by the above. Then

$$\rho \times \sigma\colon \pi_e \to \mathrm{SL}(3, \mathbb{Z}) \times \mathrm{SL}(n, \mathbb{Z}) \subseteq \mathrm{SL}(n + 3, \mathbb{Z})$$

is an embedding. This concludes the proof that fundamental groups of S^1-bundles over surfaces are linear over $\mathbb{Z}$.

Together with (C.10) this implies that fundamental groups of Seifert fibered manifolds are linear over $\mathbb{Z}$. The fact that Seifert fibered manifolds admit a geometric structure can in most cases be used to give an alternative proof of the fact that their fundamental groups are linear over $\mathbb{C}$. More precisely, if N admits a geometry X, then $\pi_1(N)$ is a discrete subgroup of $\mathrm{Isom}(X)$. By [**Boy**] the isometry groups of the following geometries are subgroups of $\mathrm{GL}(4, \mathbb{R})$: spherical geometry, $S^2 \times \mathbb{R}$, Euclidean geometry, Nil, Sol and hyperbolic geometry. Moreover, the fundamental group of an $\mathbb{H}^2 \times \mathbb{R}$–manifold is a subgroup of $\mathrm{GL}(5, \mathbb{R})$. On the other hand, the isometry group of the universal covering group of $\mathrm{SL}(2, \mathbb{R})$ is not linear (see, e.g., [**Di77**, p. 170]).

Finally, groups which are virtually polycyclic are linear over $\mathbb{Z}$ by the Auslander–Swan Theorem (see [**Swn67**], [**Aus67**, Theorem 2] and [**Weh73**, Theorem 2.5]) and (D.2). This implies in particular that fundamental groups of Sol-manifolds are linear over $\mathbb{Z}$.

(C.12) Niblo [**Nib92**, Corollary 5.1] showed that if N is Seifert fibered (in particular if N is an S^1-bundle over a surface), then $\pi_1(N)$ is double-coset separable.

It follows in particular that fundamental groups of Seifert fibered manifolds are LERF. This was first implicitly proved by Hall [**Hal49**, Theorem 5.1] if N has non-empty boundary, and by Scott [**Sco78**, Theorem 4.1], [**Sco85a**] if N is closed. We refer to [**BBS84, Nib90, Tre90, HMPR91, Lop94, GR95, Git97, LoR05, Wil07, BaC13, Pat14, GR13**] and [**RiZ93**, Theorem 2.1] for alternative proofs and extensions of these results.

(C.13) It follows from elementary arguments that if N is an S^1-bundle over a surface F such that $\pi_1(N)$ is neither solvable nor finite, then $\chi(F) < 0$, so F thus admits an essential curve c; the S^1–bundle over c is an incompressible torus.

(C.14) Suppose N is compact and orientable, and let T be an incompressible torus in N. (In particular T could be any incompressible boundary torus of N.) By (C.32) the subgroup $\pi_1(T)$ of $\pi_1(N)$ is separable, i.e., T is a separable surface. If in addition $\pi_1(N)$ is not solvable, then the torus is not a fiber surface.

(C.15) Suppose N is compact, irreducible, with non-empty incompressible boundary, and not covered by $T^2 \times I$. Cooper–Long–Reid [**CLR97**, Theorem 1.3] showed that $\pi_1(N)$ is large. (See also [**But04**, Corollary 6], [**Lac07a**, Theorem 2.1].)

Now suppose N is closed. Let Σ be a separable non-fiber surface, i.e., Σ is a connected incompressible surface in N which is not a fiber surface. By Stallings' Fibration Theorem [**Sta62**] there exists a $g \in \pi_1(N \setminus \Sigma \times (0, 1)) \setminus \pi_1(\Sigma \times \{0\})$. (See also (L.9) and [**Hem76**, Theorem 10.5].) Since $\pi_1(\Sigma \times \{0\})$ is separable by assumption, we can separate g from $\pi_1(\Sigma \times \{0\})$. A standard argument shows that in the corresponding finite cover N' of N the preimage of Σ consists of at least two non null-homologous and non-homologous orientable surfaces. Any two such surfaces give rise to a morphism from $\pi_1(N')$ onto a free group with two generators. (See also [**LoR05**, Proof of Theorem 3.2.4].)

Let π be a group. The *co-rank* $c(\pi)$ of π is defined as the maximal n such that π admits a surjective morphism onto the free group on n generators, if it exists; otherwise $c(\pi) := \infty$. If N is a 3-manifold, then $c(N) := c(\pi_1(N))$ equals the 'cut number' of N; we refer to [**Har02**, Proposition 1.1] for details. Jaco [**Ja72**, Theorem 2.3] showed that the cut number is additive under connected sum. Clearly $b_1(N) \geq c(N)$. On the other hand, Harvey [**Har02**, Corollary 3.2] showed that, given any b, there exists a closed hyperbolic 3-manifold N with $b_1(N) \geq b$ and $c(N) = 1$. (See also [**LRe02**] and [**Sik05**].) Upper bounds on $c(N)$ in terms of quantum invariants are obtained in [**GiM07**, Theorem 15.1] and [**Gil09**, Theorem 1.5].

(C.16) Higman–Neumann–Neumann [**HNN49**] showed that non-cyclic free groups are SQ-universal. Immediately from the definitions, large groups are SQ-universal.

Minasyan–Osin [**MO13**, Corollary 2.12] (extending earlier work in [**Rat87**] and [**But04**]) showed that fundamental groups of compact 3-manifolds are either virtually polycyclic or SQ-universal. Alternatively this follows later on from the discussion in Flowchart 4.

(C.17) Suppose N is compact with empty or toroidal boundary, and let $\varphi\colon \pi_1(N) \to F$ be a morphism onto a non-cyclic free group. Then $\pi_1(N)$ is homologically large. Recall that this means that given any $k \in \mathbb{N}$ there is a finite cover N' of N with

$$\operatorname{rank}_{\mathbb{Z}} \operatorname{coker}\{H_1(\partial N';\mathbb{Z}) \to H_1(N';\mathbb{Z})\} \geq k.$$

Indeed, denote by $S_1,\dots,S_m$ (respectively $T_1,\dots,T_n$) the boundary components of N which have the property that φ restricted to the boundary torus is trivial (respectively non-trivial). The image of $\pi_1(T_i) \subseteq F$ is a non-trivial infinite cyclic group generated by some $a_i \in F$. Given $k \in \mathbb{N}$, we pick a prime number p with $p \geq 2n+k$. Since F is residually p [**Iw43, Neh61b**], we can take an a morphism α from F onto a p-group P with $\alpha(a_i) \neq 1$ for $i = 1,\dots,n$. Let $F' = \operatorname{Ker}(\alpha)$ and denote by $q\colon N' \to N$ the covering of N corresponding to $\alpha \circ \varphi$. If S' is any boundary component of N' covering one of the S_i, then $\pi_1(S') \to \pi_1(N') \to F'$ is the trivial map. Using this observation we calculate

$$\begin{aligned} &\operatorname{rank}_{\mathbb{Z}} \operatorname{coker}\{H_1(\partial N';\mathbb{Z}) \to H_1(N';\mathbb{Z})\} \\ &\geq \operatorname{rank}_{\mathbb{Z}} \operatorname{coker}\{H_1(\partial N';\mathbb{Z}) \to H_1(F';\mathbb{Z})\} \\ &\geq b_1(F') - \sum_{i=1}^{n} b_1(q^{-1}(T_i)) \geq b_1(F') - 2\sum_{i=1}^{n} b_0(q^{-1}(T_i)) \\ &\geq |P|(b_1(F)-1) + 1 - 2n\tfrac{|P|}{p} \geq |P| - 2n\tfrac{|P|}{p} = \tfrac{|P|}{p}(p-2n) \geq k. \end{aligned}$$

(See also [**CLR97**, Corollary 2.9] for a related argument.)

The combination of (C.14), (C.15) and (C.17) shows in particular that if N is compact, orientable, and irreducible, if N contains an incompressible, non-boundary parallel torus and if $\pi_1(N)$ is not solvable, then $vb_1(N;\mathbb{Z}) = \infty$. This was first proved by Kojima [**Koj87**, p. 744] and Luecke [**Lue88**, Theorem 1.1].

As we just saw, if N is compact, orientable, and irreducible with non-trivial toroidal boundary, then either $N = T^2 \times I$ or $vb_1(N;\mathbb{Z}) = \infty$. The same conclusion also holds for more general types of boundary. More precisely, suppose N

is compact, orientable with incompressible boundary and no spherical boundary components, which is not a product on a boundary component (i.e., there does not exist a component Σ of ∂N such that $N \cong \Sigma \times [0,1]$). It follows from standard arguments (e.g., boundary subgroup separability, see (C.32) below) that for any n there exists a finite cover $\tilde{N}$ of N with at least n boundary components. Since none of these boundary components is spherical it follows from a Poincaré duality argument that $b_1(\tilde{N}) \geq n$. Thus indeed $vb_1(N;\mathbb{Z}) = \infty$.

(C.18) A straightforward Thurston norm argument (see [**Thu86a**] or [**CdC03**, Corollary 10.5.11] together with the argument of the first paragraph of (C.22)) shows that if N is compact, orientable, irreducible, with empty or toroidal boundary, and with $b_1(N) \geq 2$, then either N is a torus bundle (in which case $\pi_1(N)$ is solvable), or N admits a connected, incompressible, homologically essential non-fiber surface Σ. (A surface is called *homologically essential* if it represents a non-trivial homology class, such a surface is necessarily non-separating.)

Note that since Σ is homologically essential it follows from standard arguments (e.g., using Stallings' Fibration Theorem, see [**Sta62**] and (L.9)) that Σ does in fact not lift to the fiber of a surface bundle in any finite cover.

In (C.22) we will see that a generic 3-manifold is a rational homology sphere. Rivin [**Riv14**, Theorem 4.1] showed that $b_1(N) = 1$ for a generic fibered N.

(C.19) Howie [**How82**, Proof of Theorem 6.1] used Scott's Core Theorem (C.5) and the fact that a 3-manifold with non-trivial non-spherical boundary has positive first Betti number to show that if N is compact, orientable, and irreducible, and Γ is a finitely generated subgroup of infinite index of $\pi_1(N)$, then $b_1(\Gamma) \geq 1$. (See also [**HoS85**, Lemma 2] and [**Rol14b**, Section 7].)

A standard transfer argument shows that if G is a finite-index subgroup of a group H, then $b_1(G) \geq b_1(H)$. Combining these two facts it follows that if N is compact, orientable, irreducible, with $b_1(N) \geq 1$, then $b_1(\Gamma) \geq 1$ for each finitely generated subgroup Γ of $\pi_1(N)$, i.e., $\pi_1(N)$ is locally indicable.

It follows from [**Hig40**, Appendix] that a free product of finitely many groups is locally indicable if and only if each factor is locally indicable.

(C.20) Burns–Hale [**BHa72**, Corollary 2] showed that a locally indicable group is left-orderable. (The converse does not hold even for 3-manifold groups, see, e.g., [**Bem91**] for the first examples.) Note that left-orderability is not a 'green property,' i.e., there exist compact 3-manifolds with non-left-orderable fundamental groups which admit left-orderable finite-index subgroups; see, for example, [**BRW05**, Proposition 9.1] and [**DPT05**].

A free product of finitely many groups is left-orderable if and only if each factor is left-orderable, see [**Vi49, DS14**] and [**Pas77**, Theorem 13.27]. We refer to the survey paper [**Rol14a**] for more interactions between orderability of fundamental groups and topological properties of 3-manifolds.

(C.21) By [**DKR14**, Section 2.2] every left-orderable group is diffuse. But there are examples of 3-manifold groups that are diffuse but not left-orderable, see [**DKR14**, Appendix].

(C.22) Suppose N is compact, orientable, and irreducible. If N has non-empty incompressible boundary, then N is clearly Haken. It follows from [**Nemd99**, p. 221] that if $N \neq S^1 \times D^2$ and N has toroidal boundary, then each boundary component of N is incompressible and hence N is Haken. It follows from Poincaré duality that if N is closed and $b_1(N) \geq 1$, then $H_2(N;\mathbb{Z}) \neq 0$. Let Σ be an oriented surface representing a non-trivial element $\phi \in H_2(N;\mathbb{Z})$. Since N is irreducible we can assume that Σ has no spherical components and that Σ has no component which bounds a solid torus. Among all such surfaces we take a surface of maximal Euler characteristic. It follows from an extension of the Loop Theorem to properly embedded surfaces (see [**Sco74**, Corollary 3.1] and Theorem 1.3.1) that any component of such a surface is incompressible; thus, N is Haken. (See also [**Hem76**, Lemma 6.6].)

It follows from [**EdL83**, Lemma 5.1] that if ϕ is a primitive fibered class (see (F.5) for the definition), then the corresponding fiber Σ is, up to isotopy, the unique incompressible surface representing ϕ. (See also [**Gra77**, Proposition 8].) On the other hand, if ϕ is not a fibered class, then at times Σ can be represented by an incompressible surface that is unique up to isotopy (see, e.g., [**Whn73, Ly74a, Koi89, CtC93, HiS97, Kak05, GI06, Brt08, Juh08, Ban11a**]), but in general it can not (see, e.g., [**Scf67, Alf70, AS70, Ly74b, Ein76b, Ein77b, STh88, Gus81, Kak91, Kak92, Sak94, Ban11b, Alt12, Rob13, Ban13, HJS13**]).

It also follows from [**EdL83**, Lemma 5.1] that for a fiber Σ corresponding to a primitive fibered class, any incompressible surface in N disjoint from Σ is in fact a parallel copy of Σ. In general, there can be disjoint incompressible surfaces. But Haken's Endlichkeitssatz [**Hak61b**, p. 442], [**BL14**, Theorem 1.3] says that there exists an n (that depends on N) such that any collection of n disjoint incompressible surfaces contains two surfaces that are parallel.

We just showed that any orientable, irreducible N with $b_1(N) \geq 1$ is Haken. The conclusion does not hold without the assumption $b_1(N) \geq 1$. Indeed, the work of Hatcher [**Hat82**] together with [**CJR82, Men84**], [**HaTh85**, Theorem 2(b)], [**FlH82**, Theorem 1.1], [**Lop92**, Theorem A], and [**Lop93**, Theorem A] shows that almost all Dehn surgeries on many classes of 3-manifolds with toroidal boundary are non-Haken, even though most of them are hyperbolic. See also [**Thu79, Oe84, Ag03, BRT12**] for more examples of non-Haken manifolds. We point out that Jaco–Oertel [**JO84**, Theorem 4.3] (see also [**BCT14**]) specified an algorithm to decide whether a given closed irreducible 3-manifold is Haken.

Finally, let N be compact, orientable, irreducible, with non-empty boundary. A properly embedded surface Σ in N is *essential* if Σ is incompressible and not isotopic into a boundary component. It follows from the above that if $H_2(N;\mathbb{Z}) \neq 0$, then N contains an essential closed non-separating surface. On the other hand, Culler–Shalen [**CuS84**, Theorem 1] showed that if $H_2(N;\mathbb{Z}) = 0$, then N contains an essential, separating surface with non-empty boundary. Furthermore, if $H_2(N;\mathbb{Z}) = 0$, then in some cases N will contain an incompressible, closed, non-boundary parallel surface (see [**Ly71, Swp74, Sht85, Gus94, FiM99, LMo99, FiM00, MQ05, Lib09, Ozb09**]) and in some cases it will not (see [**GLit84**, Corollary 1.2], [**HaTh85**], [**Oe84**, Corollary 4], [**Lop93, Mad04, QW04, Ozb08, Ozb10**]).

(C.23) Suppose that N is compact, orientable, irreducible, with empty or toroidal boundary. By work of Allenby–Boler–Evans–Moser–Tang [**ABEMT79**, Theorems 2.9 and 4.7], if N is Haken and not a closed Seifert fibered manifold, then the Frattini group of $\pi_1(N)$ is trivial. On the other hand, if N is a closed Seifert fibered manifold and $\pi_1(N)$ is infinite, then the Frattini group of $\pi_1(N)$

is a (possibly trivial) subgroup of the infinite cyclic subgroup generated by a regular Seifert fiber (see [**ABEMT79**, Lemma 4.6]).

(C.24) By Evans–Moser [**EvM72**, Corollary 4.10], the fundamental group of an irreducible Haken 3-manifold, if non-solvable, has a non-cyclic free subgroup.

(C.25) A group which contains a non-abelian free group is non-amenable. Indeed, any subgroup of an amenable group is amenable [**Pie84**, Propositions 13.3]. On the other hand, non-cyclic free groups are not amenable [**Pie84**, Proposition 14.1]. In contrast, most 3-manifold groups are weakly amenable, see (J.5).

(C.26) Tits [**Tit72**, Corollary 1] showed that a finitely generated group which is linear over $\mathbb{C}$ is either virtually solvable or has a non-cyclic free subgroup; this dichotomy is commonly referred to as the Tits Alternative. (Recall that as in Flowchart 1 we assumed that π is neither finite nor solvable, π is also not virtually solvable by Theorem 1.11.1.)

The combination of the above and of (C.24) shows that the fundamental group of a compact 3-manifold with empty or toroidal boundary is either virtually solvable or contains a non-cyclic free group. This dichotomy is a weak version of the Tits Alternative for fundamental groups of compact 3-manifolds. We refer to (L.2) for a stronger version of this, and to [**Par92, ShW92, KZ07**] for pre-Geometrization results on the Tits Alternative.

Aoun [**Ao11**] showed that 'most' two generator subgroups of a group which is linear over $\mathbb{C}$ and not virtually solvable, are in fact free.

It follows from the above that if N is hyperbolic, then $\pi_1(N)$ has a non-cyclic free subgroup. In fact, Jaco–Shalen [**JS79**, Theorem VI.4.1] showed that if $x, y \in \pi_1(N)$ do not commute and the subgroup Γ of $\pi_1(N)$ generated by x, y has infinite index in $\pi_1(N)$, then Γ is free on the generators x, y. (See also [**JS76**, Lemma 5.4] and [**Rat87**, Corollary 4].) For closed N this result was generalized by Gitik [**Git99a**, Theorem 1].

If N is closed and hyperbolic, there are many other ways of finding non-cyclic free subgroups in $\pi_1(N)$. For example, in this case let Γ be any geometrically finite subgroup of $\pi_1(N)$ (e.g., a cyclic subgroup). It follows from work of Gromov [**Grv87**, 5.3.C] (see also [**Ar01**, Theorem 1]) that then there exists some $g \in \pi_1(N) \setminus \Gamma$ such that the canonical map $\Gamma * \langle g \rangle \to \pi_1(N)$ is injective and the image is a geometrically finite subgroup of $\pi_1(N)$. This result was extended by Martinez-Pedroza [**MP09**, Theorem 1.1].

Finally, a group is called *n-free* if every subgroup generated by at most n elements is free. For a closed, orientable, hyperbolic N such that $\pi_1(N)$ is n-free for $n \in \{3, 4, 5\}$, the results of [**ACS10**, Theorem 9.6] and [**CuS12, Guz12**] give lower bounds on the volume of N. The growth of n-freeness in a filtration of an arithmetic 3-manifold was studied in [**Bel12**].

(C.27) A consequence of the Lubotzky Alternative (cf. [**LuSe03**, Window 9, Corollary 18]) asserts that a finitely generated group which is linear over $\mathbb{C}$ either is virtually solvable or, for any prime p, has infinite virtual first $\mathbb{F}_p$-Betti number. (See also [**Lac09**, Theorem 1.3] and [**Lac11**, Section 3].)

We refer the interested reader to [**CaE11**, Example 5.7], [**Lac09**, Theorems 1.7 and 1.8], [**ShW92, Walb09**], and [**Lac11**, Section 4] for more on the growth of $\mathbb{F}_p$-Betti numbers of finite covers of hyperbolic 3-manifolds. See [**Mes90**, Proposition 3] for a pre-Geometrization result regarding the $\mathbb{F}_p$-homology of finite covers of 3-manifolds.

(C.28) Suppose N is compact. In [**AF13**] it is shown that, for all but finitely many primes p, the group $\pi := \pi_1(N)$ is virtually residually p.

For special types of 3-manifolds one can obtain more precise results:

(a) By [**AF13**, Proposition 2], if N is a graph manifold (i.e., if all its JSJ-components are Seifert fibered manifolds), then for any prime p, π is virtually residually p.

(b) The proof of [**AF13**, Proposition 4.16] shows that if N is a Seifert fibered manifold, then π has a finite-index subgroup that is residually torsion-free nilpotent. By (H.33), torsion-free nilpotent groups are residually p for any prime p. Together with (H.35) this implies that fundamental groups of Seifert fibered manifolds are virtually bi-orderable. (See also (E.2.)

(c) If N is compact, orientable, and fibered, then there is a compact, orientable surface Σ such that $\pi \cong \mathbb{Z} \ltimes F$ where $F = \pi_1(\Sigma)$. The group $\pi_1(\Sigma)$ is residually p (see, e.g., [**Gru57**] or [**Bag62**, p. 424]), and the semidirect product of a residually p group with $\mathbb{Z}$ is virtually residually p (see, e.g., [**AF13**, Corollary 4.32]). See [**AF13**, Corollary 4.32], [**Kob13**] for full details.

For 3-manifolds N such that π is linear over $\mathbb{C}$ (i.e., for hyperbolic N), it already follows from [**Pla68**] that for all but finitely many p, the group π is virtually residually p. (See also [**Weh73**, Theorem 4.7] and [**Nic13**, Theorem 3.1].)

The fundamental groups of many (arguably, most) 3-manifolds are not residually p. For example, [**Sta65**, Lemma 3.1] implies that if $J \subseteq S^3$ is a non-trivial knot, then $\pi_1(S^3 \setminus \nu J)$ is not residually nilpotent, let alone residually p.

(C.29) The well-known argument in (I.2) below can be used to show that a group which is virtually residually p is also residually finite. Residual finiteness of fundamental groups of compact 3-manifolds was first shown by Hempel [**Hem87**] and Thurston [**Thu82a**, Theorem 3.3].

Pre-Geometrization results on the residual finiteness of fundamental groups of knot exteriors were obtained by Mayland, Murasugi, and Stebe [**May72, May74, May75a, May75b, MMi76, Ste68**].

In Section 4.7 we will see that fundamental groups of compact, orientable, irreducible 3-manifolds that are not closed graph manifolds are 'virtually special.' By [**BHPa14**] this implies that fundamental groups of such groups have 'linear residual finiteness growth function.' Loosely speaking this means that non-trivial elements can be detected by 'small' finite quotients.

Residual finiteness of π implies that if we equip π with its profinite topology, then π is homeomorphic to the rationals. See [**ClS84**] for details.

Given an oriented hyperbolic 3-manifold N, the residual finiteness of $\pi_1(N)$ can be seen using congruence subgroups. By (C.7) there is an embedding $\pi_1(N) \hookrightarrow \mathrm{SL}(2, \overline{\mathbb{Q}})$ with discrete image; write $\Gamma := \rho(\pi_1(N))$. We say that a subgroup H of Γ is a *congruence subgroup of* Γ if there exists a ring R which is obtained from the ring of integers of a number field by inverting a finite number of elements and a maximal ideal $\mathfrak{m}$ of R such that $\Gamma \subseteq \mathrm{SL}(2, R)$ and

$$\mathrm{Ker}\left\{\Gamma \to \mathrm{SL}(2, R) \to \mathrm{SL}(2, R/\mathfrak{m})\right\} \leq H.$$

Congruence subgroups have finite index (see, e.g., [**Weh73**, Theorem 4.1]) and the intersection of all congruence subgroups is trivial (see, e.g., [**Mal40**] and [**Weh73**, Theorem 4.3]). This implies that $\pi_1(N) \cong \Gamma$ is residually finite.

Lubotzky [**Lub83**, p. 116] showed that in general not every finite-index subgroup of Γ is a congruence subgroup. We refer to [**Lub95, CLT09, Lac09**] and [**Lac11**, Section 3] for further results.

The fact that fundamental groups of compact 3-manifolds are residually finite together with the Loop Theorem shows in particular that a non-trivial knot admits a finite-index subgroup such that the quotient is not cyclic. Broaddus [**Brs05**] gave an explicit upper bound on the index of such a subgroup in terms of the crossing number of the knot. (See also [**Kup14**].)

(C.30) Mal'cev [**Mal40**] showed that every finitely generated residually finite group is Hopfian. (See also [**Mal65**, Theorem VII].) Here are some related properties: one says that a group π is
   (a) *co-Hopfian* if it is not isomorphic to any proper subgroup of itself;
   (b) *cofinitely Hopfian* if every endomorphism of π whose image is of finite index in π is in fact an automorphism;
   (c) *hyper-Hopfian* if every endomorphism φ of π such that $\varphi(\pi)$ is normal in π and $\pi/\varphi(\pi)$ is cyclic, is in fact an automorphism.

If Σ is a surface then $\pi_1(S^1\times\Sigma) = \mathbb{Z}\times\pi_1(\Sigma)$ is neither co-Hopfian, nor cofinitely Hopfian, nor hyper-Hopfian.
   (a) Suppose N is compact, orientable, and irreducible. If N is closed, then $\pi_1(N)$ is co-Hopfian if and only if N has no finite cover that is either a direct product $S^1\times\Sigma$ or a torus bundle over S^1 [**WY94**, Theorem 8.7]. González-Acuña–Whitten [**GW92**, Theorem 2.5] showed that if N has non-trivial toroidal boundary, then $\pi_1(N)$ is co-Hopfian if and only if $\pi_1(N) \neq \mathbb{Z}^2$ and if no non-trivial Seifert fibered piece of the JSJ-decomposition of N meets ∂N.
   (b) Bridson–Groves–Hillman–Martin [**BGHM10**, Theorems A, C] showed that fundamental groups of hyperbolic 3-manifolds are cofinitely Hopfian, and also that if $J \subseteq S^3$ is not a torus knot, then $\pi_1(S^3 \setminus \nu J)$ is cofinitely Hopfian.
   (c) Silver [**Sil96**] established that if $J \subseteq S^3$ is not a torus knot, then $\pi_1(S^3\setminus\nu J)$ is hyper-Hopfian. (See also [**BGHM10**, Corollary 7.2].)

We refer to [**GW87, Dam91, GW92, GW94, GLW94, WW94, WY99, PV00**] for more details and related results.

(C.31) See [**LyS77**, Theorem IV.4.6] and [**AFW13**] for a proof of the fact that finitely presented residually finite groups have solvable Word Problem. In fact, for 3-manifold groups a more precise statement can be made: the fundamental group of a compact 3-manifold has an exponential Dehn function [**ECHLPT92**]. Waldhausen [**Wan68b**] established the solvability of the Word Problem for fundamental groups of 3-manifolds which are virtually Haken.

(C.32) Hamilton [**Hamb01**] showed that the fundamental group of any orientable 3-manifold is AERF. Earlier results are in [**LoN91**, Theorem 2] and [**AH99**].

It follows from the above that the fundamental groups carried by embedded tori are separable. Long–Niblo [**LoN91**, Theorem 1] proved the related statement that if N is compact, orientable, irreducible, with (not necessarily toroidal) boundary, and X is a connected, incompressible subsurface of ∂N, then $\pi_1(X)$ is separable in $\pi_1(N)$.

(C.33) The Conjugacy Problem was solved for all 3-manifolds with incompressible boundary by Préaux [**Pre05, Pre06**], building on ideas of Sela [**Sel93**].

For the fundamental group of the exterior of an alternating knot a pre-Geometrization solution to the Conjugacy Problem was given in [**LyS77**, Theorem V.8.5].

(C.34) Every algorithm solving the Conjugacy Problem in a given group, applied to the conjugacy class of the identity, also solves the Word Problem.

The combination of (C.28), (C.29) and the above thus shows that the word problem is solvable for $\pi_1(N)$ if N is a compact 3-manifold. A pre-Geometrization solution to the Word Problem for fundamental groups of certain knot complements was given by Weinbaum [**Web71**, Theorem C].

(C.35) By Wilton–Zalesskii [**WZ10**, Theorem A], closed orientable prime 3-manifolds are efficient. Suppose N is prime with toroidal boundary, and denote by W the result of gluing exteriors of hyperbolic knots to the boundary components of N. It follows from Proposition 1.6.2 that the JSJ-tori of W consist of the JSJ-tori of N and the boundary tori of N. Since W is efficient, by [**WZ10**, Theorem A] it follows that N is also efficient.

See also [**AF13**, Chapter 5] for a discussion of the question whether closed orientable prime 3-manifolds are, for all but finitely many primes p, virtually p-efficient. (Here p-efficiency is the natural analogue of efficiency for the pro-p-topology; cf. [**AF13**, Section 5.1].)

(C.36) The Whitehead group of the fundamental group of a compact, orientable, non-spherical irreducible 3-manifold is trivial. This follows from the Geometrization Theorem together with the work of Farrell–Jones [**FJ86**, Corollary 1], Waldhausen [**Wan78a**, Theorem 17.5], Farrell–Hsiang [**FaH81**] and Plotnick [**Plo80**]. We also refer to [**FJ87**] for extensions of this result.

Using this fact and building on [**Tur88**], Kreck–Lück [**KrL09**, Theorem 0.7] showed that every orientation preserving homotopy equivalence $M \to N$ between closed, oriented, connected 3-manifolds, where $\pi_1(N)$ is torsion-free, is homotopic to a homeomorphism.

By [**Coh73**, § 22], homotopy equivalent CW-complexes M and M' are simple homotopy equivalent if $\mathrm{Wh}(\pi_1(M'))$ is trivial. It follows in particular that two compact, orientable, non-spherical irreducible 3-manifolds which are homotopy equivalent are in fact simple homotopy equivalent. On the other hand, homotopy equivalent lens spaces are not necessarily simple homotopy equivalent. See [**Mil66, Coh73, Rou11**] and [**Kir97**, p. 119] for more details.

Continuing earlier investigations by Roushon [**Rou08a, Rou08b**], Bartels–Farrell–Lück [**BFL14**] showed that the fundamental group of any 3-manifold (not necessarily compact) satisfies the Farrell–Jones Conjecture from algebraic K-theory. As pointed out in [**BFL14**, p. 4], the Farrell–Jones Conjecture for fundamental groups of 3-manifolds implies in particular the following, for each 3-manifold group π:

(a) an alternative proof that $\mathrm{Wh}(\pi)$ is trivial if π is torsion-free;
(b) if π is torsion-free, then it satisfies the Kaplansky Conjecture, i.e., the group ring $\mathbb{Z}[\pi]$ has no non-trivial idempotents [**Kay57, Kay70**];
(c) the Novikov Conjecture holds for π.

Matthey–Oyono-Oyono–Pitsch [**MOP08**, Theorem 1.1] showed that the fundamental group of any orientable 3-manifold satisfies the Baum–Connes Conjecture, which gives an alternative proof for the Novikov and Kaplansky Conjectures for 3-manifold groups [**MOP08**, Theorem 1.13].

(C.37) Lott–Lück [**LoL95**, Theorem 0.1] showed that if N is compact, irreducible, non-spherical, with empty or toroidal boundary, then $b_i^{(2)}(N) = b_i^{(2)}(\pi) = 0$ for any i. See also [**Lü02**, Section 4.2]. We refer to the above references for the calculation of L^2-Betti numbers of any compact 3-manifold.

(C.38) Let X be a topological space, $\alpha\colon \pi_1(X) \to \Gamma$ be a morphism to a residually finite group, and $\{\Gamma_n\}$ be a cofinal normal filtration of Γ. Lück [**Lü94**, Theorem 0.1] showed that

$$b_1^{(2)}(X,\alpha) = \lim_{n\to\infty} \frac{b_1\big(\operatorname{Ker}(\pi_1(X) \xrightarrow{\alpha} \Gamma \to \Gamma/\Gamma_n)\big)}{[\Gamma : \Gamma_n]}.$$

This extends earlier work of Gromov [**Grv93**, pp. 20, 231], Kazhdan [**Kaz75**].

Combining this result with (C.37) we see that if N is a compact, irreducible, non-spherical 3-manifold with empty or toroidal boundary, then

$$\lim_{\tilde{N}} \frac{b_1(\tilde{N}; R)}{[N : \tilde{N}]} = 0.$$

We refer to [**Far98**] for a generalization of this result, to [**ClW03**, Theorem 0.1] for more information on the rate of convergence of the limit, and to [**ABBGNRS11**, Théorème 0.1], [**ABBGNRS12**, Corollary 1.4] for a generalization for closed hyperbolic 3-manifolds. Here the assumption that the finite covers are regular is necessary: Girão (see proof of [**Gir14**, Theorem 3.1]) gave an example of a hyperbolic 3-manifold with non-empty boundary together with a cofinal (not necessarily normal) filtration of $\{\pi_n\}$ of $\pi = \pi_1(N)$ with

$$\lim_{n\to\infty} \frac{b_1(\pi_n)}{[\pi : \pi_n]} > 0.$$

See [**BeG04**] for more on limits of Betti numbers in finite irregular covers.

The study of the growth of various complexities of groups (first Betti number, rank, size of torsion homology, etc.) in filtrations of 3-manifold groups has garnered a lot of interest in recent years. See [**BD13, BE06, BV13, BBW02, CD06, Gir14, Gir13, KiS12, Lü13, Lü15, Pf13, Rai12b, Rai13, Sen11, Sen12, Fri14**] for more results.

Remark. In Flowchart 1, statements (C.1)–(C.5) don't rely on the Geometrization Theorem. Statement (C.6) is a variation on the Geometrization Theorem, whereas (C.7)–(C.27) gain their relevance from the Geometrization Theorem. The general statements (C.28)–(C.38) rely directly on the Geometrization Theorem. In particular the results of Hempel [**Hem87**] and Hamilton [**Hamb01**] were proved for 3-manifolds 'for which geometrization works.' After Perelman, these results hold in the above generality.

In most cases it is clear that the properties in Flowchart 1 written in green satisfiy indeed the 'green condition' of (B.4). Thus it suffices to give the following justifications.

(D.1) Theorem 1.9.3 implies that if N' is a finite cover of a compact, orientable, irreducible 3-manifold N with empty or toroidal boundary, then N' is hyperbolic (Seifert fibered, admits non-trivial JSJ-decomposition) if and only if N has the same property.

(D.2) Let π be a group containing subgroup π' of finite index. Suppose π' is linear over the commutative ring R, witnessed by a faithful representation $\pi' \to \mathrm{GL}(n,R)$. Then π acts faithfully on $R[\pi] \otimes_{R[\pi']} R^n \cong R^{n[\pi:\pi']}$ by left-multiplication. Hence π is also linear over R.

(D.3) Niblo [**Nib92**, Proposition 2.2] showed that a finite-index subgroup of π is double-coset separable if and only if π is double-coset separable.

(D.4) Suppose π has a finite-index subgroup π' which is LERF; then π is LERF itself. Indeed, let Γ be a finitely generated subgroup of π. Then the subgroup $\Gamma \cap \pi'$ of π' is separable, i.e., closed (in the profinite topology of) π'. It then follows that $\Gamma \cap \pi'$ is closed in π. Finally Γ, which can be written as a union of finitely many translates of $\Gamma \cap \pi'$, is also closed in π, i.e., Γ is separable in π.

(D.5) In [**Nemc73**] it is shown that if π is a group that admits a finite-index subgroup that is SQ-universal, then π is also SQ-universal.

(D.6) By [**Pie84**, Propositions 13.4] a finite-index supergroup of an amenable group is again amenable.

3.3. Additional results and implications

There are a few arrows and results on 3-manifold groups which can be proved using the Geometrization Theorem, and which we left out of the flowchart. Again, we let N be a 3-manifold.

(E.1) Wilton [**Wil08**, Corollary 2.10] determined the closed 3-manifolds with residually free fundamental group. In particular, it is shown there that if N is compact, orientable, prime, with empty or toroidal boundary, and $\pi_1(N)$ residually free, then N is the product of a circle with a connected surface.

(E.2) Boyer–Rolfsen–Wiest [**BRW05**, Corollary 1.6] established the virtual bi-orderability of the fundamental groups of Seifert fibered manifolds. Perron–Rolfsen [**PR03**, Theorem 1.1], [**PR06**, Corollary 2.4] and Chiswell–Glass–Wilson [**CGW14**, Corollary 2.5] gave many examples of fibered 3-manifolds (i.e., 3-manifolds which fiber over S^1) with bi-orderable fundamental group. In the other direction, Smythe proved that the fundamental group of the trefoil complement is not bi-orderable (see [**Neh74**, p. 228]). Thus not all fundamental groups of fibered 3-manifolds are bi-orderable. See [**CR12, NaR14, CDN14**] and again [**CGW14**, Corollary 2.5] for many more examples, including some fibered hyperbolic 3-manifolds whose fundamental groups are not bi-orderable.

(E.3) A Poincaré duality argument shows that if N is closed and not orientable, then $b_1(N) \geq 1$; see, e.g., [**BRW05**, Lemma 3.3] for full details.

(E.4) Teichner [**Tei97**] showed that if the lower central series of the fundamental group π of a closed 3-manifold stabilizes, then the maximal nilpotent quotient of π is the fundamental group of a closed 3-manifold (and such groups were determined in [**Tho68**, Theorem N]). The lower central series and nilpotent quotients of 3-manifold groups were also studied by Cochran–Orr [**CoO98**, Corollary 8.2], Cha–Orr [**ChO13**, Theorem 1.3], Freedman–Hain–Teichner [**FHT97**, Theorem 3], Putinar [**Pur98**], and Turaev [**Tur82**].

Kawauchi (see [**Kaw89a**, Corollary 4.3] and [**Kaw89b**, Property V]) showed that if N is compact, orientable, with empty or toroidal boundary, then there exists a degree 1 map $N' \to N$ from a hyperbolic 3-manifold N' which, for each n, induces an isomorphism $\pi_1(N')/\pi_1(N')^{(n)} \to \pi_1(N)/\pi_1(N)^{(n)}$. We refer to [**Kaw93, Kaw94, Kaw97**] for related results.

(E.5) Groves–Manning [**GrM08**, Corollary 9.7] and independently Osin [**Osi07**, Theorem 1.1] showed that if N is a hyperbolic 3-manifold with non-empty boundary, then $\pi_1(N)$ is fully residually the fundamental group of a closed hyperbolic 3-manifold. More precisely, if $A \subseteq \pi_1(N)$ is finite, they showed that there exists a hyperbolic Dehn filling M of N such that the induced map $\pi_1(N) \to \pi_1(M)$ is injective when restricted to A.

(E.6) A group is said to have *Property U* if it has uncountably many maximal subgroups of infinite index. Margulis–Soifer [**MrS81**, Theorem 4] showed that every finitely generated group which is linear over $\mathbb{C}$ and not virtually solvable has Property U. Using the linearity of free groups, one can use this to show that every large group also has Property U. Tracing through Flowchart 1 yields that the fundamental group π of every compact, orientable, aspherical 3-manifold with empty or toroidal boundary has Property U, unless π is solvable. It follows from [**GSS10**, Corollary 1.2] that every maximal subgroup of infinite index of the fundamental group of a hyperbolic 3-manifold is infinitely generated.

(E.7) As pointed out in the introduction, in higher dimensions, topology puts no restrictions on the finitely presented groups occurring as fundamental groups of manifolds: for each finitely presented group π and $n \geq 4$, there is a closed n-manifold with fundamental group π. (See [**CZi93**, Theorem 5.1.1], [**De11**, Kapitel III], [**GfS99**, Theorem 1.2.33], [**SeT34**, p. 180] or [**SeT80**, Section 52].)

On the other hand, if one adds further geometric or topological restrictions on the class of manifolds under consideration, one often obtains further restrictions on the type of fundamental groups which can occur.

For example, let π be the fundamental group of a Kähler manifold. If π also happens to be the fundamental group of a closed 3-manifold N, then π is finite. Indeed, by Gromov [**Grv89**] the fundamenal group of a Kähler manifold not a free product of non-trivial groups. By the Kneser Conjecture 2.1.1 this implies that N is prime. Kotschick [**Kot12**, Theorem 4] showed that $vb_1(N) = 0$. It follows from (C.14), (C.15), and (C.17) together with (H.2), (H.5) and (H.13) that π is indeed finite. This result was first obtained by Dimca–Suciu [**DiS09**], with an alternative proof in [**BMS12**, Theorem 2.26]. See [**CaT89, DPS11, FS12, BiM12, Kot13, BiM14**] for other approaches and extensions of these results to quasiprojective groups.

We refer to [**Tau92**, p. 165], [**Gom95**, Theorem 0.1], [**ABCKT96**, Corollary 1.66], [**GfS99**, Theorem 10.2.10], [**FS12**], [**PP12**] and [**ANW13**, Theorem 1] for further results where additional assumptions on the class of manifolds do (not) preclude 3-manifold groups to occur.

(E.8) Ruberman [**Rub01**, Theorem 2.4] compared the behavior of the Atiyah–Patodi–Singer η-invariant [**APS75a, APS75b**] as well as the Chern–Simons invariants [**ChS74**] under finite coverings to give an obstruction for a group to be a 3-manifold group.

(E.9) Using connected sums of lens spaces one shows that any finite abelian group appears as the first homology of a closed orientable 3-manifold. In fact by [**KKo0**, Theorem 6.1] any non-singular symmetric bilinear pairing $H \times H \to \mathbb{Q}/\mathbb{Z}$ over a finite abelian group H appears as the linking pairing of a closed rational homology sphere. On the other hand Reznikov [**Rez97**] and Cavendish [**Cav14**] showed that not every finite metabelian group appears as $\pi_1(N)/\pi_1(N)^{(2)}$ for a closed orientable 3-manifold N. Other potential restrictions on the type of finite solvable groups that can appear as $\pi_1(N)/\pi_1(N)^{(k)}$ for a closed orientable 3-manifold N are in [**Rou04**, Conjecture (0.2)] and [**Cav14**, Question 1].

(E.10) Suppose N is compact, orientable, with no spherical boundary components. De la Harpe and Préaux [**dlHP11**, Proposition 8] showed that if N is neither a Seifert manifold nor a Sol-manifold, then $\pi_1(N)$ is a 'Powers group,' which implies that $\pi_1(N)$ is C^*–simple, by [**Pow75**]. (A group is called *C^*-simple* if it is infinite and if its reduced C^*-algebra has no non-trivial two-sided ideals. We refer to [**Dan96**] for background.)

(E.11) Suppose N is closed, orientable, irreducible, and has n hyperbolic JSJ-components. Weidmann [**Wei02**, Theorem 2] showed that the minimal number of generators of $\pi_1(N)$ is bounded below by $n+1$.

(E.12) Suppose N is compact, orientable, and every loop in N is freely homotopic to a loop in a boundary component. Brin–Johannson–Scott [**BJS85**, Theorem 1.1] showed that there exists a boundary component F such that $\pi_1(F) \to \pi_1(N)$ is surjective. (See also [**MMt79**, §2] with $\rho = 1$.)

(E.13) Let S be a finite generating set of a group π. The *growth function* of (π, S) is

$$n \mapsto a_n(\pi, S) := \#\{\text{elements in } \pi \text{ with word length} \leq n\},$$

where the word length is taken with respect to S. The *exponential growth rate* of (π, S) is then defined as

$$\omega(\pi, S) := \lim_{k\to\infty} \sqrt[n]{\# a_n(\pi, S)}.$$

The *uniform exponential growth rate* of π is defined as

$$\omega(\pi) := \inf\{\omega(\pi, S) : S \text{ finite generating set of } \pi\}.$$

By work of Leeb [**Leb95**, Theorem 3.3] and di Cerbo [**dCe09**, Theorem 2.1] there is a $C > 1$ such that $\omega(\pi_1(N)) > C$ for any closed irreducible 3-manifold N which is not a graph manifold. This result builds on and extends work of Milnor [**Mil68**], Avez [**Av70**], Besson–Courtois–Gallot [**BCG11**] and Bucher–de la Harpe [**BdlH00**]. The growth function was studied in more detail for certain torus knots in [**Sho94, JKS95, Gll99, Mnna11, NTY14**] and [**Mnna12**, Section 14.1], and for Sol-manifolds in [**Pun06, Par07**].

(E.14) The Lickorish-Wallace Theorem states that each closed 3-manifold is the boundary of a smoooth 4-manifold; see [**Lic62, Wac60**] and see also [**Rol90**, p. 277] and [**ELL13**]. Hausmann [**Hau81**, p. 122] showed that given any closed 3-manifold N there exists in fact a smooth 4-manifold W such that $\pi_1(N) \to \pi_1(W)$ is injective. (See also [**FR12**].)

CHAPTER 4

The Work of Agol, Kahn–Markovic, and Wise

The Geometrization Theorem resolves the Poincaré Conjecture and, more generally, the classification of 3-manifolds with finite fundamental group. For 3-manifolds with infinite fundamental group, the Geometrization Theorem can be viewed as asserting that the key problem is to understand hyperbolic 3-manifolds. Even after Perelman's proof, many questions remained about the topology of this central type of 3-manifold, the most important of which had been articulated by Thurston. In this chapter we discuss the resolution of these questions.

We first discuss the Tameness Theorem (Section 4.1), which was proved independently by Agol [**Ag04**] and by Calegari–Gabai [**CaG06**], and implies an essential dichotomy for finitely generated subgroups of hyperbolic 3-manifolds. Then (in Sections 4.2–4.6) we turn to the Virtually Compact Special Theorem of Agol [**Ag13**], Kahn–Markovic [**KM12a**] and Wise [**Wis12a**]. This theorem, together with the Tameness Theorem and further work of Agol [**Ag08**] and Haglund [**Hag08**] and many others, answers many hitherto intractable questions about hyperbolic 3-manifolds. For example, it implies that every hyperbolic 3-manifold is virtually Haken. In Section 4.9 we summarize earlier work on virtual properties of hyperbolic 3-manifolds. We will survey consequences of the Tameness and Virtually Compact Special theorems for the fundamental groups of 3-manifolds in Chapter 5, and for their subgroups in Chapter 6.

A 'compact special' group is the fundamental group of a compact (non-positively curved) 'special cube complex,' a topological space with a very combinatorial flavor. This type of group is intimately intertwined with the more familiar notion of 'right-angled Artin group.' These are introduced in Section 4.3. Wise showed that hyperbolic 3-manifolds with non-empty boundary and more generally, 'most' Haken hyperbolic 3-manifolds, are virtually compact special; see Section 4.4. Agol built on Wise's work to show that in fact *all* hyperbolic 3-manifolds are virtually compact special, a result which in particular implies the famous Virtually Haken conjecture. Both Wise's and Agol's work concerns cube complexes. The link with 3-manifolds is provided by a construction of Sageev. Kahn–Markovic's proof of the Surface Subgroup Conjecture in fact made it possible to apply Sageev's construction to show that all hyperbolic 3-manifolds are cubulable. These results are explained in Sections 4.5 and 4.6.

In generalizing the Virtually Compact Special Theorem beyond the hyperbolic case, a connection with non-positive curvature emerges (see Section 4.7). In Section 4.8, we discuss the case of compact, orientable 3-manifolds with non-toroidal boundary. Finally, in Section 4.9, we survey some of the enormous body of work addressing Thurston's questions that were at last resolved by the Virtually Compact Special Theorem.

4.1. The Tameness Theorem

In 2004, Agol [**Ag04**] and Calegari–Gabai [**CaG06**, Theorem 0.4] independently proved the following theorem, which was first conjectured by Marden [**Man74**] in 1974:

THEOREM 4.1.1 (Tameness Theorem). *Let N be a hyperbolic 3-manifold, not necessarily of finite volume, whose fundamental group is finitely generated. Then N is topologically tame, i.e., homeomorphic to the interior of a compact 3-manifold.*

We refer to [**Cho06, Som06a, Cal07, Cay08, Gab09, Bow10, Man07**] for further details regarding the statement and alternative approaches to the proof. We especially refer to [**Cay08**, Section 6] for a detailed discussion of earlier results leading towards the proof of the Tameness Theorem. It is not known whether the conclusion of the Tameness Theorem also holds if N is a non-positively curved manifold (see, e.g., [**Bek13**, Question 2.6]), but some positive evidence is given by [**Bow10**, Theorem 1.4], [**Cay08**, Theorem 9.2] and [**BGS85**, Appendix 2]. On the other hand the conclusion of the Tameness Theorem does not hold for all 3-manifolds, see, e.g., [**Whd35b**], [**STu89**, Example 3], and [**FrG07**, Remark 0.5].

In the context of this book, the main application of the Tameness Theorem is the Subgroup Tameness Theorem below. To formulate it requires a few more definitions.

(1) A *surface group* is the fundamental group of a closed, orientable surface of genus at least 1.
(2) Let Γ be a *Kleinian group*, i.e., a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$. If Γ acts cocompactly on the convex hull of its limit set, then Γ is called *geometrically finite*; see [**LoR05**, Chapter 3] for details. (Note that a geometrically finite Kleinian group is necessarily finitely generated.) Now let N be a hyperbolic 3-manifold. We can identify $\pi_1(N)$ with a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$, up to conjugation and complex conjugation; see [**Shn02**, Section 1.6] and (C.7). A subgroup of $\pi_1(N)$ is *geometrically finite* if, viewed as subgroup of $\mathrm{PSL}(2,\mathbb{C})$, it is a geometrically finite Kleinian group. We refer to [**Bow93**] for a discussion of various different equivalent definitions of 'geometrically finite.' An incompressible surface $\Sigma \subseteq N$ is *geometrically finite* if $\pi_1(\Sigma)$ is a geometrically finite subgroup of $\pi_1(N)$.
(3) A 3-manifold is *fibered* if it admits the structure of a surface bundle over S^1. By a *surface fiber* in a 3-manifold N we mean the fiber of a surface bundle $N \to S^1$. Let Γ be a subgroup of $\pi_1(N)$. We say that Γ is a *surface fiber subgroup* of N if there exists a surface fiber Σ such that $\Gamma = \pi_1(\Sigma)$, and that Γ is a *virtual surface fiber subgroup* if N admits a finite cover $N' \to N$ such that $\Gamma \subseteq \pi_1(N')$ and Γ is a surface fiber subgroup of N'.

Now we can state the Subgroup Tameness Theorem, which follows from combining the Tameness Theorem with Canary's Covering Theorem. (See [**Cay94**, Section 4], [**Cay96**] and [**Cay08**, Corollary 8.1].)

THEOREM 4.1.2 (Subgroup Tameness Theorem). *Let N be a hyperbolic 3-manifold and let Γ be a finitely generated subgroup of $\pi_1(N)$. Then either*

(1) Γ *is a virtual surface fiber group, or*
(2) Γ *is geometrically finite.*

The importance of this theorem will become fully apparent in Sections 5 and 6.

4.2. The Virtually Compact Special Theorem

In his landmark 1982 article [**Thu82a**], Thurston posed twenty-four questions, which illustrated the limited understanding of hyperbolic 3-manifolds at that point. These questions guided research into hyperbolic 3-manifolds in the following years. Otal's discussion [**Ot14**] of Thurston's original paper shows that huge progress towards answering these questions has been made since. For example, Perelman's proof of the Geometrization Theorem answered Thurston's Question 1 and the proof by Agol and Calegari–Gabai of the Tameness Theorem answered Question 5.

By early 2012, all but five of Thurston's questions had been answered. Of the remaining open problems, Question 23 plays a special rôle: Thurston [**Thu82a**] conjectured that not all volumes of hyperbolic 3-manifolds are rationally related, i.e., there exist closed hyperbolic 3-manifolds such that the ratio of volumes is not a rational number. (See [**Mil82**] for a related conjecture.) This is a very difficult question which in nature is much closer to deep problems in number theory than to topology or differential geometry. We list the other four questions (with the original numbering):

QUESTIONS 4.2.1. (Thurston, 1982)

(15) *Are fundamental groups of hyperbolic* 3*-manifolds LERF?*
(16) *Is every hyperbolic* 3*-manifold virtually Haken?*
(17) *Does every hyperbolic* 3*-manifold have a finite-sheeted cover with positive first Betti number?*
(18) *Is every hyperbolic* 3*-manifold virtually fibered?*

(It is clear that a positive answer to Question 18 implies a positive answer to Question 17, and in (C.22) we saw that a positive answer to Question 17 implies a positive answer to Question 16.) There has been a tremendous effort to resolve these four questions over the last three decades. (See Section 4.9 for an overview of previous results.) Nonetheless, progress had been slow for the better part of the period. In fact opinions on Question 18 were split. Regarding this particular question, Thurston himself famously wrote 'this dubious-sounding question seems to have a definite chance for a positive answer' [**Thu82a**, p. 380]. According to [**Gab86**] the 'question was upgraded in 1984' to the 'Virtual Fibering Conjecture,' i.e., it was conjectured by Thurston that the question should be answered in the affirmative.

A stunning burst of collective creativity during the years 2007–2012 lead to the following theorem. It was proved by Agol [**Ag13**], Kahn–Markovic [**KM12a**] and Wise [**Wis12a**], with major contributions coming from Agol–Groves–Manning [**Ag13**], Bergeron–Wise [**BeW12**], Haglund–Wise [**HaW08, HaW12**], Hsu–Wise [**HsW12**], and Sageev [**Sag95, Sag97**].

THEOREM 4.2.2 (Virtually Compact Special Theorem). *The fundamental group of each hyperbolic* 3*-manifold N is virtually compact special.*

REMARKS.

(1) We give the definition of 'virtually compact special' in Section 4.3. In that section we will also state the theorem of Haglund–Wise (see Corollary 4.3.4) which gives an alternative formulation of the Virtually Compact Special Theorem in terms of subgroups of right-angled Artin groups.
(2) In the case that N is closed and admits a geometrically finite surface, a proof was first given by Wise [**Wis12a**, Theorem 14.1]. Wise also gave a proof in the case that N has non-empty boundary (see Theorem 4.4.8). Finally, for the case that N is closed and does not admit a geometrically finite surface, the decisive ingredients of the proof were given by the work of Kahn–Markovic [**KM12a**] and Agol [**Ag13**]. The latter builds heavily on the ideas and results of [**Wis12a**]. See Flowchart 2 on p. 66 for further details.

(3) Recall that, according to our conventions, a hyperbolic 3-manifold is assumed to be of finite volume. In fact the theorems of Agol [**Ag13**, Theorems 9.1 and 9.2] and Wise [**Wis12a**, Theorem 14.29] deal precisely with that case. But from these theorems one can also deduce that the fundamental group of any compact hyperbolic 3-manifold with (possibly non-toroidal) incompressible boundary is virtually compact special. This is well known to the experts, but as far as we are aware does not appear in the literature. In Section 4.8 below we explain how to deduce the infinite-volume case from Wise's results.

We discuss consequences of the Virtually Compact Special Theorem in detail in Section 5, but as an *amuse-bouche* we mention that it gives affirmative answers to Thurston's Questions 15–18. More precisely, Theorem 4.2.2 together with the Tameness Theorem, work of Haglund [**Hag08**], Haglund–Wise [**HaW08**] and Agol [**Ag08**] yields:

COROLLARY 4.2.3. *If N is a hyperbolic 3-manifold, then*

(1) $\pi_1(N)$ *is LERF;*
(2) N *is virtually Haken;*
(3) $vb_1(N) = \infty$*; and*
(4) N *is virtually fibered.*

Flowchart 2 summarizes the various contributions to the proof of Theorem 4.2.2. It can also be viewed as a guide to the next sections. We use the following color code.

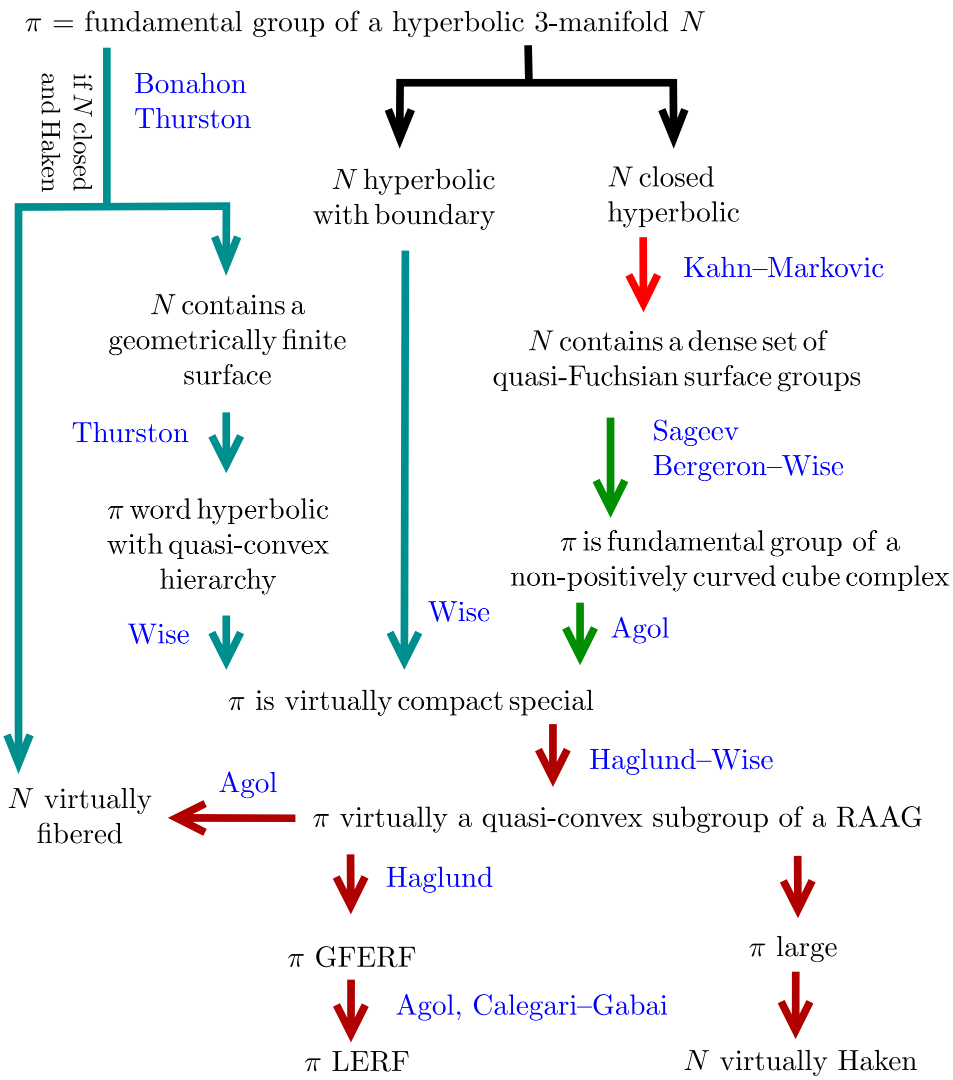
(1) Turquoise arrows correspond to Section 4.4.
(2) The red arrow is treated in Section 4.5.
(3) The green arrows are covered in Section 4.6.
(4) Finally, the brown arrows correspond to the consequences of Theorem 4.2.2. They are treated in detail in Section 5.

4.3. Special cube complexes

The idea of applying non-positively curved cube complexes to the study of 3-manifolds originated with the work of Sageev [**Sag95**]. The definition of a *special* cube complex given by Haglund–Wise was a major step forward, and sparked the recent surge of activity [**HaW08**]. In this section, we give rough definitions that are designed to give a flavor of the material. The reader is referred to [**HaW08**] for a precise treatment. For most applications, Corollary 4.3.3 or Corollary 4.3.4 can be taken as a definition of a 'special group' and a 'compact special group.' See also [**Bea14, Sag12, Cal13b, Ag14**] for an exposition of some of the subsequent definitions, results and proofs.

A *cube complex* is a finite-dimensional cell complex X in which each cell is a cube and the attaching maps are combinatorial isomorphisms. We also impose the condition, whose importance was brought to the fore by Gromov, that X should admit a locally CAT(0) (i.e., non-positively curved) metric. One of the attractions of cube complexes is that this condition can be phrased purely combinatorially. Before we state Gromov's theorem we point out that the link of a vertex in a cube complex naturally has the structure of a simplicial complex. Recall that a simplicial complex X is a *flag* if every subcomplex that is isomorphic to the boundary of an n-simplex (for $n \geq 2$) is the boundary of an n-simplex in X.

THEOREM 4.3.1 (Gromov's Link Condition). *A cube complex admits a non-positively curved metric if and only if the link of each vertex is flag.*



Flowchart 2. The Virtually Compact Special Theorem.

This theorem is due originally to Gromov [**Grv87**]. See also [**BrH99**, Theorem II.5.20] for a proof, as well as many more details about CAT(0) metric spaces and cube complexes. The next concept is due to Salvetti [**Sal87**].

Example (Salvetti complexes). Let Σ be a (finite) graph. We build a cube complex S_Σ as follows:

(1) S_Σ has a single 0-cell x_0;
(2) S_Σ has one (oriented) 1-cell e_v for each vertex v of Σ;
(3) S_Σ has a square 2-cell with boundary reading $e_u e_v \bar{e}_u \bar{e}_v$ whenever u and v are joined by an edge in Σ;
(4) for $n \geq 3$, the n-skeleton is defined inductively—attach an n-cube to any subcomplex isomorphic to the boundary of n-cube which does not already bound an n-cube.

It is an easy exercise to check that S_Σ satisfies Gromov's Link Condition and hence is non-positively curved.

DEFINITION. Given a finite graph Σ with vertices $v_1, \dots, v_n$, the corresponding *right-angled Artin group* (*RAAG*) A_Σ is defined as

$$A_\Sigma = \big\langle v_1, \dots, v_n \,\big|\, [v_i, v_j] = 1 \text{ if } v_i \text{ and } v_j \text{ are connected by an edge of } \Sigma \big\rangle.$$

Note that here the definition of A_Σ specifies a generating set. Clearly the fundamental group of the Salvetti complex S_Σ is canonically isomorphic to the RAAG A_Σ.

Right-angled Artin groups were introduced by Baudisch [**Bah81**] under the name *semi-free groups*, but they are also sometimes referred to as *graph groups* or *free partially commutative groups*. We refer to [**Cha07**] for a very readable survey on RAAGs.

Cube complexes have natural immersed codimension-one subcomplexes, called *hyperplanes*. Identifying an n-cube C in X with $[-1, 1]^n$, a hyperplane of C is any intersection of C with a coordinate hyperplane of $\mathbb{R}^n$. We then glue together hyperplanes in adjacent cubes whenever they meet, to get the hyperplanes of $\{Y_i\}$ of X, which naturally immerse into X. Pulling back the cubes in which the cells of Y_i land defines an I-bundle N_i over Y_i, which also has a natural immersion $\iota_i \colon N_i \to X$. This I-bundle has a natural boundary ∂N_i, which is a 2-to-1 cover of Y_i, and we let $N_i^\circ = N_i \setminus \partial N_i$.

Henceforth, although it will sometimes be convenient to consider non-compact cube complexes, we will always assume that the cube complexes we consider have only finitely many hyperplanes.

Using this language, here is a list of pathologies for hyperplanes in cube complexes.

(1) A hyperplane Y_i is *one-sided* if $N_i \to Y_i$ is not a product bundle; otherwise Y_i is *two-sided*.
(2) A hyperplane Y_i is *self-intersecting* if the restriction of ι_i to Y_i is not injective.
(3) A hyperplane Y_i is *directly self-osculating* if there are vertices $x \neq y$ in the same component of ∂N_i such that $\iota_i(x) = \iota_i(y)$ but, for some small neighborhoods $B_\varepsilon(x)$ and $B_\varepsilon(y)$, the restriction of ι_i to $(B_\varepsilon(x) \sqcup B_\varepsilon(y)) \cap N_i^\circ$ is injective.
(4) A pair of distinct hyperplanes Y_i, Y_j is *inter-osculating* if Y_i, Y_j intersect and osculate; that is, the map $Y_i \sqcup Y_j \to X$ is not an embedding and there are vertices $x \in \partial N_i$ and $y \in \partial N_j$ such that $\iota_i(x) = \iota_j(y)$ but, for some small neighborhoods $B_\varepsilon(x)$ and $B_\varepsilon(y)$, the restriction of $\iota_i \sqcup \iota_j$ to $(B_\varepsilon(x) \cap N_i^\circ) \sqcup (B_\varepsilon(y) \cap N_j^\circ)$ is an injection.

In Figure 1 we give a schematic illustration of directly self-osculating and inter-osculating hyperplanes in a cube complex.

DEFINITION (Haglund–Wise [**HaW08**]). A cube complex is *special* if none of the above pathologies occur. (This is the definition of *A-special* from [**HaW08**]. Their definition of a *special* cube complex is slightly less restrictive. But these two definitions agree up to passing to finite covers, so the two notions of 'virtually special' coincide.)

DEFINITION. The *hyperplane graph* of a cube complex X is the graph $\Sigma(X)$ with vertex-set equal to the hyperplanes of X, and with two vertices joined by an edge if and only if the corresponding hyperplanes intersect.

Suppose every hyperplane of X is two-sided; then there is a natural *typing map* $\phi_X \colon X \to S_{\Sigma(X)}$, which we now describe. Each 0-cell of X maps to the unique 0-cell x_0 of $S_{\Sigma(X)}$. Each 1-cell e crosses a unique hyperplane Y_e of X; ϕ_X maps e to

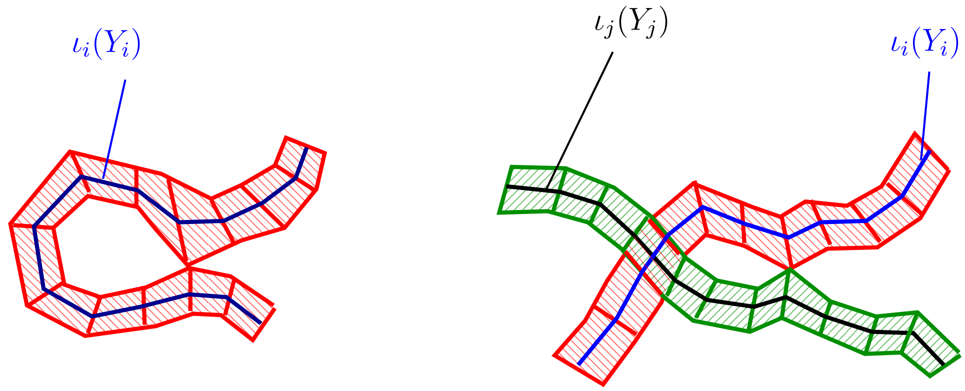


Figure 1. Directly self-osculating and inter-osculating hyperplanes.

the 1-cell e_{Y_e} of $S_{\Sigma(X)}$ that corresponds to the hyperplane Y_e, and the two-sided-ness hypothesis ensures that orientations can be chosen consistently. Finally, ϕ_X is defined inductively on higher dimensional cubes: a higher-dimensional cube C is mapped to the unique cube of $S_{\Sigma(X)}$ with boundary $\phi_X(\partial C)$.

The key observation of [**HaW08**] is that pathologies (2)–(4) above correspond exactly to the failure of the map ϕ_X to be a local isometry. We sketch the argument. For each 0-cell x of X, the typing map ϕ_X induces a map of links $\phi_{X*}\colon \mathrm{lk}(x) \to \mathrm{lk}(x_0)$. This map ϕ_{X*} embeds $\mathrm{lk}(x)$ as an isometric subcomplex of $\mathrm{lk}(x_0)$. Indeed, if ϕ_x identifies two 0-cells of $\mathrm{lk}(x)$, then we have a self-intersection or a direct self-osculation; likewise, if there are 0-cells u, v of $\mathrm{lk}(x)$ that are not joined by an edge but $\phi_{X*}(u)$ and $\phi_{X*}(u)$ are joined by an edge in $\mathrm{lk}(x_0)$, then there is an inter-osculation.

Theorem 4.3.2 (Haglund–Wise). *A non-positively curved cube complex X is special if and only if there is a graph Σ and a local isometry $X \to S_\Sigma$.*

This is one direction of [**HaW08**, Theorem 4.2]; the other direction of the theorem is a straightforward consequence of the results of [**HaW08**].

Let Σ be a graph and $\phi\colon X \to S_\Sigma$ be a local isometry. Lifting ϕ to universal covers, we obtain a genuine isometric embedding $\widetilde{X} \hookrightarrow \widetilde{S}_\Sigma$. So ϕ induces an injection $\pi_1(X) \to A_\Sigma$. Since a covering space of a special cube complex is itself a special cube complex, Theorem 4.3.2 thus yields a characterization of subgroups of RAAGs.

Definition. A finitely generated group is called *special* (respectively, *compact special*) if it is the fundamental group of a non-positively curved special cube complex (respectively, a compact, non-positively curved special cube complex).

Corollary 4.3.3. *Every special group is a subgroup of a RAAG. Conversely, every subgroup of a RAAG is the fundamental group of a special non-positively curved cube complex.*

Proof. Haglund–Wise [**HaW08**, Theorem 1.1] showed that every special group has a finite-index subgroup which is a subgroup of a RAAG. Indeed, suppose that π is the fundamental group of a special cube complex X. Take a graph Σ and a local isometry $\phi\colon X \to S_\Sigma$, by Theorem 4.3.2. The induced map on universal covers $\widetilde{\phi}\colon \widetilde{X} \to \widetilde{S}_\Sigma$ is then an isometry onto a convex subcomplex of $\widetilde{S}_\Sigma$ [**HaW08**, Lemma 2.11]. It follows that ϕ_* is injective.

For the partial converse, if π is a subgroup of a RAAG A_Σ, then π is the fundamental group of a covering space X of S_Σ; the Salvetti complex S_Σ is special and so, by [**HaW08**, Corollary 3.8], is X. □

Arbitrary subgroups of RAAGs may exhibit quite wild behavior. However, if the cube complex X is compact, then $\pi_1(X)$ turns out to be a quasi-convex subgroup of a RAAG, and hence much better behaved.

DEFINITION. Let X be a geodesic metric space. A subspace Y of X is said to be *quasi-convex* if there exists $\kappa \geq 0$ such that any geodesic in X with endpoints in Y is contained within the κ-neighborhood of Y.

DEFINITION. Let π be a group with a fixed generating set S. A subgroup of π is said to be *quasi-convex* (with respect to S) if it is a quasi-convex subspace of $\mathrm{Cay}_S(\pi)$, the Cayley graph of π with respect to the generating set S.

In general the notion of quasi-convexity depends on the choice of generating set S. The definition of a RAAG as given above specifies a generating set; we always take this given choice of generating set when we talk about a quasi-convex subgroup of a RAAG.

COROLLARY 4.3.4. *A group is compact special if and only if it is a quasi-convex subgroup of a right-angled Artin group.*

PROOF. Let π be the fundamental group of a compact special cube complex X. Just as in the proof of Corollary 4.3.3, there is a graph Σ and a map of universal covers $\tilde{\phi}\colon \widetilde{X} \to \widetilde{S}_\Sigma$ that maps $\widetilde{X}$ isometrically onto a convex subcomplex of $\widetilde{S}_\Sigma$. It follows from [**Hag08**, Corollary 2.29] that $\widetilde{\phi}_*\pi_1(\widetilde{X})$ is a quasi-convex subgroup of $\pi_1(S_\Sigma) = A_\Sigma$ because $\pi_1(\widehat{X})$ acts cocompactly on $\widetilde{X}$.

For the converse let π be a subgroup of a RAAG A_Σ. As in the proof of Corollary 4.3.3, π is the fundamental group of a covering space X of S_Σ. By [**Hag08**, Corollary 2.29], π acts cocompactly on a convex subcomplex $\widetilde{Y}$ of the universal cover of S_Σ. The quotient $Y = \widetilde{Y}/\pi$ is a locally convex, compact subcomplex of X and so is special, by [**HaW08**, Corollary 3.9]. □

4.4. Haken hyperbolic 3-manifolds: Wise's Theorem

In this section, we discuss Wise's work on hyperbolic groups with a quasi-convex hierarchy. One of its most striking consequences is that Haken hyperbolic 3-manifolds are virtually fibered. The starting point for this implication is the following theorem of Bonahon [**Bon86**] and Thurston, which is a special case of the Tameness Theorem. (See also [**CEG87, CEG06**].) Geometrically finite surfaces were introduced in Section 4.1.

THEOREM 4.4.1 (Bonahon–Thurston). *Let N be a closed hyperbolic 3-manifold and let $\Sigma \subseteq N$ be an incompressible connected surface. Then either Σ lifts to a surface fiber in a finite cover, or Σ is geometrically finite.*

Hence a closed hyperbolic Haken manifold is virtually fibered or admits a geometrically finite surface. Also, note that by the argument of (C.18) and Theorem 4.4.1, every hyperbolic 3-manifold N with $b_1(N) \geq 2$ contains a geometrically finite surface.

Let N be a closed, hyperbolic 3-manifold that contains a geometrically finite surface. Thurston proved that N in fact admits a hierarchy of geometrically finite surfaces [**Cay94**, Theorem 2.1]. In order to link up with Wise's results we need to recast

Thurston's result in the language of geometric group theory. The following notion was introduced by Gromov [**Grv81b, Grv87**]. See [**BrH99**, Section III.Γ.2], and the references therein, for details.

Definition. A group is called *word-hyperbolic* if it acts properly discontinuously and cocompactly by isometries on a Gromov-hyperbolic space.

For a word-hyperbolic group π, the quasi-convexity of a subgroup of π does not depend on the choice of generating set [**BrH99**, Corollary III.Γ.3.6], so we may speak unambiguously of a quasi-convex subgroup of π.

Next, we introduce the class $\mathcal{QH}$ of *groups with a quasi-convex hierarchy.*

Definition. The class $\mathcal{QH}$ is defined to be the smallest class of finitely generated groups that is closed under isomorphism and satisfies the following properties.

(1) $\{1\} \in \mathcal{QH}$.
(2) If $A, B \in \mathcal{QH}$ and the inclusion map $C \hookrightarrow A *_C B$ is a quasi-isometric embedding, then $A *_C B \in \mathcal{QH}$.
(3) If $A \in \mathcal{QH}$ and the inclusion map $C \hookrightarrow A*_C$ is a quasi-isometric embedding, then $A*_C \in \mathcal{QH}$.

A finitely generated subgroup of a word-hyperbolic group is quasi-isometrically embedded if and only if it is quasi-convex, which justifies the terminology. (See, e.g., [**BrH99**, Corollary III.Γ.3.6].)

The next proposition makes it possible to go from hyperbolic 3-manifolds to the purely group-theoretic realm.

Proposition 4.4.2. *Let N be a closed hyperbolic 3-manifold. Then*

(1) $\pi = \pi_1(N)$ *is word-hyperbolic;*
(2) *a subgroup of π is geometrically finite if and only if it is quasi-convex;*
(3) *if N has a hierarchy of geometrically finite surfaces, then $\pi_1(N) \in \mathcal{QH}$.*

Proof. For the first statement, note that $\mathbb{H}^3$ is Gromov-hyperbolic and so fundamental groups of closed hyperbolic manifolds are word-hyperbolic; see [**BrH99**] for details. We refer to [**Swp93**, Theorem 1.1 and Proposition 1.3] and [**KaS96**, Theorem 2] for proofs of the second statement. The second statement implies the third. □

We thus obtain a reinterpretation of the aforementioned theorem of Thurston:

Theorem 4.4.3 (Thurston). *The fundamental group of each closed, hyperbolic 3-manifold containing a geometrically finite surface is word-hyperbolic and in $\mathcal{QH}$.*

The main theorem of [**Wis12a**], Theorem 13.3, concerns word-hyperbolic groups with a quasi-convex hierarchy.

Theorem 4.4.4 (Wise). *Word-hyperbolic groups in $\mathcal{QH}$ are virtually compact special.*

We immediately obtain the following corollary.

Corollary 4.4.5. *If N is a closed hyperbolic 3-manifold that contains a geometrically finite surface, then $\pi_1(N)$ is virtually compact special.*

The proof of Theorem 4.4.4 is beyond the scope of this book. However, we will state two of the most important ingredients here. Recall that a subgroup H of a group G is called *malnormal* if $gHg^{-1} \cap H = \{1\}$ for every $g \in G \setminus H$. A finite set $\mathcal{H}$ of subgroups of G is called *almost malnormal* if for all $g \in G$ and $H, H' \in \mathcal{H}$ with $g \notin H$ or $H \neq H'$, one has $|gHg^{-1} \cap H'| < \infty$.

The first ingredient is the Malnormal Special Combination Theorem of Haglund–Hsu–Wise, which is a gluing theorem for virtually compact special groups. The statement we give is Theorem 11.3 of [**Wis12a**], which is a consequence of the main theorems of [**HaW12**] and [**HsW12**].

Theorem 4.4.6 (Haglund–Hsu–Wise). *Let $G = A *_C B$ (respectively $G = A*_C$) be word-hyperbolic, where C is almost malnormal and quasiconvex and A, B are (respectively A is) virtually compact special. Then G is virtually compact special.*

The second ingredient is Wise's Malnormal Special Quotient Theorem [**Wis12a**, Theorem 12.3]. (An alternative proof of this theorem is in [**AGM14**, Corollary 2.8].) This asserts the profound fact that the result of a (group-theoretic) Dehn filling on a virtually compact special word-hyperbolic group is still virtually compact special, for all sufficiently deep (in a suitable sense) fillings.

Theorem 4.4.7 (Wise). *Suppose π is a word-hyperbolic and virtually compact special group, and $\{H_1, \dots, H_n\}$ is an almost malnormal family of subgroups of π. Then there are finite-index subgroups K_i of H_i such that, for all finite-index subgroups L_i of K_i, the quotient $\pi/\langle\langle L_1, \dots, L_n\rangle\rangle$ is word-hyperbolic and virtually compact special.*

Wise also proved a generalization of Theorem 4.4.4 to the case of certain relatively hyperbolic groups, from which he deduces the corresponding result in the cusped case [**Wis12a**, Theorem 16.28 and Corollary 14.16].

Theorem 4.4.8 (Wise). *The fundamental group of each non-closed hyperbolic 3-manifold is virtually compact special.*

This last theorem relies on extending some of Wise's techniques from the word-hyperbolic case to the relatively hyperbolic case. Some foundational results for the relatively hyperbolic case were proved in [**HrW14**].

4.5. Quasi-Fuchsian surface subgroups: the work of Kahn and Markovic

As discussed in Section 4.4, Wise's work applies to hyperbolic 3-manifolds with a geometrically finite hierarchy. A non-Haken 3-manifold, on the other hand, by definition has no hierarchy. Likewise, although Haken hyperbolic 3-manifolds without a geometrically finite hierarchy are virtually fibered by Theorem 4.4.1, Thurston's Question 15 (LERF), as well as other important open problems such as largeness, do not follow from Wise's theorems in this case.

The starting point for dealing with hyperbolic 3-manifolds without a geometrically finite hierarchy is provided by Kahn and Markovic's proof of the Surface Subgroup Conjecture. More precisely, as a key step towards answering Thurston's question in the affirmative, Kahn–Markovic [**KM12a**] showed that the fundamental group of any closed hyperbolic 3-manifold contains a surface group. In fact they proved a significantly stronger statement. In order to state their theorem precisely, we need two additional definitions. In the following, N is a closed hyperbolic 3-manifold.

(1) We refer to [**KAG86**, p. 4] and [**KAG86**, p. 10] for the definition of a *quasi-Fuchsian surface group*. A surface subgroup Γ of $\pi_1(N)$ is quasi-Fuchsian if and only if it is geometrically finite [**Oh02**, Lemma 4.66]. (If N has cusps then we need to add the condition that Γ has no 'accidental' parabolic elements.)

(2) We fix an identification of $\pi_1(N)$ with a discrete subgroup of $\operatorname{Isom}(\mathbb{H}^3)$. We say that N *contains a dense set of quasi-Fuchsian surface groups* if for each great circle C of $\partial\mathbb{H}^3 = S^2$ there exists a sequence of π_1-injective immersions $\iota_i\colon \Sigma_i \to N$ of surfaces Σ_i such that the following hold:
   (a) for each i the group $(\iota_i)_*(\pi_1(\Sigma_i))$ is a quasi-Fuchsian surface group,
   (b) the sequence $(\partial\Sigma_i)$ converges to C in the Hausdorff metric on $\partial\mathbb{H}^3$.

We are now able to state the theorem of Kahn–Markovic [**KM12a**]. See also [**Ber12**] and [**Han14**]; this particular formulation is [**Ber12**, Théorème 5.3].

THEOREM 4.5.1 (Kahn–Markovic). *Every closed hyperbolic* 3*-manifold contains a dense set of quasi-Fuchsian surface groups.*

The theme that closed hyperbolic 3-manifolds contain a wealth of quasi-Fuchsian surface was further developed in [**KM12b**] and [**LMa13**].

4.6. Agol's Theorem

The following theorem of Bergeron–Wise [**BeW12**, Theorem 1.4], building extensively on work of Sageev [**Sag95, Sag97**], makes it possible to approach hyperbolic 3-manifolds via non-positively curved cube complexes.

THEOREM 4.6.1 (Sageev, Bergeron–Wise). *Let N be a closed hyperbolic* 3*-manifold which contains a dense set of quasi-Fuchsian surface groups. Then $\pi_1(N)$ is also the fundamental group of a compact non-positively curved cube complex.*

In the previous section we saw that Kahn–Markovic showed that every closed hyperbolic 3-manifold satisfies the hypothesis of the theorem.

The next theorem was conjectured by Wise [**Wis12a**] and proved by Agol [**Ag13**].

THEOREM 4.6.2 (Agol). *If the fundamental group π of a compact, non-positively curved cube complex is word-hyperbolic, then π is virtually compact special.*

The proof of Theorem 4.6.2 relies heavily on results in the appendix to [**Ag13**], which are due to Agol, Groves and Manning. The results of this appendix extend the techniques of [**AGM09**] to word-hyperbolic groups with torsion, and combine them with the Malnormal Special Quotient Theorem 4.4.7. We refer to [**Bea14**, Section 8] for a summary of Agol's proof and to [**Ber14**] for another account of this proof.

The combination of Theorems 4.5.1, 4.6.1 and 4.6.2 now implies Theorem 4.2.2 for closed hyperbolic 3-manifolds.

4.7. 3-manifolds with non-trivial JSJ-decomposition

Although a grasp of the hyperbolic case is key to an understanding of all 3-manifolds, a good understanding of hyperbolic 3-manifolds and of Seifert fibered manifolds does not necessarily immediately lead to answers to questions on 3-manifolds with non-trivial JSJ-decomposition. For example, by (C.7) the fundamental group of a hyperbolic 3-manifold is linear over $\mathbb{C}$, but it is still an open question whether the fundamental group of *any* closed irreducible 3-manifold is linear. (See Section 7.3.1 below.)

In the following we say that a 3-manifold with empty or toroidal boundary is *non-positively curved* if its interior admits a complete non-positively curved Riemannian metric. Furthermore, a compact, orientable, irreducible 3-manifold with empty or toroidal boundary is called a *graph manifold* if all of its JSJ-components are Seifert fibered manifolds. In particular Seifert fibered manifolds are graph manifolds. These concepts are related the following theorem due to Leeb [**Leb95**].

THEOREM 4.7.1 (Leeb). *Let N be an irreducible 3-manifold with empty or toroidal boundary. If N is not a closed graph manifold, then N is non-positively curved.*

The question of which closed graph manifolds are non-positively-curved was treated in detail by Buyalo and Svetlov [**BuS05**]. The following theorem is due to Liu [**Liu13**].

THEOREM 4.7.2 (Liu). *Let N be an aspherical graph manifold. Then $\pi_1(N)$ is virtually special if and only if N is non-positively curved.*

REMARKS.

(1) Liu [**Liu13**], building on the ideas and results of [**Wis12a**], proved Theorem 4.7.2 in the case where the JSJ-decomposition of N is non-trivial. The case that N is a Seifert fibered 3-manifold is well known to the experts and follows 'by inspection.' More precisely, let N be an aspherical Seifert fibered 3-manifold. If $\pi = \pi_1(N)$ is virtually special, then by the arguments of Section 5 (see (H.5), (H.17) and (H.18)) the group π virtually retracts onto each of its infinite cyclic subgroups. This implies easily that π is virtually special if and only if its underlying geometry is either Euclidean or $\mathbb{H} \times \mathbb{R}$. On the other hand it is well known that these are precisely the geometries of aspherical Seifert fibered 3-manifolds supporting a non-positively curved metric (see, e.g., [**ChE80, Eb82, Leb95**]).

(2) By Theorem 4.7.1 graph manifolds with non-empty boundary are non-positively curved. Liu thus showed in particular that fundamental groups of graph manifolds with non-empty boundary are virtually special; this was also obtained by Przytycki–Wise [**PW14a**].

(3) The fundamental group of many graph manifolds are not virtually special:
   (a) There exist closed graph manifolds with non-trivial JSJ-decompositions that are not virtually fibered (see, e.g., [**LuW93**, p. 86] and [**Nemd96**, Theorem D]), and hence by (H.5), (H.17) and (H.20) are not non-positively curved and their fundamental groups are not virtually special; see also [**BuK96a, BuK96b**], [**Leb95**, Example 4.2] and [**BuS05**].
   (b) The fundamental groups of non-trivial torus bundles are not virtually special by (H.5), (H.18) and (H.19) (cf. [**Ag08**, p. 271]).
   (c) By [**Liu13**, Section 2.2] and [**BuS05**] there exist fibered graph manifolds with non-trivial JSJ-decomposition which are not torus bundles, and which are not virtually special.

(4) Hagen–Przytycki [**HaP13**, Theorem B] showed that if N is a graph manifold such that $\pi_1(N)$ is virtually *cocompactly* cubulated (e.g., if $\pi_1(N)$ is virtually *compact* special), then N is 'chargeless.' Since many, arguably most, graph manifolds have non-zero charges, it follows that the fundamental groups of many non-positively curved graph manifolds (closed or with boundary) are not virtually cocompactly cubulated, let alone virtually compact special.

Przytycki–Wise [**PW12**, Theorem 1.1], building on [**Wis12a**], proved the following theorem, which complements the Virtually Compact Special Theorem of Agol, Kahn–Markovic and Wise, and Liu's Theorem.

THEOREM 4.7.3 (Przytycki–Wise). *The fundamental group of an irreducible* 3*-manifold with empty or toroidal boundary, which is neither hyperbolic nor a graph manifold, is virtually special.*

The combination of the Virtually Compact Special Theorem of Agol, Kahn–Markovic and Wise, the results of Liu and Przytycki–Wise and the theorem of Leeb now gives us the following succinct and beautiful statement:

THEOREM 4.7.4. *A compact, orientable, aspherical* 3*-manifold N with empty or toroidal boundary is non-positively curved if and only if $\pi_1(N)$ is virtually special.*

REMARKS.

(1) The connection between $\pi_1(N)$ being virtually special and N being non-positively curved that we presented is very indirect. It is an interesting question whether one can find a more direct connection between these two notions.
(2) By the Virtually Compact Special Theorem, the fundamental groups of hyperbolic 3-manifolds actually are virtually *compact* special. It seems to be unknown whether fundamental groups of non-positively curved irreducible non-hyperbolic 3-manifolds are also virtually compact special.

4.8. 3-manifolds with more general boundary

For simplicity of exposition, we have only considered compact 3-manifolds with empty or toroidal boundary. However, the Virtually Special Theorems above apply equally well in the case of general boundary, and in this section we give some details. We emphasize that we make no claim to the originality of any of the results of this section.

The main theorem of [**PW12**] also applies in the case with general boundary, and so we have the following addendum to Theorem 4.7.4.

THEOREM 4.8.1 (Przytycki–Wise). *The fundamental group of a compact, orientable, aspherical* 3*-manifold N with non-empty boundary is virtually special.*

REMARK. Compressing the boundary and doubling along a suitable subsurface, one may also deduce that $\pi_1(N)$ is a subgroup of a RAAG directly from Theorem 4.7.4.

Invoking suitably general versions of the torus decomposition (e.g., [**Bon02**, Theorem 3.4] or [**Hat**, Theorem 1.9]) and the Geometrization Theorem for manifolds with boundary (e.g., [**Kap01**, Theorem 1.43]), the proof of Theorem 4.7.1 applies equally well in this setting, and one obtains the following statement. (See [**Bek13**, Theorem 2.3] for further details.)

THEOREM 4.8.2. *The interior of a compact, orientable, aspherical* 3*-manifold with non-empty boundary admits a complete, non-positively curved, Riemannian metric.*

REMARK. Bridson [**Brd01**, Theorem 4.3] proved that the interior of a compact, orientable, aspherical 3-manifold N with non-empty boundary admits an *incomplete*, non-positively curved, Riemannian metric that extends to a non-positively curved (i.e., locally CAT(0)) metric on the whole of N.

Combining Theorems 4.8.1 and 4.8.2, the hypotheses on the boundary in Theorem 4.7.4 can be removed. For completeness, we state the most general result here.

Theorem 4.8.3 (Agol, Liu, Przytycki, Wise). *Let N be a compact, orientable, aspherical 3-manifold with possibly empty boundary. Then $\pi_1(N)$ is virtually special if and only if N is non-positively curved.*

Next, we turn to the case of a hyperbolic 3-manifold with *infinite volume*. Recall from the discussion preceding Convention 1.7 that these are precisely the manifolds that admit a complete hyperbolic structure which are either homeomorphic to $T^2 \times I$ or that have a non-toroidal boundary component. Appealing to the theory of Kleinian groups, the (implicit) hypotheses of Theorem 4.2.2 can be relaxed. We begin with a classical result about Kleinian groups [**Cay08**, Theorem 11.1].

Theorem 4.8.4. *Let $N \neq T^2 \times I$ be a hyperbolic 3-manifold of infinite volume. The fundamental group of N has a geometrically finite representation as a Kleinian group in which only the fundamental groups of toroidal boundary components are parabolic.*

Now we can apply a theorem of Brooks [**Brk86**, Theorem 2] to deduce that $\pi_1(N)$ can be embedded in the fundamental group of a hyperbolic 3-manifold of finite volume.

Theorem 4.8.5. *Let $N \neq T^2 \times I$ be a hyperbolic 3-manifold of infinite volume. There exists a hyperbolic 3-manifold M of finite volume such that $\pi_1(N)$ embeds into $\pi_1(M)$ as a geometrically finite subgroup and only the fundamental groups of toroidal boundary components are parabolic.*

Now we obtain the following theorem, which says that $\pi_1(N)$ is virtually compact special (cf. Theorem 14.29 and the paragraph before Corollary 14.33 of [**Wis12a**]).

Theorem 4.8.6 (Wise). *Let N be a hyperbolic 3-manifold of infinite volume. The group $\pi_1(N)$ is virtually compact special.*

Proof. Let M be as in Theorem 4.8.5. By Theorem 4.4.8, $\pi_1(M)$ is virtually compact special. We argue that $\pi_1(N)$ is virtually compact special as well. Indeed, $\pi_1(N)$ is a geometrically finite, and hence relatively quasiconvex, subgroup of $\pi_1(M)$; see (L.18). As $\pi_1(N)$ contains any cusp subgroup that it intersects non-trivially, it is in fact a *fully* relatively quasiconvex subgroup of $\pi_1(M)$, and is therefore virtually compact special by [**CDW12**, Proposition 5.5] or [**SaW12**, Theorem 1.1]. □

As a consequence of Theorem 4.8.1 and the discussion in Section 5, we obtain some properties of fundamental groups of 3-manifolds with general boundary:

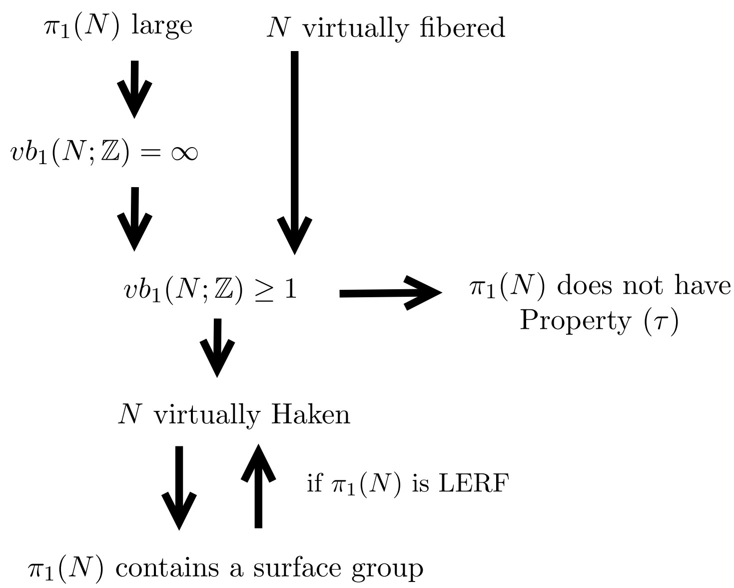
Corollary 4.8.7. *Let N be a compact, orientable, aspherical 3-manifold with non-empty boundary with at least one non-toroidal boundary component. Then $\pi_1(N)$ is*

(1) *linear over $\mathbb{Z}$;*
(2) *RFRS (see (F.4) for a definition); and*
(3) *LERF provided N is hyperbolic.*

Proof. It follows from Theorem 4.8.1 that $\pi_1(N)$ is virtually special. Linearity over $\mathbb{Z}$ and RFRS now both follow from Theorem 4.8.1: see (H.5), (H.17) and (H.31) in Section 5 for details. If N is hyperbolic, then by Theorem 4.8.5, $\pi_1(N)$ is a subgroup of $\pi_1(M)$, where M is a hyperbolic 3-manifold of finite volume. Because $\pi_1(M)$ is LERF (H.11), it follows that $\pi_1(N)$ is also LERF. □

4.9. Summary of previous research on the virtual conjectures

Questions 15–18 of Thurston, stated in Section 4.2 above, have been of central concern in 3-manifold topology over the last 30 years. The study of these questions lead to various other questions and conjectures. Perhaps the most important of these is the Lubotzky–Sarnak Conjecture (see [**Lub96a**, Conjecture 4.2]), that there is no closed hyperbolic 3-manifold with fundamental group having Property (τ). (We refer to [**Lub94**, Definition 4.3.1] and [**LuZ03**] for the definition of Property (τ).)



Flowchart 3. Virtual properties of 3-manifolds

In Flowchart 3 we list various (virtual) properties of 3-manifold groups and logical implications between them. Some of the implications are obvious, and two implications follow from (C.17) and (C.22). Note that if a 3-manifold N contains a surface group, then there is a π_1-injective morphism $\pi_1(\Sigma) \to \pi_1(N)$, for a closed surface Σ with genus at least 1. If in addition $\pi_1(N)$ is LERF, then there is a finite cover of N such that the immersion lifts to an embedding. (See [**Sco78**, Lemma 1.4] for details.) Finally note that by [**Lub96a**, p. 444], if $vb_1(N;\mathbb{Z}) \geq 1$, then $\pi_1(N)$ does not have Property (τ).

Now we will survey some of the work in the past on Thurston's questions and the properties of Flowchart 3. The literature is so extensive that we cannot hope to achieve completeness. Beyond the summary below we also refer to the survey papers by Long–Reid [**LoR05**] and Lackenby [**Lac11**] for further details and references.

We arrange this survey by grouping references under the question that they address.

Question 4.9.1 (Surface Subgroup Conjecture). *Let N be a closed hyperbolic 3-manifold. Does $\pi_1(N)$ contain a (quasi-Fuchsian) surface group?*

The following papers attack Question 4.9.1.

(1) Cooper–Long–Reid [**CLR94**, Theorem 1.5] showed that if N is a closed hyperbolic 3-manifold which fibers over S^1, then there exists a π_1-injective immersion of a quasi-Fuchsian surface into N. We note one important consequence: if N

is a hyperbolic 3-manifold such that $\pi_1(N)$ is LERF and contains a surface subgroup, then $\pi_1(N)$ is large (cf. (C.15)).

(2) The work of [**CLR94**] was extended by Masters [**Mas06b**, Theorem 1.1], which in turn allowed Dufour [**Duf12**, p. 6] to show that if N is a closed hyperbolic 3-manifold which is virtually fibered, then $\pi_1(N)$ is also the fundamental group of a compact non-positively curved cube complex. This proof does not require the Surface Subgroup Theorem 4.5.1 of Kahn–Markovic [**KM12a**].

(3) Cooper–Long [**CoL01**] and Li [**Li02**] showed that given a hyperbolic 3-manifold with one boundary component, almost all Dehn fillings result in a 3-manifold containing a surface group. This result was extended by Bart [**Bar01, Bar04**], Wu [**Wu04**] and Easson [**Ea06**].

(4) Masters–Zhang [**MaZ08, MaZ09**] showed that any cusped hyperbolic 3-manifold contains a quasi-Fuchsian surface group which implies that the same conclusion holds for 'most' Dehn fillings. (See also [**BaC14**].)

(5) Lackenby [**Lac10**, Theorem 1.2] showed that closed arithmetic hyperbolic 3-manifolds contain surface groups.

(6) Bowen [**Bowe04**] attacked the Surface Subgroup Conjecture with methods which foreshadowed the approach taken by Kahn–Markovic [**KM12a**].

QUESTION 4.9.2 (Virtually Haken Conjecture). *Is every closed hyperbolic* 3*-manifold virtually Haken?*

Here is a summary of approaches towards the Virtually Haken Conjecture.

(1) Thurston [**Thu79**] showed that all but finitely many Dehn fillings of the figure-8 knot complement are not Haken. For this reason, there has been considerable interest in the Virtually Haken Conjecture for fillings of 3-manifolds. A lot of work in this direction was done by various people: Aitchison–Rubinstein [**AiR99b**], Aitchison–Matsumoti–Rubinstein [**AMR97, AMR99**], Baker [**Bak88, Bak89, Bak90, Bak91**], Boyer–Zhang [**BrZ00**], Cooper–Long [**CoL99**] (building on [**FF98**]), Cooper–Walsh [**CoW06a, CoW06b**], Hempel [**Hem90**], Kojima–Long [**KLg88**], Masters [**Mas00, Mas07**], Masters–Menasco–Zhang [**MMZ04, MMZ09**], Morita [**Moa86**], as well as by X. Zhang [**Zha05**] and Y. Zhang [**Zhb12**].

(2) Hempel [**Hem82, Hem84, Hem85a**] and Wang [**Wag90**], [**Wag93**, p. 192] studied the Virtually Haken Conjecture for 3-manifolds which admit an orientation reversing involution.

(3) Long [**Lo87**] showed that every hyperbolic 3-manifold which admits a totally geodesic immersion of a closed surface is virtually Haken. (See also [**Zha05**, Corollary 1.2].)

(4) We refer to Millson [**Mis76**], Clozel [**Cl87**], Labesse–Schwermer [**LaS86**], Xue [**Xu92**], Li–Millson [**LiM93**], Rajan [**Raj04**], Reid [**Red07**] and Schwermer [**Scr04, Scr10**] for details of approaches to the Virtually Haken Conjecture for arithmetic hyperbolic 3-manifolds using number-theoretic methods.

(5) Reznikov [**Rez97**] studied hyperbolic 3-manifolds N with $vb_1(N) = 0$.

(6) Experimental evidence towards the validity of the conjecture was provided by Dunfield–Thurston [**DnTb03**].

(7) We refer to Lubotzky [**Lub96b**] and Lackenby [**Lac06, Lac07b, Lac09**] for work towards the stronger conjecture that fundamental groups of hyperbolic 3-manifolds are large. (See Question 4.9.5 below.)

QUESTION 4.9.3. *Let N be a hyperbolic 3-manifold. Is $\pi_1(N)$ LERF?*

The following papers gave evidence for an affirmative answer to Question 4.9.3. Note that by the Subgroup Tameness Theorem, $\pi_1(N)$ is LERF if and only if every geometrically finite subgroup is separable, i.e., $\pi_1(N)$ is *GFERF*. See (H.11) for details.

(1) Let N be a compact hyperbolic 3-manifold and Σ a totally geodesic immersed surface in N. Long [**Lo87**] proved that $\pi_1(\Sigma)$ is separable in $\pi_1(N)$.
(2) The first examples of hyperbolic 3-manifolds with LERF fundamental groups were given by Gitik [**Git99b**].
(3) Agol–Long–Reid [**ALR01**] showed that geometrically finite subgroups of Bianchi groups are separable.
(4) Wise [**Wis06**] showed that the fundamental group of the figure-8 knot complement is LERF.
(5) Agol–Groves–Manning [**AGM09**] showed that fundamental groups of hyperbolic 3-manifolds are LERF if every word-hyperbolic group is residually finite.
(6) After the definition of special complexes was given in [**HaW08**], it was shown that various classes of hyperbolic 3-manifolds had virtually special fundamental groups, and hence were LERF (and virtually fibered). The following were shown to be virtually compact special:
   (a) ‘standard’ arithmetic 3-manifolds [**BHW11**];
   (b) certain branched covers of the figure-8 knot [**Ber08**, Theorem 1.1];
   (c) manifolds built from gluing all-right ideal polyhedra, such as augmented link complements [**CDW12**].
(7) As we saw in the previous sections, the fact that a hyperbolic 3-manifold group can be cubulated played an essential rôle in showing that the group is GFERF. Aitchison–Rubinstein [**AiR89, AiR92**] showed that alternating link complements can be cubulated.

QUESTION 4.9.4 (Lubotzky–Sarnak Conjecture). *Let N be a closed hyperbolic 3-manifold. Is it true that $\pi_1(N)$ does not have Property (τ)?*

The following represents some of the major work on the Lubotzky–Sarnak Conjecture. We also refer to [**Lac11**, Section 7] and [**LuZ03**] for further details.

(1) Lubotzky [**Lub96a**] stated the conjecture and proved that certain arithmetic 3-manifolds have positive virtual first Betti number, extending the above-mentioned work of Millson [**Mis76**] and Clozel [**Cl87**].
(2) Lackenby [**Lac06**, Theorem 1.7] showed that the Lubotzky–Sarnak Conjecture, together with a conjecture about Heegaard gradients, implies the Virtually Haken Conjecture.
(3) Long–Lubotzky–Reid [**LLuR08**] proved that the fundamental group of every hyperbolic 3-manifold has Property (τ) with respect to some cofinal regular filtration of $\pi_1(N)$.
(4) Lackenby–Long–Reid [**LaLR08b**] proved that if the fundamental group of a hyperbolic 3-manifold N is LERF, then $\pi_1(N)$ does not have Property (τ).

QUESTIONS 4.9.5. *Let N be a hyperbolic 3-manifold with $b_1(N) \geq 1$.*

(1) *Does N admit a finite cover N' with $b_1(N') \geq 2$?*
(2) *Is $vb_1(N) = \infty$?*
(3) *Is $\pi_1(N)$ large?*

The virtual Betti numbers of hyperbolic 3-manifolds in particular were studied by the following authors:

(1) Cooper–Long–Reid [**CLR97**, Theorem 1.3] showed that if N is a compact, irreducible 3-manifold with non-trivial incompressible boundary, then either N is covered by $N = T^2 \times I$, or $\pi_1(N)$ is large. (See also [**But04**, Corollary 6] and [**Lac07a**, Theorem 2.1].)
(2) Cooper–Long–Reid [**CLR07**, Theorem 1.3], Venkataramana [**Ven08**, Corollary 1] and Agol [**Ag06**, Theorem 0.2] proved that if N is an arithmetic 3-manifold, then $vb_1(N) \geq 1 \Rightarrow vb_1(N) = \infty$. Further work of Lackenby–Long–Reid [**LaLR08a**] shows that if $vb_1(N) \geq 1$, then $\pi_1(N)$ is large.
(3) Long and Oertel [**LO97**, Theorem 2.5] gave many examples of fibered 3-manifolds with $vb_1(N; \mathbb{Z}) = \infty$. Masters [**Mas02**, Corollary 1.2] showed that $vb_1(N; \mathbb{Z}) = \infty$ for every fibered 3-manifold N whose fiber has genus 2.
(4) Kionke–Schwermer [**KiS12**] showed that certain arithmetic hyperbolic 3-manifolds admit a cofinal tower with rapid growth of first Betti numbers.
(5) Cochran and Masters [**CMa06**] studied the growth of Betti numbers in abelian covers of 3-manifolds with Betti number equal to two or three.
(6) Button [**But11a**] gave computational evidence towards the conjecture that the fundamental group of any hyperbolic 3-manifold N with $b_1(N) \geq 1$ is large.
(7) Koberda [**Kob12a, Kob14**] gave a detailed study of Betti numbers of finite covers of fibered 3-manifolds.

QUESTION 4.9.6 (Virtually Fibered Conjecture). *Is every hyperbolic 3-manifold virtually fibered?*

The following papers deal with the Virtually Fibered Conjecture:

(1) An affirmative answer was given for specific classes of 3-manifolds (e.g., certain knot and link complements) by Agol–Boyer–Zhang [**ABZ08**], Aitchison–Rubinstein [**AiR99a**], DeBlois [**DeB10**], Gabai [**Gab86**], Guo–Zhang [**GZ09**], Reid [**Red95**], Leininger [**Ler02**], and Walsh [**Wah05**].
(2) Button [**But05**] gave computational evidence towards an affirmative answer to the Virtually Fibered Conjecture.
(3) Sakuma [**Sak81**], Brooks [**Brk85**], and Montesinos [**Mon86, Mon87**] showed that every closed, orientable 3-manifold admits a 2-fold *branched* cover which is fibered and hyperbolic.
(4) Long–Reid [**LoR08b**] showed that arithmetic hyperbolic 3-manifolds which are fibered admit in fact finite covers with arbitrarily many fibered faces in the Thurston norm ball. (See also Dunfield–Ramakrishnan [**DR10**] and [**Ag08**, Theorem 7.1].)
(5) Lackenby [**Lac06**, p. 320] gave an approach to the Virtually Fibered Conjecture using 'Heegaard gradients.' (See also [**Lac11**].) This idea was further developed by Lackenby [**Lac04**], Maher [**Mah05**] and Renard [**Ren10**]. The latter author gave another approach [**Ren14a, Ren14b**] to the conjecture.
(6) Agol [**Ag08**, Theorem 5.1] showed that aspherical 3-manifolds with virtually RFRS fundamental groups are virtually fibered. (See (F.4) for the definition of RFRS.) The first examples of 3-manifolds with virtually RFRS fundamental groups were given by Agol [**Ag08**, Corollary 2.3], Bergeron [**Ber08**, Theorem 1.1], Bergeron–Haglund–Wise [**BHW11**], and Chesebro–DeBlois–Wilton [**CDW12**].

CHAPTER 5

Consequences of the Virtually Compact Special Theorem

In Chapter 4 we explained the content and the significance of the Virtually Compact Special Theorem of Agol, Kahn–Markovic and Wise. In the present chapter we will outline the numerous consequences of this theorem. As in Chapter 3, we will organize most results in a flowchart (Flowchart 4). Section 5.1 contains the definitions of the properties discussed in the flowchart, Section 5.2 contains the justifications for the implications in the flowchart, and Section 5.3 lists some additional material not contained in the flowchart. Finally, Section 5.4 contains detailed proofs of a few statements made earlier in the chapter.

5.1. Definitions and Conventions

In this section we summarize various consequences of the fundamental group of a 3-manifold being virtually (compact) special. As in Section 3 we present the results in a flowchart, see p. 84. We start out with additional definitions needed for Flowchart 4. Again the definitions are roughly in the order that they appear in the flowchart. In the items below we let π be a group.

(F.1) One says that π *virtually retracts* onto a subgroup $A \subseteq \pi$ if there exists a finite-index subgroup $\pi' \subseteq \pi$ that contains A and a morphism $\pi' \to A$ which is the identity on A. In this case, we say that *A is a virtual retract of* π.

(F.2) We call π *conjugacy separable* if for any two non-conjugate elements $g, h \in \pi$ there exists a morphism α from π onto a finite group in which $\alpha(g)$ and $\alpha(h)$ are non-conjugate. A group is called *hereditarily conjugacy separable* if any of its finite-index (not necessarily normal) subgroups is conjugacy separable.

(F.3) For a hyperbolic 3-manifold N, we say that $\pi_1(N)$ is *GFERF* if all geometrically finite subgroups are separable.

(F.4) One calls π *residually finite rationally solvable* (*RFRS*) if there exists a sequence $\{\pi_n\}$ of subgroups of π such that $\bigcap_n \pi_n = \{1\}$ and for any n,
 (1) the subgroup π_n is normal and of finite index in π;
 (2) $\pi_n \supseteq \pi_{n+1}$, and the natural surjection $\pi_n \to \pi_n/\pi_{n+1}$ factors through $\pi_n \to H_1(\pi_n;\mathbb{Z})/\text{torsion}$.

(F.5) Let N be a compact, orientable 3–manifold. From Section 4.1 recall that N is said to be *fibered* if it admits the structure of a surface bundle over S^1. We call $\phi \in H^1(N;\mathbb{R})$ is *fibered* if it can be represented by a non-degenerate closed 1-form.

By [**Tis70**] an integral class $\phi \in H^1(N;\mathbb{Z}) = \operatorname{Hom}(\pi_1(N),\mathbb{Z})$ is fibered if and only if there exists a surface bundle $p\colon N \to S^1$ such that the induced map $p_*\colon \pi_1(N) \to \pi_1(S^1) = \mathbb{Z}$ coincides with ϕ.

(F.6) A torsion-free group π satisfies the *Atiyah Conjecture* if given any finite CW-complex X and given any epimorphism $\alpha\colon \pi_1(X) \to \pi$ all the L^2-Betti numbers

$b_i^{(2)}(X,\alpha)$ are integers. We refer to [**At76**, p. 72], [**Ecn00**, p. 209] and [**Lü02**, Chapter 10] for details.

(F.7) A group has the *finitely generated intersection property* (*f.g.i.p.*) if the intersection of any two of its finitely generated subgroups is finitely generated.

(F.8) The group π is called *poly-free* if there is a finite sequence of subgroups

$$\pi = \pi_0 \rhd \pi_1 \rhd \pi_2 \rhd \cdots \rhd \pi_n = \{1\}$$

of π such that for any $i \in \{0,\dots,n-1\}$ the quotient group π_i/π_{i+1} is a (not necessarily finitely-generated) free group.

(F.9) We denote the profinite completion of π by $\widehat{\pi}$. (We refer to [**RiZ10**, Section 3.2] and [**Win98**, Section 1] for the definition and for the main properties.) The group π is called *good* if the natural morphism $H^*(\widehat{\pi};A) \to H^*(\pi;A)$ is an isomorphism for any finite abelian group A with a π-action. (See [**Ser97**, D.2.6 Exercise 2].)

(F.10) The unitary dual of a group is defined to be the set of equivalence classes of its irreducible unitary representations. The unitary dual can be viewed as a topological space with respect to the Fell topology. One says that π has *Property FD* if the finite representations of π are dense in its unitary dual. We refer to [**BdlHV08**, Appendix F.2] and [**LuSh04**] for details.

(F.11) The class of *elementary-amenable groups* is the smallest class of groups which contains all abelian and all finite groups, which is extension closed and which is closed under directed unions.

(F.12) The group π has the *torsion-free elementary-amenable factorization property* if any homomorphism to a finite group factors through a homomorphism to a torsion-free elementary-amenable group.

(F.13) The group π is called *potent* if for any non-trivial $g \in \pi$ and any n there exists a morphism α of π onto a finite group such that $\alpha(g)$ has order n.

(F.14) A subgroup of π is called *characteristic* if it is preserved by every automorphism of π. Every characteristic subgroup of π is normal. One calls π *characteristically potent* if for all $g \in \pi \setminus \{1\}$ and any n there exists a finite-index characteristic subgroup $\pi' \subseteq \pi$ such that $g\pi'$ has order n in π/π'.

(F.15) The group π is called *weakly characteristically potent* if for any $g \in \pi\setminus\{1\}$ there exists an m such that for any n there is a characteristic finite-index subgroup $\pi' \subseteq \pi$ such that $g\pi'$ has order mn in π/π'.

(F.16) Suppose π is torsion-free. We say that $g_1,\dots,g_n \in \pi$ are *independent* if distinct pairs of elements do not have conjugate non-trivial powers; that is, if there are $k,l \in \mathbb{Z}\setminus\{0\}$ and $c \in \pi$ with $cg_i^k c^{-1} = g_j^l$, then $k=l$. The group π is called *omnipotent* if given any independent $g_1,\dots,g_n \in \pi$ there exists an $m \geq 1$ such that for any $l_1,\dots,l_n \in \mathbb{N}$ there exists a morphism α from π to a finite group such that the order of $\alpha(g_i)$ is $l_i m$. This definition was introduced by Wise in [**Wis00**, Definition 3.2].

(F.17) See [**Lub94**, Definition 4.3.1] and [**LuZ03**] for the definition of Property (τ).

Flowchart 4 is supposed to be read in the same manner as Flowchart 1. For the reader's convenience we recall some of the conventions.

(G.1) In Flowchart 4, N is a compact, orientable, irreducible 3-manifold such that its boundary consists of a (possibly empty) collection of tori. Moreover we assume throughout that $\pi := \pi_1(N)$ is neither solvable nor finite. Without these extra assumptions some of the implications do not hold. For example the fundamental group of the 3-torus T^3 is a RAAG, but $\pi_1(T^3)$ is not large.

(G.2) In the flowchart the top arrow splits into several arrows. In this case exactly one of the possible three conclusions holds.

(G.3) Red arrows indicate that the conclusion holds *virtually*, e.g., Arrow 20 says that if $\pi = \pi_1(N)$ is RFRS, then N is virtually fibered.

(G.4) If a property $\mathcal{P}$ of groups is written in green, then the following conclusion always holds: If N is a compact, orientable, irreducible 3-manifold with empty or toroidal boundary such that the fundamental group of a finite (not necessarily regular) cover of N has Property $\mathcal{P}$, then $\pi_1(N)$ also has Property $\mathcal{P}$. In (I.1)–(I.7) below we will show that the various properties written in green do indeed have the above property.

(G.5) The Conventions (F.3) and (F.4) imply that a concatenation of red and black arrows which leads to a green property means that the initial group also has the green property.

(G.6) An arrow with a condition next to it indicates that this conclusion only holds if this extra condition is satisfied.

5.2. Justifications

Now we give the justifications for the implications of Flowchart 4. As in Flowchart 1 we strive for maximal generality. Unless we say otherwise, we will therefore only assume that N is a connected 3-manifold and each justification can be read independently of all the other steps.

(H.1) Suppose N is compact, orientable, irreducible, with empty or toroidal boundary. It follows from the Geometrization Theorem 1.7.6 that N is either hyperbolic or a graph manifold, or the JSJ-decomposition of N is non-trivial with at least one hyperbolic JSJ-component.

(H.2) The Virtually Compact Special Theorem proved by Agol [**Ag13**], Kahn–Markovic [**KM12a**], and Wise [**Wis12a**] implies that if N is hyperbolic, then $\pi_1(N)$ is virtually compact special. We refer to Section 4 for details.

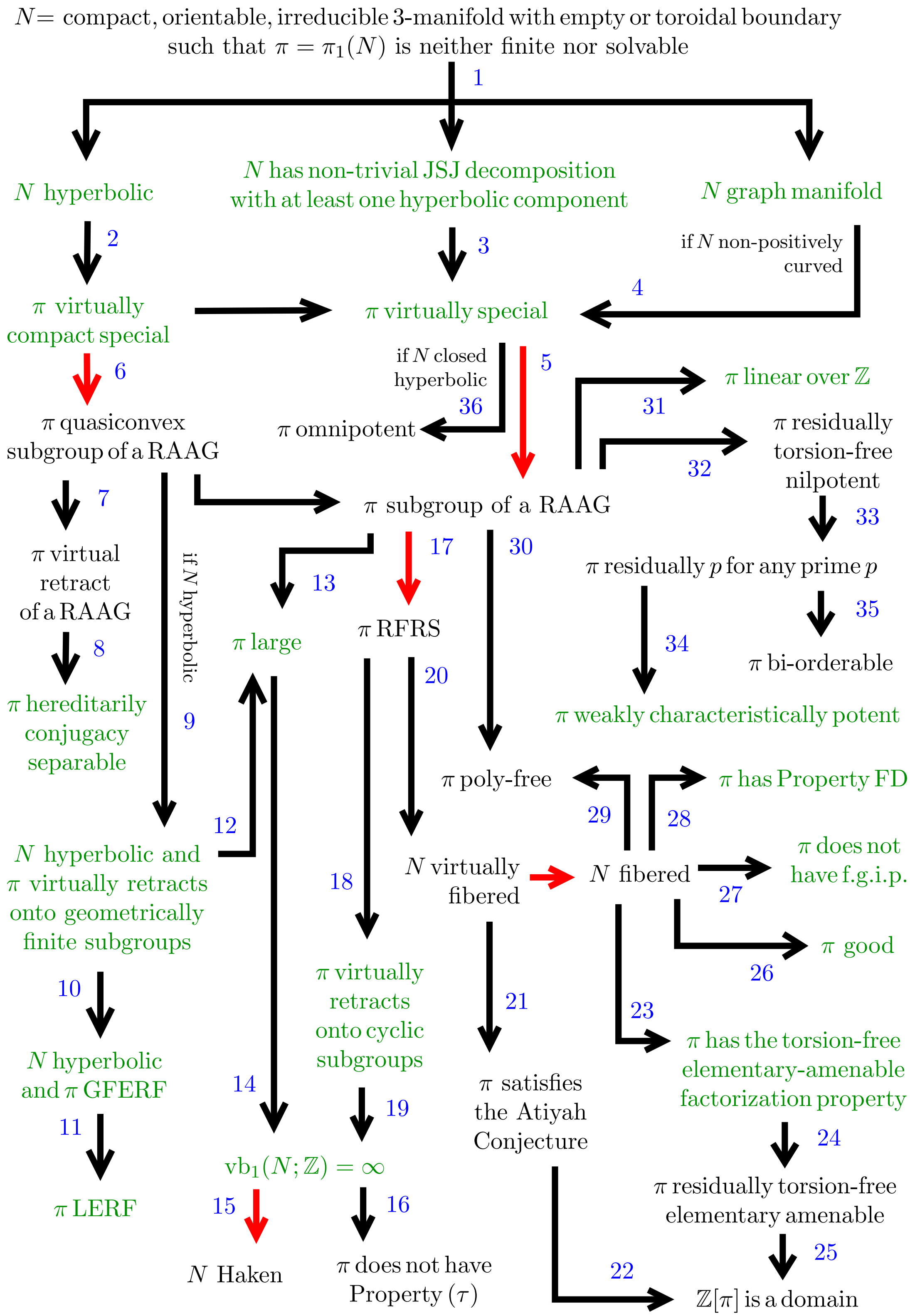
(H.3) Suppose N is irreducible with empty or toroidal boundary, and that it is neither hyperbolic nor a graph manifold. Then $\pi_1(N)$ is virtually special by the theorem of Przytycki–Wise. (See [**PW12**, Theorem 1.1] and Theorem 4.7.3.)

(H.4) Suppose N is an aspherical graph manifold. Liu [**Liu13**] (see Theorem 4.7.2) showed that $\pi_1(N)$ is virtually special if and only if N is non-positively curved. By Theorem 4.7.1 a graph manifold with non-empty boundary is non-positively curved. Przytycki–Wise [**PW14a**] gave an alternative proof that fundamental groups of graph manifolds with non-empty boundary are virtually special.

As mentioned in Section 4.7, by Hagen–Przytycki [**HaP13**, Theorem B] the fundamental group of many (arguably, 'most') non-positively curved graph manifolds (closed or with boundary) are not virtually *compact* special.

(H.5) In Corollary 4.3.3 we saw that a group is virtually special if and only if it is virtually a subgroup of a RAAG. In fact a somewhat stronger statement holds: as Agol [**Ag14**, Theorem 9.1] points out, the proof of Haglund–Wise [**HaW08**] gives the slightly stronger statement that a virtually special group π fits into a short exact sequence

$$1 \to \Gamma \to \pi \to G \to 1$$



Flowchart 4. Consequences of being virtually (compact) special.

where Γ is a RAAG and G is finite.

(H.6) In Corollary 4.3.4 we saw that a group is virtually compact special if and only if it is virtually a quasiconvex subgroup of a RAAG.

(H.7) Haglund [**Hag08**, Theorem F] showed that quasi-convex subgroups of RAAGs are virtual retracts. In fact, he proved more generally that quasi-convex subgroups of Right-Angled Coxeter Groups are virtual retracts, generalizing earlier results of Scott [**Sco78**] (the reflection group of the right-angled hyperbolic pentagon) and Agol–Long–Reid [**ALR01**, Theorem 3.1] (reflection groups of arbitrary right-angled hyperbolic polyhedra).

(H.8) Minasyan [**Min12**, Theorem 1.1] showed that any RAAG is hereditarily conjugacy separable. It follows immediately that virtual retracts of RAAGs are hereditarily conjugacy separable.

The combination of Minasyan's result with (H.2), (H.6), (H.7) and (I.1) implies that fundamental groups of hyperbolic 3-manifolds are conjugacy separable. In (J.1) we will see that this is a key ingredient in the proof of Hamilton–Wilton–Zalesskii [**HWZ13**] that the fundamental group of any closed, orientable, irreducible 3-manifold is conjugacy separable.

(H.9) Suppose that N is hyperbolic and $\pi = \pi_1(N)$ is a quasi-convex subgroup of a RAAG A_Σ. Let Γ be a geometrically finite subgroup of π. The idea is that Γ should be a quasi-convex subgroup of A_Σ. One could then apply [**Hag08**, Theorem F] to deduce that Γ is a virtual retract of A_Σ and hence of π. However, it is not true in full generality that a quasi-convex subgroup of a quasi-convex subgroup is again quasi-convex, and so a careful argument is needed. In the closed case, the required technical result is [**Hag08**, Corollary 2.29]. In the cusped case, in fact it turns out that Γ may not be a quasi-convex subgroup of A_Σ. Nevertheless, it is possible to circumvent this difficulty. Now we give detailed references.

Suppose first that N is closed. It follows from Proposition 4.4.2 that then π is word-hyperbolic and Γ is a quasi-convex subgroup of π (see (L.18)). The group π acts by isometries on $\widetilde{S}_\Sigma$, the universal cover of the Salvetti complex of A_Σ. Fix a base 0-cell $x_0 \in \widetilde{S}_\Sigma$. The 1-skeleton of $\widetilde{S}_\Sigma$ is the Cayley graph of A_Σ with respect to its standard generating set, and so, by hypothesis, the orbit πx_0 is a quasi-convex subset of $\widetilde{S}_\Sigma^{(1)}$. By [**Hag08**, Corollary 2.29], π acts cocompactly on some convex subcomplex $\widetilde{X} \subseteq \widetilde{S}_\Sigma$. Using the Morse Lemma for geodesics in hyperbolic spaces [**BrH99**, Theorem III.D.1.7], the orbit Γx_0 is a quasi-convex subset of $\widetilde{X}^{(1)}$. Using [**Hag08**, Corollary 2.29] again, it follows that Γ acts cocompactly on a convex subcomplex $\widetilde{Y} \subseteq \widetilde{X}$. The complex $\widetilde{Y}$ is also a convex subcomplex of $\widetilde{S}_\Sigma$, which by a final application of [**Hag08**, Corollary 2.29] implies that Γx_0 is a quasi-convex subset of $\widetilde{S}_\Sigma^{(1)}$, or, equivalently, that Γ is a quasi-convex subgroup of A_Σ. Hence, by [**Hag08**, Theorem F], Γ is a virtual retract of A_Σ and hence of π.

If N is not closed, then π is not word-hyperbolic, but in any case it is *relatively hyperbolic* and Γ is a *relatively quasi-convex* subgroup. (See (L.18) below for a reference.) One can show that in this case Γ is again a virtual retract of A_Σ and hence of π. The argument is rather more involved than the argument in the word-hyperbolic case; in particular, it is not necessarily true that Γ is a quasi-convex subgroup of A_Σ. See [**CDW12**, Theorem 1.3] for the details; the proof again relies on [**Hag08**, Theorem F] together with work of Manning–Martinez-Pedrosa [**MMP10**]. See also [**SaW12**, Theorem 7.3] for an alternative argument.

This result about retraction onto subgroups is foreshadowed by the classical theorem of Hall [**Hal49**] that any finitely generated subgroup G of a free group F is virtually a subfactor, i.e., there exists a finite-index subgroup F' which contains G and a subgroup H such that $F' = G * H$. (See also [**Bus69**, Theorem 1] and [**Sta85**, Corollary 6.3].)

(H.10) The following well-known argument shows that a virtual retract G of a residually finite group π is separable (cf. [**Hag08**, Section 3.4]): Let $\rho\colon \pi_0 \to G$ be a retraction onto G from a subgroup π_0 of finite index in π. Consider the map $g \mapsto \delta(g) := g^{-1}\rho(g)\colon \pi_0 \to \pi_0$. It is easy to check that δ is continuous in the profinite topology on π_0, and so $G = \delta^{-1}(1)$ is closed. That is to say, G is separable in π_0, and hence in π. In particular, this shows that if N is compact and $\pi = \pi_1(N)$ virtually retracts onto geometrically finite subgroups, then π is GFERF (since by (C.29), $\pi = \pi_1(N)$ is residually finite).

(H.11) Suppose N is hyperbolic and $\pi = \pi_1(N)$. If π is GFERF and Γ is a finitely generated subgroup of π; then Γ is separable: by the Subgroup Tameness Theorem (see Theorem 4.1.2) we only have to deal with the case that Γ is a virtual surface fiber group, but an elementary argument shows that in that case Γ is separable. (See (L.11) for more details.)

In fact (regardless whether π is GFERF or not), π is in fact double-coset separable, which means that any subset of the form SgT is separable, where S and T are finitely generated subgroups of π and g is an element of π. Indeed, let $S, T \subseteq \pi$ be finitely generated subgroups and $g \in \pi$. If S or T is a virtual fiber subgroup, then a direct argument using part (ii) of [**Nib92**, Proposition 2.2] shows that SgT is separable in π. If neither one of S or T is a virtual fiber subgroup, then this follows from the Subgroup Tameness Theorem (see Theorem 4.1.2), from (H.2), from work of Wise [**Wis12a**, Theorem 16.23] and from work of Hruska [**Hr10**, Corollary 1.6].

We just showed that the fundamental group of each hyperbolic 3-manifold is LERF. In contrast, fundamental groups of graph manifolds are in general not LERF; see, e.g., [**BKS87**], [**Mat97a**, Theorem 5.5], [**Mat97b**, Theorem 2.4], [**RuW98**], [**NW01**, Theorem 4.2], [**Liu14**], [**Wod15**] and Section 7.2.1 below. In fact, there are finitely generated surface subgroups of graph manifold groups that are not contained in any proper subgroup of finite index [**NW98**, Theorem 1]. On the other hand, Przytycki–Wise [**PW14a**, Theorem 1.1] showed that if Σ is an oriented incompressible surface embedded in a graph manifold N, then $\pi_1(\Sigma)$ is separable in $\pi_1(N)$. See Section 7.2.1 for a more detailed discussion on (non-) separability of subgroups of 3-manifold groups.

(H.12) Suppose N is hyperbolic such that $\pi = \pi_1(N)$ virtually retracts onto each of its geometrically finite subgroups. Let $F \subseteq \pi$ be a geometrically finite noncyclic free subgroup, such as a Schottky subgroup. (Every non-elementary Kleinian group contains a Schottky subgroup, as was first observed by Myrberg [**Myr41**].) Then by assumption there exists a finite-index subgroup of π with a surjective morphism onto F. This shows that fundamental groups of hyperbolic 3-manifolds are large.

The combination with (C.15) shows that the fundamental group of any compact 3-manifold N with empty or toroidal boundary is large, as long as $\pi_1(N)$ is not virtually polycyclic.

(H.13) Antolín–Minasyan [**AM11**, Corollary 1.6] showed that every (not necessarily finitely generated) subgroup of a RAAG is either free abelian of finite rank or maps onto a non-cyclic free group. This implies directly the fact that if $\pi_1(N)$ is virtually special, then either $\pi_1(N)$ is virtually solvable or $\pi_1(N)$ is large. (Recall that in the flowchart we assume throughout that $\pi_1(N)$ is neither finite nor solvable, and that $\pi_1(N)$ is also not virtually solvable by Theorem 1.11.1.)

The combination of (H.13) together with (C.6), (C.13), (C.14), (C.15) shows that the fundamental group π of a compact 3-manifold with empty or toroidal boundary is either virtually polycyclic or large. Together with (C.38) this shows that π is either virtually polycyclic or SQ-universal. As mentioned in (C.38), this dichotomy was also proved by Minasyan–Osin [**MO13**] without the Virtually Compact Special Theorem.

The result of Antolín–Minasyan also implies that every 2-generator subgroup of a RAAG is either free abelian or free. This result was first proved by Baudisch [**Bah81**, Theorem 1.2] and reproved by several other authors, see [**KiK13**, Corollary 4], [**Scri10**] and [**Car14**].

(H.14) We already saw in (C.17) that a group π which is large is homologically large, in particular, $vb_1(\pi;\mathbb{Z}) = \infty$.

Hence if N is compact, irreducible, with empty or toroidal boundary such that $\pi_1(N)$ is not virtually solvable, then given any free abelian group A there exists a finite cover $\tilde{N}$ of N such that A embeds into $H_1(\tilde{N};\mathbb{Z})$. For closed hyperbolic 3-manifolds this was considerably strengthened by Sun [**Sun13**, Corollary 1.6] who showed that given any closed hyperbolic 3-manifold N and given any finitely generated abelian group A there exists a finite cover $\tilde{N}$ of N such that A is a subsummand of $H_1(\tilde{N};\mathbb{Z})$. (See also [**Sun14**, Remark 1.5].)

(H.15) In (C.22) we showed that if N is compact, irreducible, with $vb_1(N;\mathbb{Z}) \geq 1$, then N is virtually Haken. (In (C.19) and (C.20) we also saw that $\pi_1(N)$ is virtually locally indicable and virtually left-orderable.)

(H.16) It follows from [**Lub96a**, p. 444] that $vb_1(\pi;\mathbb{Z}) \geq 1$ implies that π does not have Property (τ).

If N is compact, orientable, aspherical, with empty or toroidal boundary, then tracing through the arguments of Flowchart 1 and Flowchart 4 shows that $vb_1(N) \geq 1$. It follows from [**Lub96a**, p. 444] that $\pi_1(N)$ does not have Property (τ). This answers in particular the Lubotzky-Sarnak Conjecture in the affirmative: there is no hyperbolic 3-manifold with fundamental group having Property (τ). (See [**Lub96a**] and [**Lac11**] and also Section 4.9 for details.)

Every group which has Kazhdan's Property (T) also has Property (τ); see, e.g., [**Lub96a**, p. 444] for details and [**BdlHV08**] for background on Property (T). Thus the fundamental group of a compact, orientable, aspherical 3-manifold with empty or toroidal boundary does not satisfy Kazhdan's Property (T). This result was first obtained by Fujiwara [**Fuj99**].

(H.17) Agol [**Ag08**, Theorem 2.2] showed that RAAGs are virtually RFRS. It is clear that a subgroup of a RFRS group is again RFRS. A close inspection of Agol's proof using [**DJ00**, Section 1] in fact implies that a RAAG is already RFRS. We will not make use of this fact.

(H.18) Let π be RFRS and $g \in \pi$. It follows easily from the definition that we can take a finite-index subgroup π' of π containing g such that g represents a non-trivial element $[g]$ in $H' := H_1(\pi';\mathbb{Z})/\text{torsion}$. Take a finite-index subgroup H'' of H' which contains $[g]$ and such that $[g]$ is a primitive element in H''. We then have a morphism $\varphi\colon H'' \to \mathbb{Z}$ such that $\varphi([g]) = 1$. The map

$$\operatorname{Ker}\{\pi' \to H'/H''\} \to H'' \xrightarrow{\varphi} \mathbb{Z} \xrightarrow{1\mapsto g} \langle g \rangle$$

is a virtual retraction onto the cyclic group generated by g.

The above, together with the argument of (D.4) and (H.10), shows that cyclic subgroups of virtually RFRS groups are separable. If we combine this observation with (H.2), (H.3) and (H.4) we see that cyclic subgroups of compact, orientable, prime 3-manifolds with empty or toroidal boundary which are not closed graph manifolds are separable. This was proved earlier for all 3-manifolds by E. Hamilton [**Hamb01**].

In contrast, in Proposition 5.4.7 we show that there exist Seifert fibered manifolds, and also graph manifolds with non-trivial JSJ-decomposition, whose fundamental group does not virtually retract onto all cyclic subgroups.

(H.19) Let π be an infinite torsion-free group which is not virtually abelian and which retracts virtually onto its cyclic subgroups. An argument using the transfer map shows $vb_1(\pi;\mathbb{Z}) = \infty$. (See, e.g., [**LoR08a**, Theorem 2.14] for a proof.)

(H.20) Suppose N is compact, aspherical, with empty or toroidal boundary, such that $\pi_1(N)$ is RFRS. Agol [**Ag08**, Theorem 5.1] showed that N is virtually fibered. (See also [**FKt14**, Theorem 5.1].) In fact, Agol proved a more refined statement. If $\phi \in H^1(N;\mathbb{Q})$ is a non-fibered class, then there exists a finite solvable cover $p\colon N' \to N$ (in fact a cover which corresponds to one of the π_i in the definition of RFRS) such that $p^*(\phi) \in H^1(N';\mathbb{Q})$ lies on the boundary of a fibered cone of the Thurston norm ball of N'. (We refer to [**Thu86a**] and Section 5.4.3 for background on the Thurston norm and fibered cones.)

Let N be irreducible with empty or toroidal boundary, and assume N is not a graph manifold. We show in Proposition 5.4.12 below that there exist finite covers of N with arbitrarily many fibered faces; see also [**Ag08**, Theorem 7.2] for the hyperbolic case.

Agol [**Ag08**, Theorem 6.1] also proved a corresponding theorem for 3-manifolds with non-toroidal boundary. More precisely, if (N,γ) is a connected taut sutured manifold such that $\pi_1(N)$ is virtually RFRS, then there exists a finite-sheeted cover $(\tilde{N},\tilde{\gamma})$ of (N,γ) with a depth-one taut-oriented foliation. See [**Gab83a, Ag08, CdC03**] for background and precise definitions.

Let $N \to S^1$ be a fibration with surface fiber Σ. We have a short exact sequence

$$1 \to \Gamma := \pi_1(\Sigma) \to \pi = \pi_1(N) \to \mathbb{Z} = \pi_1(S^1) \to 1.$$

This sequence splits, so π is isomorphic to the semidirect product

$$\mathbb{Z} \ltimes_\varphi \Gamma := \big\langle \Gamma, t \,|\, tgt^{-1} = \varphi(g), g \in \Gamma \big\rangle, \quad \text{for some } \varphi \in \operatorname{Aut}(\Gamma).$$

Let $\varphi, \psi \in \operatorname{Aut}(\Gamma)$. It is straightforward to show that there exists an isomorphism $\mathbb{Z} \ltimes_\varphi \Gamma \to \mathbb{Z} \ltimes_\psi \Gamma$ commuting with the canonical projections to $\mathbb{Z}$ exactly if there are $h \in \Gamma$, $\alpha \in \operatorname{Aut}(\Gamma)$ with $h\varphi(g)h^{-1} = (\alpha \circ \psi \circ \alpha^{-1})(g)$ for all $g \in \Gamma$.

In Section 7.4 below we will see that a 'random' closed, orientable 3-manifold is a rational homology sphere, in particular not fibered. Dunfield–D. Thurston [**DnTb06**] also give evidence that a 'random' 3-manifold N with $b_1(N) = 1$ is not fibered.

Finally, it follows from the combination of (H.4), (H.5), (H.17), and (H.20) that non-positively curved graph manifolds (e.g., graph manifolds with non-empty boundary, see [**Leb95**, Theorem 3.2]) are virtually fibered. Wang–Yu [**WY97**, Theorem 0.1] proved directly that graph manifolds with non-empty boundary are virtually fibered. (See also [**Nemd96**].) Svetlov [**Sv04**] proved that non-positively curved graph manifolds are virtually fibered.

(H.21) Let $\mathcal{C}$ denote the smallest class of groups which contains all free groups, which is closed under directed unions and which satisfies $G \in \mathcal{C}$ whenever G admits a normal subgroup $H \in \mathcal{C}$ such that G/H is elementary amenable. By work of Linnell [**Lil93**, Theorem 1.5] (see also [**Lü02**, Theorem 10.19]) any group in $\mathcal{C}$ satisfies the Atiyah conjecture.

The fundamental group of an oriented compact surface with boundary is free and thus lies in $\mathcal{C}$. The fundamental group of an oriented closed surface $F \neq S^2$ lies in $\mathcal{C}$ since the kernel of any epimorphism $\pi_1(F) \to \mathbb{Z}$ is free. Since the fundamental group of a compact, orientable 3-manifold is the semidirect product of $\mathbb{Z}$ with the fundamental group of a compact, oriented surface it follows that such a group also lies in $\mathcal{C}$. Finally, again by the above theorem of Linnell the fundamental group of any virtually fibered compact 3-manifold satisfies the Atiyah Conjecture.

(H.22) If π is a torsion-free group that satisfies the Atiyah Conjecture, then by [**Lü02**, Lemma 10.15] the group ring $\mathbb{C}[\pi]$ is a domain, in particular $\mathbb{Z}[\pi]$ is a domain.

The Kaplansky Conjecture (see [**Lü02**, Conjecture 10.14] and [**Lil98**]) predicts more generally that the complex group ring of any torsion-free group is a domain.

(H.23) The proof of [**Scv14**, Lemma 2.3] (see also [**Ag11, Fak75**]) implies that fundamental groups of compact, orientable surfaces have the torsion-free elementary-amenable factorization property. It then follows from [**Scv14**, Lemma 2.4] that fundamental groups of compact, orientable fibered 3-manifold have the torsion-free elementary-amenable factorization property.

By [**Scv14**, Lemmas 2.1 and 2.2] and [**Scv14**, Corollary 2.6] virtual retracts of a RAAG also have the torsion-free elementary-amenable factorization property.

(H.24) It follows immediately from the definitions that a group that is residually finite and that has the torsion-free elementary-amenable factorization property is residually torsion-free elementary-amenable.

(H.25) We say that a ring R has fully residually a property if given any finite collection of elements $a_1, \dots, a_s \in R$ there exists a ring epimomorphism $\alpha\colon R \to S$ to a ring S with property $\mathcal{P}$ such that $\alpha(a_1), \dots, \alpha(s_s)$ are all non-zero. It is straightforward to see that a ring that is fully residually a domain is a domain.

Kropholler–Linnell–Moody [**KLM88**, Theorem 1.4] showed that the group ring of a torsion-free elementary-amenable group is a domain. Let π be a group that is residually torsion-free elementary-amenable. It is straightforward to see that π is also fully residually torsion-free elementary-amenable. It follows easily from the above that $\mathbb{Z}[\pi]$ is fully residually a domain, hence a domain. (Similarly, if π is locally indicable or left-orderable, then $\mathbb{Z}[\pi]$ is a domain by [**RoZ98**, Proposition 6], [**Lil93**, Theorem 4.3], and [**Hig40**, Theorem 12]).

The combination of the above with the other results mentioned in the diagram thus implies that the group ring of the fundamental group of a compact, orientable, irreducible 3-manifold with empty or toroidal boundary that is not a closed graph manifold is a domain.

If Γ is an amenable group, then $\mathbb{Z}[\Gamma]$ admits an Ore localization $\mathbb{K}(\Gamma)$. (See, e.g., [**Lü02**, Section 8.2.1] for a survey on Ore localizations and see [**Ta57**] and [**DLMSY03**, Corollary 6.3] for a proof of the statement.) If $\mathbb{Z}[\Gamma]$ is a domain, then the natural morphism $\mathbb{Z}[\Gamma] \to \mathbb{K}(\Gamma)$ is injective, and we can view $\mathbb{Z}[\Gamma]$ as a subring of the skew field $\mathbb{K}(\Gamma)$, with $\mathbb{K}(\Gamma)$ flat over $\mathbb{Z}[\Gamma]$.

It follows from the above that the group ring of a residually torsion-free elementary-amenable group is residually a ring that admits an Ore localization. Morphisms from $\mathbb{Z}[\pi]$ to group rings with Ore localizations played a major rôle in the work of Cochran–Orr–Teichner [**COT03**], Cochran [**Coc04**], Harvey [**Har05**] and also in [**Fri07, FST15, FTi15**].

(H.26) If N is fibered, then $\pi_1(N)$ is the semidirect product of $\mathbb{Z}$ with the fundamental group of a compact surface, and so $\pi_1(N)$ is good by Propositions 3.5 and 3.6 of [**GJZ08**].

Let N be compact, orientable, irreducible, with empty or toroidal boundary. Wilton–Zalesskii [**WZ10**, Corollary C] showed that if N is a graph manifold, then $\pi_1(N)$ is good. It follows from (I.5), (H.2), (H.3), (H.5), (H.17) and (H.20) that if N is not a graph manifold, then $\pi_1(N)$ is good. (For hyperbolic 3-manifolds this can be proved alternatively using the fact that right-angled Artin groups are good, see [**Lor08**, Theorem 1.4] or [**Ag14**, Theorem 9.3], that finite-index subgroups of good groups are good, see [**GJZ08**, Lemma 3.2], and that retracts of good groups are good, see [**Scv14**, Lemma 2.4], together with (I.5), (H.2), (H.6) and (H.7)).

Cavendish [**Cav12**, Section 3.5, Lemma 3.7.1], building on the results of Wise, showed that the fundamental group of any compact 3-manifold is good.

We state two consequences of being good. First of all, if G is a good, residually finite group of finite cohomological dimension, then its profinite completion is torsion-free [**Red13**, Corollary 7.6]. Secondly, any residually finite group G that is good is in fact *highly residually finite* (also called *super residually finite*), i.e., any finite extension $1 \to K \to H \to G \to 1$ of G by a residually finite group K is again residually finite; see [**Lor08**, Proposition 2.14] and also [**Red13**, Corollary 7.11].

(H.27) Suppose N is virtually fibered. Jaco–Evans [**Ja80**, p. 76] showed that $\pi_1(N)$ does not have the f.g.i.p., unless $\pi_1(N)$ is virtually solvable.

Combining this result with the ones above and with work of Soma [**Som92**], we obtain: Suppose N is compact, orientable, irreducible, with empty or toroidal boundary. Then $\pi_1(N)$ has the f.g.i.p. if and only if $\pi_1(N)$ is finite or solvable. Indeed, if $\pi_1(N)$ is finite or solvable, then $\pi_1(N)$ is virtually polycyclic and so every subgroup is finitely generated. (See [**Som92**, Lemma 2] for details.) If N is Seifert fibered and $\pi_1(N)$ is neither finite nor solvable, then $\pi_1(N)$ does not have the f.g.i.p. (See again [**Som92**, Proposition 3] for details.) It follows from the combination of (H.2), (H.5), (H.17), (H.20) and the above mentioned result of Jaco–Evans that if N is hyperbolic, then $\pi_1(N)$ does not have the f.g.i.p. either. Finally, if N has non-trivial JSJ-decomposition, then by the above already the fundamental group of a JSJ-component does not have the f.g.i.p., hence $\pi_1(N)$ also does not have the f.g.i.p.

Examples of 3-manifolds with non-toroidal boundary having fundamental group with the f.g.i.p. are given in [**Hem85b**, Theorem 1.3].

(H.28) If N is fibered, then $\pi_1(N)$ is a semidirect product of $\mathbb{Z}$ with the fundamental group of a compact surface, and so $\pi_1(N)$ has Property FD by [**LuSh04**, Theorem 2.8]. It follows from (I.6), (H.2), (H.3), (H.5), (H.17), and (H.20) that if N is compact, orientable, irreducible, with empty or toroidal boundary, which is not a closed graph manifold, then $\pi_1(N)$ has Property FD.

(H.29) Suppose N is fibered. Then there exists a compact surface Σ and a surjective morphism $\pi_1(N) \to \mathbb{Z}$ whose kernel equals $\pi_1(\Sigma)$. If Σ has boundary, then $\pi_1(\Sigma)$ is free, and if Σ is closed, then the kernel of any surjective morphism $\pi_1(\Sigma) \to \mathbb{Z}$ is a free group. In both cases we see that $\pi_1(N)$ is poly-free.

(H.30) Hermiller–Šunić [**HeS07**, Theorem A] showed that any RAAG is poly-free. It is clear that any subgroup of a poly-free group is also poly-free.

(H.31) Hsu–Wise [**HsW99**, Corollary 3.6] showed that any RAAG is linear over $\mathbb{Z}$. (See also [**DJ00**, p. 231].) The idea of the proof is that any RAAG embeds in a right angled Coxeter group, and these are known to be linear over $\mathbb{Z}$ (see for example [**Bou81**, Chapitre V, §4, Section 4]).

(H.32) The lower central series (π_n) of a group π is defined inductively via $\pi_1 := \pi$ and $\pi_{n+1} = [\pi, \pi_n]$. If π is a RAAG, then $\bigcap_n \pi_n = \{1\}$, and the successive quotients π_n/π_{n+1} are free abelian groups. This was proved by Duchamp–Krob [**DK92**, pp. 387 and 389], see also [**Dr83**, Section III]. This implies that any RAAG (and hence any subgroup of a RAAG) is residually torsion-free nilpotent.

(H.33) Gruenberg [**Gru57**, Theorem 2.1] showed that every torsion-free nilpotent group is residually p for any prime p.

(H.34) Any group that is residually p for all primes p is characteristically potent; see for example [**BuM06**, Proposition 2.2]. In particular such a group is weakly characteristically potent. We refer to [**ADL11**, Section 10] for more information and references on potent groups.

(H.35) Rhemtulla [**Rh73**] showed that a group that is residually p for infinitely many primes p, is in fact bi-orderable.

The combination of (H.32) and (H.33) with the above result of Rhemtulla [**Rh73**] implies that RAAGs are bi-orderable. This result was also proved directly by Duchamp–Thibon [**DpT92**]. This also follows from (H.32) and a theorem of B. Neumann [**Nema49**, Theorem 3.1].

Boyer–Rolfsen–Wiest [**BRW05**, Question 1.10] asked whether fundamental groups of compact, irreducible 3-manifold with empty or toroidal boundary are virtually bi-orderable. Chasing through the flowchart we see that the question has an affirmative answer for manifolds that are non-positively curved. By Theorem 4.7.1 it thus remains to address the question for graph manifolds that are not non-positively curved.

(H.36) Theorem 14.26 of [**Wis12a**] asserts that word-hyperbolic groups (in particular fundamental groups of closed hyperbolic 3-manifolds, see Proposition 4.4.2) which are virtually special are omnipotent.

Wise observes in [**Wis00**, Corollary 3.15] that if π is an omnipotent, torsion-free group and if $g, h \in \pi$ are two elements with g not conjugate to $h^{\pm 1}$, then there exists a morphism α from π to a finite group such that the orders of $\alpha(g)$ and $\alpha(h)$ are different. This can be viewed as a strong form of conjugacy separability for pairs of elements g, h with g not conjugate to $h^{\pm 1}$.

Wise states that a corresponding result holds in the cusped case [**Wis12a**, Remark 14.27]. However, cusped hyperbolic manifolds in general do not satisfy the definition of omnipotence given in (F.16). Indeed, it is easy to see that $\mathbb{Z}^2$ is not omnipotent [**Wis00**, Remark 3.3], and also that a retract of an omnipotent group is omnipotent. However, there are many examples of cusped hyperbolic 3-manifolds N such that $\pi_1(N)$ retracts onto a cusp subgroup (see, for instance, (H.9)), so $\pi_1(N)$ is not omnipotent.

Most of the 'green properties' are either green by definition or for elementary reasons. Furthermore, we already showed on page 58 that several group theoretic properties are 'green' It thus suffices to justify the following statements. Below we let π be a group.

(I.1) In Theorem 5.4.1 we show that the fundamental group of a compact, orientable, irreducible 3-manifold with empty or toroidal boundary, which has a hereditarily conjugacy separable subgroup of finite index, is also hereditarily conjugacy separable.

(I.2) The same argument as in (D.4) shows that the fundamental group of a hyperbolic 3-manifold, having a finite-index subgroup which is GFERF, is GFERF.

(I.3) Long–Reid [**LoR05**, Proof of Theorem 4.1.4] (or [**LoR08a**, Proof of Theorem 2.10]) showed that the ability to retract onto linear subgroups of π extends to finite-index supergroups of π. We get the following conclusions:
  (a) If a finite-index subgroup of π retracts onto cyclic subgroups, then π also retracts onto cyclic subgroups (since cyclic subgroups are linear).
  (b) If N is hyperbolic and has a finite cover N' such that $\pi_1(N')$ retracts onto geometrically finite subgroups, then the above and linearity of $\pi_1(N)$ imply that $\pi_1(N)$ also retracts onto geometrically finite subgroups.

(I.4) Suppose π is a torsion-free group that has a finite classifying space and which is good. If π admits a finite index subgroup which has the torsion-free elementary-amenable factorization property, then it follows from [**FST15**] (which builds heavily on work of Linnell–Schick [**LiS07**]) that π also has the torsion-free elementary-amenable factorization property.

By the discussion in (H.26) the fundamental group of any compact 3-manifold is good. It thus follows from (C.2) and (C.3) and from the above that having the torsion-free elementary-amenable factorization property is a 'green property' in the sense of (G.4).

(I.5) It follows from [**GJZ08**, Lemma 3.2] that a group is good if it admits a finite-index subgroup which is good.

(I.6) By [**LuSh04**, Corollary 2.5], a group with a finite-index subgroup which has Property FD also has Property FD.

(I.7) Suppose we have a finite-index subgroup π' of π that is weakly characteristically potent; then π is also weakly characteristically potent. To see this, since subgroups of weakly characteristically potent groups are weakly characteristically potent, we can assume that π' is in fact a characteristic subgroup of π, by a standard argument. Now let $g \in \pi$. Let $k \in \mathbb{N}$, $k \geq 1$, be minimal such that $g^k \in \pi'$. Since π' is weakly characteristically potent there exists an $r' \in \mathbb{N}$ such that for any n there exists a characteristic finite-index subgroup π_n of π' such that $g^k\pi_n$ has order $r'n$ in π'/π_n. Now we let $r = r'k$. Note that π_n is a normal subgroup of π since $\pi_n \subseteq \pi'$ is characteristic. Clearly $g^{rn} = 1 \in \pi/\pi_n$. Furthermore, if $g^m \in \pi_n$, then $g^m \in \pi'$, hence $m = km'$, and thus m divides $rn = kr'n$. Finally note that π_n is characteristic in π since $\pi_n \subseteq \pi'$ and $\pi' \subseteq \pi$ are characteristic. This shows that π is weakly characteristically potent.

5.3. Additional results and implications

In this short section we give a list of further results and alternative arguments which we left out of Flowchart 4. We let N be a 3-manifold and π be a group.

(J.1) Hamilton–Wilton–Zalesskii [**HWZ13**, Theorem 1.2] showed that if N is closed, orientable, irreducible, such that the fundamental group of each JSJ piece of N is conjugacy separable, then $\pi_1(N)$ is conjugacy separable. By doubling along the boundary and appealing to Lemma 1.4.3, the same result holds for compact, orientable, irreducible 3-manifolds with toroidal boundary.

It follows from (H.2), (H.6), (H.7), (H.8), and (I.1) that the fundamental group of each hyperbolic 3-manifold is conjugacy separable. Also, the fundamental group of each compact, orientable Seifert fibered manifold is conjugacy separable; see [**Mao07, AKT05, AKT10**]. The aforementioned result from [**HWZ13**] implies that the fundamental group of any compact, orientable, irreducible 3-manifold with empty or toroidal boundary is conjugacy separable.

Finally note that the argument of [**LyS77**, Theorem IV.4.6] also shows that if a finitely presented group π is conjugacy separable (see (F.2) for the definition), then the conjugacy problem is solvable for π. The above results thus give another solution to the Conjugacy Problem first solved by Préaux (see (C.33)).

(J.2) Droms [**Dr87**, Theorem 2] showed that a RAAG corresponding to a finite graph G is the fundamental group of a 3-dimensional manifold if and only if each connected component of G is either a tree or a triangle. Gordon [**Gon04**] (extending work of Brunner [**Bru92**] and Hermiller–Meier [**HM99**, Proposition 5.7]) furthermore determined which Artin groups appear as fundamental groups of 3-manifolds.

(J.3) Bridson [**Brd13**, Corollary 5.2] (see also [**Kob12b**, Proposition 1.3]) showed that if π has a subgroup of finite index that embeds into a RAAG, then there are infinitely many closed surfaces whose mapping class groups embed π. By the above results this applies to fundamental groups of compact, closed, orientable, irreducible 3-manifolds with empty or toroidal boundary which are not closed graph manifolds. Thus 'most' 3-manifold groups can be viewed as subgroups of mapping class groups.

(J.4) Suppose π is finitely generated. Given a proper subgroup Γ of π, we call (Γ, π) a *Grothendieck pair* if the natural inclusion $\Gamma \hookrightarrow \pi$ induces an isomorphism of profinite completions. One says that π is *Grothendieck rigid* if (π, Γ) is never a Grothendieck pair for each finitely generated proper subgroup Γ of π.

Platonov and Tavgen′ exhibited a residually finite group which is not Grothendieck rigid [**PT86**]. Answering a question of Grothendieck [**Grk70**, p. 384], Bridson and Grunewald [**BrGd04**] gave an example of a Grothendieck pair (π, Γ) where both π and Γ are residually finite and finitely presented.

If π is LERF, then π is Grothendieck rigid: if Γ is finitely generated and a proper subgroup of π then the inclusion map $\Gamma \hookrightarrow \pi$ does not induce a surjection on profinite completions. It thus follows from the above (see in particular (C.12) and (H.11)) that fundamental groups of Seifert fibered manifolds and hyperbolic 3-manifolds are Grothendieck rigid. This result was first obtained by Long and Reid [**LoR11**]. Cavendish [**Cav12**, Proposition 3.7.1] used (H.26) to show that the fundamental group of any closed prime 3-manifold is Grothendieck rigid.

(J.5) By (C.25), the fundamental groups of most compact 3-manifolds are not amenable. On the other hand, Theorems 4.2.2, 4.7.2, and 4.7.3, together with work of Mizuta [**Miz08**, Theorem 3] and Guentner–Higson [**GuH10**] imply that the fundamental group of each compact, irreducible 3-manifold which is not a closed graph manifold is 'weakly amenable.' For closed hyperbolic 3-manifolds this also follows from [**Oza08**].

(J.6) In [**FN14**] it is shown, building on the results of this section and on [**Nag14**, Proposition 4.18], that if $N \neq S^3$ is compact, orientable, irreducible 3-manifold with empty or toroidal boundary, then there exists some unitary representation $\alpha\colon \pi_1(N) \to U(k)$ such that all corresponding twisted homology groups $H_*^\alpha(N;\mathbb{C}^k)$ vanish.

(J.7) A group is *free-by-cyclic* if it is isomorphic to a semidirect product $\mathbb{Z} \ltimes F$ of the infinite cyclic group $\mathbb{Z}$ with a finitely generated non-cyclic free group F. If F is a free group on two generators, then any automorphism of F can be realized topologically by an automorphism of a once-punctured torus (see, e.g., [**LyS77**, Proposition 4.5]); in particular, $\mathbb{Z} \ltimes F$ is the fundamental group of a 3-manifold. On the other hand, if F is a free group of rank greater ≥ 4, then for 'most' automorphisms of F this is not true; see, e.g., [**Sta82**, p. 22], [**Ger83**, Theorem 3.9], [**BeH92**, Theorem 4.1]. The question to which extent properties of fundamental groups of fibered 3-manifolds with boundary carry over to the more general case of free-by-cyclic groups has been studied by many authors: [**AlR12, KR14, DKL12, DKL13, AHR13**]. Hagen–Wise [**HnW13, HnW14**], using work of Agol [**Ag13**], showed that if ϕ is an irreducible automorphism of a free group F such that $\pi = \mathbb{Z} \ltimes_\phi F$ is word-hyperbolic, then π is virtually compact special. Button [**But13**] used this result and work in [**BF92, Brm00, But08, But10**] to show that all free-by-cyclic groups are large.

5.4. Proofs

In this section we collect the proofs of several statements that were mentioned in the previous sections.

5.4.1. Conjugacy separability. Clearly a subgroup of a residually finite group is itself residually finite, and it is also easy to prove that a group with a residually finite subgroup of finite index is itself residually finite. In contrast, the property of conjugacy separability (see (F.2)) is more delicate. Goryaga [**Goa86**] gave an example of a non-conjugacy-separable group with a conjugacy separable subgroup of finite index. In the other direction, Martino–Minasyan [**MMn12**, Theorem 1.1] constructed examples of conjugacy separable groups with non-conjugacy-separable subgroups of finite index.

For this reason, one defines a group to be *hereditarily conjugacy separable* if every finite-index subgroup is conjugacy separable. In the 3-manifold context, hereditary conjugacy separability passes to finite-index supergroups:

THEOREM 5.4.1. *Let N be a compact, orientable, irreducible 3-manifold with toroidal boundary, and let π' be a finite-index subgroup of $\pi = \pi_1(N)$. If π' is hereditarily conjugacy separable, then so is π.*

The proof of this theorem is based on the following useful criterion due to Chagas–Zalesskii [**ChZ10**, Proposition 2.1].

PROPOSITION 5.4.2. *Let π be a finitely generated group and π' be a conjugacy separable normal subgroup of finite index of π. Let $g \in \pi$ and $m \geq 1$ with $g^m \in \pi'$. Suppose*

(1) *$C_\pi(g^m)$ is conjugacy separable; and*
(2) *$\widehat{C_{\pi'}(g^m)} = \overline{C_{\pi'}(g^m)} = C_{\widehat{\pi'}}(g^m)$.*

Then whenever $h \in \pi$ is not conjugate to g, there is a morphism α from π to a finite group such that $\alpha(g)$ is not conjugate to $\alpha(h)$.

To show Theorem 5.4.1, we may assume that N does not admit Sol geometry, as polycyclic groups are known to be conjugacy separable by a theorem of Remeslennikov [**Rev69**]. Furthermore, we may assume that π' is normal, corresponding to a regular covering map $N' \to N$ of finite degree. In particular, π' is also the fundamental group of a compact, orientable, irreducible 3-manifold with toroidal boundary. We summarize the structure of the centralizers of elements of π in the following proposition, which is an immediate consequence of Theorems 2.5.1 and 2.5.2.

PROPOSITION 5.4.3. *Let N be a compact, orientable, irreducible 3-manifold with toroidal boundary that does not admit* Sol *geometry, and let $g \in \pi = \pi_1(N)$, $g \neq 1$. Then either $C_\pi(g)$ is free abelian or there is a Seifert fibered piece N' of the JSJ-decomposition of N such that $C_\pi(g)$ is a subgroup of index at most two in $\pi_1(N')$.*

Now we will prove three lemmas about centralizers. These enable us to apply the result of Chagas–Zalesskii to finish the proof of Theorem 5.4.1. In all three lemmas, we let N, π, and g be as in the preceding proposition.

LEMMA 5.4.4. *The centralizer $C_\pi(g)$ is conjugacy separable.*

PROOF. This is clear if $C_\pi(g)$ is free abelian. Otherwise, $C_\pi(g)$ is the fundamental group of a Seifert fibered manifold and hence conjugacy separable by a theorem of Martino [**Mao07**]. □

Let G be a group. Recall that we denote by $\widehat{G}$ the profinite completion of G. (We again refer to [**RiZ10**, Section 3.2] and [**Win98**, Section 1] for the definition and for the main properties.) For a subgroup H of G, let $\overline{H}$ denote the closure of H in $\widehat{G}$.

LEMMA 5.4.5. *The canonical map $\widehat{C_\pi(g)} \to \overline{C_\pi(g)}$ is an isomorphism.*

PROOF. We need to prove that the profinite topology on π induces the full profinite topology on $C_\pi(g)$. To this end, it is enough to prove that every finite-index subgroup H of $C_\pi(g)$ is separable in π.

If $C_\pi(g)$ is free abelian, then so is H, so H is separable by the main theorem of [**Hamb01**]. Therefore, suppose $C_\pi(g)$ is a subgroup of index at most two in $\pi_1(M)$, where M is a Seifert fibered vertex space of N, so H is a subgroup of finite index in $\pi_1(M)$. By (C.35), the group π induces the full profinite topology on $\pi_1(M)$, and $\pi_1(M)$ is separable in π. It follows that H is separable in π. □

The final condition is a direct consequence of Proposition 3.2 and Corollary 12.2 of [**Min12**], together with the hypothesis that π' is hereditarily conjugacy separable.

LEMMA 5.4.6. *The inclusion* $\overline{C_\pi(g)} \to C_{\widehat{\pi}}(g)$ *is surjective.*

Now we are in a position to prove Theorem 5.4.1.

PROOF OF THEOREM 5.4.1. As mentioned earlier, we may assume that N does not admit Sol geometry. Let π' be a hereditarily conjugacy separable subgroup of $\pi = \pi_1(N)$ of finite index. By replacing π' with the intersection of its conjugates, we may assume that π' is normal. Let $g, h \in \pi$ be non-conjugate and $m \geq 1$ with $g^m \in \pi'$. By Lemma 5.4.4, $C_\pi(g^m)$ is conjugacy separable. Let N' be a finite-sheeted covering of N with $\pi' = \pi_1(N')$. Lemma 5.4.5 applied to π' in place of π shows that the first equality in condition (2) of Proposition 5.4.2 holds, and Lemma 5.4.6 shows that the second equality holds. Proposition 5.4.2 now yields a morphism α from π to a finite group such that $\alpha(g)$, $\alpha(h)$ are non-conjugate. Thus π is conjugacy separable. □

5.4.2. Non-virtually-fibered graph manifolds and retractions onto cyclic subgroups. There exist Seifert fibered manifolds which are not virtually fibered, and also graph manifolds with non-trivial JSJ-decomposition which are not virtually fibered (see, e.g., [**LuW93**, p. 86] and [**Nemd96**, Theorem D]). The following proposition shows that such examples also have the property that their fundamental groups do not virtually retract onto cyclic subgroups.

PROPOSITION 5.4.7. *Let N be a non-spherical graph manifold. If N is not virtually fibered, then $\pi_1(N)$ has a cyclic subgroup onto which it does not virtually retract.*

PROOF. We show the contrapositive: Suppose that $\pi_1(N)$ virtually retracts onto all its cyclic subgroups; we will prove that N is virtually fibered. By the last remark in (C.10) the manifold N is finitely covered by a 3-manifold each of whose JSJ-components is an S^1-bundle over a surface. We can therefore without loss of generality assume that N itself is already of that form.

We first consider the case that N is a Seifert fibered manifold, i.e., that N is an S^1-bundle over a surface Σ. As mentioned in Section 1.5, the assumption that N is non-spherical implies that the regular fiber generates an infinite cyclic subgroup of $\pi_1(N)$. It is well known, and straightforward to see, that if $\pi_1(N)$ retracts onto this infinite cyclic subgroup, then N is a product $S^1 \times \Sigma$; in particular, N is fibered.

Now we consider the case that N has a non-trivial JSJ-decomposition. We denote the JSJ-components of N by $M_1, \dots, M_n$. By hypothesis, each M_i is an S^1-bundle over a surface with non-empty boundary, so each M_i is in fact a product. We denote by f_i the Seifert fiber of M_i; each f_i generates an infinite cyclic subgroup of $\pi_1(N)$.

Since $\pi_1(N)$ virtually retracts onto cyclic subgroups, for each i we can find a finite-sheeted covering space $\widetilde{N}_i$ of N such that $\tilde{\pi}_i = \pi_1(\widetilde{N}_i)$ retracts onto $\langle f_i \rangle$. In particular, the image of f_i in $H_1(\widetilde{N}_i; \mathbb{Z})$/torsion is non-zero. Let $\widetilde{N}$ be a regular finite-sheeted cover of N that covers every $\widetilde{N}_i$. (For instance, $\pi_1(\widetilde{N})$ could be the intersection of all the conjugates of $\bigcap_{i=1}^n \tilde{\pi}_i$.)

Let $\tilde{f}$ be a Seifert fiber of a JSJ-component of $\widetilde{N}$. Up to the action of the deck group of $\widetilde{N} \to N$, $\tilde{f}$ covers the lift of some f_i in $\widetilde{N}_i$. Thus $\tilde{f}$ is non-zero in $H_1(\widetilde{N}; \mathbb{Z})$/torsion. So we can take a group morphism $\phi\colon H_1(\widetilde{N}; \mathbb{Z}) \to \mathbb{Z}$ which is non-trivial on the Seifert fibers of all JSJ-components of $\widetilde{N}$. Since each JSJ-component is a product, the restriction of ϕ

to each JSJ-component of $\widetilde{N}$ is a fibered class. By [**EN85**, Theorem 4.2], we conclude that $\widetilde{N}$ fibers over S^1. ☐

5.4.3. (Fibered) faces of the Thurston norm ball of finite covers. *In this subsection we let N be a compact, orientable 3–manifold.* Recall that we say that $\phi \in H^1(N;\mathbb{R})$ is *fibered* if ϕ can be represented by a non-degenerate closed 1-form. The *Thurston norm* of $\phi \in H^1(N;\mathbb{Z})$ is defined as

$$\|\phi\|_T = \min\left\{\chi_-(\Sigma) : \Sigma \subseteq N \text{ properly embedded surface dual to } \phi\right\}.$$

Here, given a surface Σ with connected components $\Sigma_1,\dots,\Sigma_n$, we define $\chi_-(\Sigma) = \sum_{i=1}^n \max\{-\chi(\Sigma_i),0\}$. Thurston [**Thu86a**, Theorems 2 and 5] (see also [**Kap01**, Chapter 2], [**CdC03**, Chapter 10], and [**Oe86**, p. 259]) proved the following results:

(1) $\|-\|_T$ is a seminorm on $H^1(N;\mathbb{Z})$ which can be extended to a seminorm, also denoted by $\|-\|_T$, on $H^1(N;\mathbb{R})$.
(2) The norm ball $B_N := \left\{\phi \in H^1(N;\mathbb{R}) : \|\phi\|_T \le 1\right\}$ is a finite-sided rational polytope.
(3) There exist open top-dimensional faces $F_1,\dots,F_m$ of B_N such that

$$\left\{\phi \in H^1(N;\mathbb{R}) : \phi \text{ fibered}\right\} = \bigcup_{i=1}^m \mathbb{R}^{>0} F_i.$$

These faces are called the *fibered faces* of B_N and the cones on these faces are called the *fibered cones* of B_N.

In general the Thurston norm is degenerate, e.g., for 3-manifolds with homologically essential tori. On the other hand the Thurston norm of a hyperbolic 3-manifold is non-degenerate, since a hyperbolic 3-manifold admits no homologically essential surfaces of non-negative Euler characteristic.

We start out with the following fact.

PROPOSITION 5.4.8. *Let $p\colon N' \to N$ be a finite cover. Then $\phi \in H^1(N;\mathbb{R})$ is fibered if and only if $p^*\phi \in H^1(N';\mathbb{R})$ is fibered. Moreover,*

$$\|p^*\phi\|_T = [N' : N]\cdot\|\phi\|_T \quad \textit{for any class } \phi \in H^1(N;\mathbb{R}).$$

In particular, the map $p^*\colon H^1(N;\mathbb{R}) \to H^1(N';\mathbb{R})$ is, up to a scale factor, an isometry, and it maps fibered cones into fibered cones.

PROOF. The first statement for rational cohomology classes is an immediate consequence of Stallings' Fibration Theorem (see [**Sta62**] and (L.9)), and the second statement follows from work of Gabai [**Gab83a**, Corollary 6.13]. (See also [**Pes93**] for an alternative proof.) The first statement for real classes is now an immediate consequence of the the above two facts and general facts on fibered cones of 3-manifolds. ☐

Now we can prove the following proposition.

PROPOSITION 5.4.9. *Suppose N is irreducible with empty or toroidal boundary, and N is not a graph manifold. Then for each n, there is a finite cover of N whose Thurston norm ball has at least n faces.*

If N is closed and hyperbolic, then the proposition relies on the Virtually Compact Special Theorem (Theorem 4.2.2). In the other cases it follows from the work of Cooper–Long–Reid [**CLR97**] and classical facts on the Thurston norm.

Proof. We first suppose that N is hyperbolic. It follows from (H.2), (H.6), (H.13), and (C.17) that N admits a finite cover N' with $b_1(N') \geq n$. Since the Thurston norm of a hyperbolic 3-manifold is non-degenerate it follows that $B_{N'}$ has at least $n+1$ faces.

Now we suppose that N is not hyperbolic. By hypothesis of the proposition, we can take a hyperbolic JSJ-component X of N which is hyperbolic and which necessarily has non-empty boundary. It follows from [**CLR97**, Theorem 1.3] (see also (C.15)) that $\pi_1(X)$ is large. Hence by (C.17) we can take a finite cover X' of X with non-peripheral homology of rank at least n.

A standard argument using (C.35) shows that we can take a finite cover N' of N which has a hyperbolic JSJ-component Y covering X'. An elementary argument also shows that Y has non-peripheral homology of rank $\geq n$. We consider the natural morphism $p\colon H_2(Y;\mathbb{R}) \to H_2(Y, \partial Y;\mathbb{R})$ and set $V := \operatorname{Im} p$. Using Poincaré Duality, the Universal Coefficient Theorem and the information on the non-peripheral homology, we see that $\dim(V) \geq n$.

Now we consider the natural morphism $q\colon H_2(Y;\mathbb{R}) \to H_2(N', \partial N';\mathbb{R})$ and set $W := \operatorname{Im} q$. Since p is the composition of q and the restriction map $H_2(N', \partial N';\mathbb{R}) \to H_2(Y, \partial Y;\mathbb{R})$, we see that $\dim W \geq \dim V \geq n$. Since N is hyperbolic, it follows that the Thurston norm of Y is non-degenerate, in particular non-degenerate on V. By [**EN85**, Proposition 3.5] the Thurston norm of $p_*\phi$ in Y agrees with the Thurston norm of $q_*\phi$ in N'. Thus the Thurston norm of N' is non-degenerate on W; in particular, $B_{N'}$ has at least $n+1$ faces. □

We say that $\phi \in H^1(N;\mathbb{R})$ is *quasi-fibered* if ϕ lies on the closure of a fibered cone of the Thurston norm ball of N. Now we can formulate Agol's Virtually Fibered Theorem [**Ag08**, Theorem 5.1].

Theorem 5.4.10 (Agol). *Suppose N is irreducible with empty or toroidal boundary such that $\pi_1(N)$ is virtually RFRS. Then for each $\phi \in H^1(N;\mathbb{R})$ there exists a finite cover $p\colon N' \to N$ such that $p^*\phi$ is quasi-fibered.*

The following is now a straightforward consequence of Agol's theorem.

Proposition 5.4.11. *Suppose N is irreducible with empty or toroidal boundary. Suppose B_N has n faces, $\pi_1(N)$ is virtually RFRS, and N is not a graph manifold. Then there exists a finite cover N' of N such that $B_{N'}$ has at least n fibered faces.*

The proof of the proposition is precisely that of [**Ag08**, Theorem 7.2]. We therefore give just a very quick outline of the proof. We pick classes ϕ_i ($i = 1, \dots, n$) in $H^1(N;\mathbb{R})$ which lie in n faces. For $i = 1, \dots, n$ we then apply Theorem 5.4.10 to the class ϕ_i and we obtain a finite cover $N_i' \to N$ such that for each i, the pull-back of ϕ_i is quasi-fibered.

Denote by $p\colon N' \to N$ the cover corresponding to $\bigcap_i \pi_1(N_i')$. It follows from Proposition 5.4.8 that pull-backs of quasi-fibered classes are quasi-fibered, and that pull-backs of B_N lie on faces of $B_{N'}$. Thus $p^*\phi_1, \dots, p^*\phi_n$ lie on closures of fibered faces of N', so N' has at least n fibered faces. □

It is a natural question to ask in how many different ways a (virtually) fibered 3-manifold (virtually) fibers. We recall the following facts:

(1) If $\phi \in H^1(N;\mathbb{Z})$ is a fibered class, then by [**EdL83**, Lemma 5.1] there exists, up to isotopy, a unique surface bundle representing ϕ.
(2) It follows from the description of fibered cones on p. 97 that being fibered is an open condition in $H^1(N;\mathbb{R})$. We refer to [**Nemd79**] and [**BNS87**, Theorem A]

for a group-theoretic proof for classes in $H^1(N;\mathbb{Q})$, to [**To69**] and [**Nemb76**] for earlier results and to [**HLMA06**] for an explicit discussion for a particular example. If $b_1(N) \geq 2$ and if the Thurston norm is not identically zero, then a basic Thurston norm argument shows that N admits fibrations with connected fibers of arbitrarily large genus. (See, e.g., [**But07**, Theorem 4.2] for details).

A deeper question is whether a 3-manifold admits (virtually) many distinct fibered faces. The following proposition is an immediate consequence of Propositions 5.4.9 and 5.4.11 together with (H.2), (H.3), (H.5), and (H.17).

PROPOSITION 5.4.12. *Suppose N is irreducible with empty or toroidal boundary, and N is not a graph manifold. Then for each n, N has a finite cover whose Thurston norm ball has at least n fibered faces.*

REMARKS.

(1) Suppose N is irreducible with empty or toroidal boundary such that $\pi_1(N)$ is virtually RFRS but not virtually abelian. According to [**Ag08**, Theorem 7.2], N has finite covers with arbitrarily many faces in the Thurston norm ball. At this level of generality the statement does not hold. As an example consider the product manifold $N = S^1 \times \Sigma$ where Σ is a compact, orientable surface. Any finite cover of N is again a product. It is well known (see, e.g. [**McM02**]) that the Thurston norm ball of such a product manifold has just two faces.

(2) It would be interesting to find criteria which decide whether a given graph manifold has virtually arbitrarily many faces in the Thurston norm ball.

CHAPTER 6

Subgroups of 3-manifold groups

In this chapter we collect properties of finitely generated infinite-index subgroups of 3-manifold groups in a flowchart (Flowchart 5), which can be found on p. 103. The organization is similar to previous chapters. Section 6.1 contains the definitions of the properties discussed in the flowchart, Section 6.2 contains the justifications for the implications in the flowchart, and Section 6.3 lists some additional material not contained in the flowchart. Since the studies of 3-manifold groups and of their subgroups are closely intertwined, the content of this chapter partly overlaps with the results mentioned in the previous chapters.

6.1. Definitions and Conventions

Most of the definitions required for understanding Flowchart 5 have been introduced in the previous chapters. Therefore we need to introduce only the following new definitions. We let N be a 3-manifold and π be a group.

(K.1) Let X be a connected subspace of N and Γ be a subgroup of $\pi_1(N)$. We say that Γ *is carried by* X if $\Gamma \subseteq \operatorname{Im}\{\pi_1(X) \to \pi_1(N)\}$ (up to conjugation).

(K.2) Suppose π is finitely generated, and let Γ be a finitely generated subgroup of π. We say that the *membership problem is solvable for* Γ if for all generators $g_1, \dots, g_n$ for π there exists an algorithm which can determine whether or not an input word in $g_1, \dots, g_n$ defines an element of Γ. We refer to [**CZi93**, Section D.1.1.9] and to [**AFW13**] for more about this concept.

(K.3) Let $\Sigma \subseteq N$ be a connected compact surface. We call Σ a *semifiber* if N is the union of two twisted I-bundles over the non-orientable surface Σ along their S^0-bundles. (In the literature usually the surface given by the S^0-bundle is referred to as a 'semifiber.') Note that if Σ is a semifiber, then N has a double cover p such that $p^{-1}(\Sigma)$ consists of two components, each of which is a surface fiber.

(K.4) Let Γ be a subgroup of π. The notion of the *width of* Γ *in* π was defined in [**GMRS98**] as follows. First we say that $g_1, \dots, g_n \in \pi$ are *essentially distinct* (with respect to Γ) if $\Gamma g_i \neq \Gamma g_j$ whenever $i \neq j$. Conjugates of Γ by essentially distinct elements are called *essentially distinct conjugates.* Then the *width* of Γ in π is defined as the maximal $n \in \mathbb{N} \cup \{\infty\}$ such that there exists a collection of n essentially distinct conjugates of Γ with the property that the intersection of any two elements of the collection is infinite. The width of Γ is 1 if Γ is malnormal. If Γ is normal and infinite, then the width of Γ equals its index.

(K.5) Let Γ be a subgroup of π. The commensurator subgroup of Γ is

$$\operatorname{Comm}_\pi(\Gamma) := \big\{ g \in \pi : \Gamma \cap g\Gamma g^{-1} \text{ has finite index in } \Gamma \big\}.$$

(K.6) Let $\mathcal{P}$ be a property of subgroups of a given group. A subgroup Γ of π is *virtually* $\mathcal{P}$ (in π) if π admits a (not necessarily normal) subgroup π' of finite index which contains Γ and such that Γ, viewed as subgroup of π', satisfies $\mathcal{P}$.

As in Flowcharts 1 and 4 we use the convention that if an arrows splits into several arrows, then exactly one of the possible conclusions holds. Furthermore, if an arrow is decorated with a condition, then the conclusion holds if that condition is satisfied.

6.2. Justifications

In Flowchart 5 we put several restrictions on the 3-manifold N which we consider. Below, in the justifications for the arrows in Flowchart 5, we will only assume that N is connected, without any other blanket restrictions on N. Before we give the justifications we point out that only (L.15) depends on the Virtually Compact Special Theorem.

(L.1) It follows from the Sphere Theorem (see Theorem 1.3.2) that each compact, irreducible 3-manifold with empty or toroidal boundary, whose fundamental group has a non-trivial finite subgroup, is spherical. See (C.3) for details.

(L.2) Suppose N is compact and let Γ be a finitely generated subgroup of $\pi = \pi_1(N)$. Then Γ either
  (a) is virtually solvable, or
  (b) contains a non-cyclic free subgroup.

  (In other words, π satisfies the 'Tits Alternative.') Indeed, by Scott's Core Theorem (C.5) applied to the covering of N corresponding to Γ, there exists a compact 3-manifold M with $\pi_1(M) = \Gamma$. It follows easily from Theorem 1.2.1, Lemma 1.4.2, Lemma 1.11.2, combined with (C.24) and (C.26), that if $\pi_1(M)$ is not virtually solvable, then it contains a non-cyclic free subgroup.

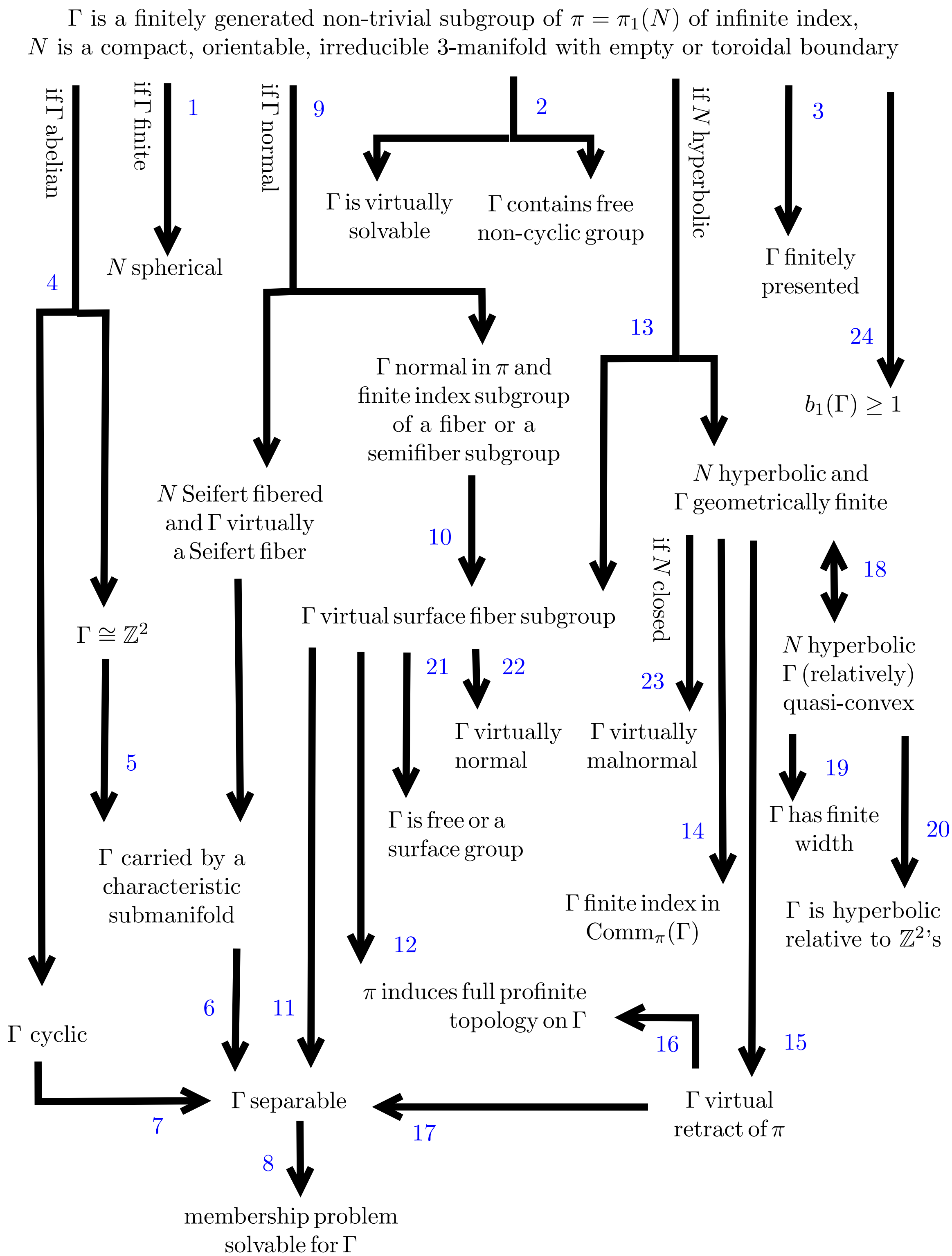
(L.3) Scott [**Sco73b**] proved that any finitely generated 3-manifold group is also finitely presented. See (C.5) for more information.

(L.4) Suppose N is compact, orientable, irreducible, with empty or toroidal boundary. Let Γ be an abelian subgroup of $\pi_1(N)$. It follows from Theorems 2.5.1 and 2.5.2, or alternatively from the remark after the proof of Theorem 1.11.1 and the Core Theorem (C.5), that Γ is either cyclic, $\Gamma \cong \mathbb{Z}^2$, or $\Gamma \cong \mathbb{Z}^3$. In the latter case it follows from the discussion in Section 1.11 that N is the 3-torus and that Γ is a finite-index subgroup of $\pi_1(N)$.

(L.5) Suppose N is compact, orientable, irreducible, with empty or toroidal boundary, and let Γ be a subgroup of $\pi_1(N)$ isomorphic to $\mathbb{Z}^2$. Then there exists a singular map $f\colon T \to N$, where T is the 2-torus, such that $f_*(\pi_1(T)) = \Gamma$. It follows from the Characteristic Pair Theorem 1.7.7 that Γ is carried by a characteristic submanifold.

  The above statement is also known as the 'Torus Theorem.' It was announced by Waldhausen [**Wan69**]; the first proof was given by Feustel [**Feu76a**, p. 29], [**Feu76b**, p. 56]. We refer to [**Cal14a**, Theorem 5.1] for a proof due to Casson and to [**Wan69, CF76, Milb84, Sco80, Sco84**] for information on the closely related 'Annulus Theorem.' Both theorems can be viewed as predecessors of the Characteristic Pair Theorem.

(L.6) Suppose $N$is compact, orientable, with empty or toroidal boundary. Let M be a characteristic submanifold of N and Γ be a finitely generated subgroup of $\pi_1(M)$. Then Γ is separable in $\pi_1(M)$ by Scott's Theorem [**Sco78**, Theorem 4.1] (see also (C.12)), and $\pi_1(M)$ is separable in $\pi_1(N)$ by Wilton–Zalesskii [**WZ10**, Theorem A] (see also (C.35)). It follows that Γ is separable in $\pi_1(N)$. The same argument also generalizes to hyperbolic JSJ-components with LERF fundamental groups. More precisely, if M is a hyperbolic JSJ-component of N such



Flowchart 5. Subgroups of 3-manifolds groups.

that $\pi_1(M)$ is LERF and if Γ is a finitely generated subgroup of $\pi_1(M)$, then Γ is separable in $\pi_1(N)$.

(L.7) E. Hamilton [**Hamb01**] showed that the fundamental group of any compact, orientable 3-manifold N is abelian subgroup separable. In particular, any cyclic subgroup of $\pi_1(N)$ is separable. See also (C.32) for more information.

It follows from (H.18) that if $\pi_1(N)$ is virtually RFRS, then every infinite cyclic subgroup Γ of $\pi_1(N)$ is a virtual retract of $\pi_1(N)$; by (L.17), this gives another proof that Γ is separable.

(L.8) The argument of [**LyS77**, Theorem IV.4.6] can be used to show that if Γ is a finitely generated separable subgroup of a finitely presented group, then the membership problem for Γ is solvable.

In [**FrW13**] it is shown, building on the Virtually Compact Special Theorem and work of Kapovich–Miasnikov–Weidmann [**KMW05**], that the membership problem is in fact solvable for any finitely generated subgroup of fundamental groups of any compact 3-manifold.

(L.9) Suppose N is compact and orientable, and let Γ be a normal finitely generated non-trivial subgroup of $\pi_1(N)$ of infinite index. Work of Hempel–Jaco [**HJ72**, Theorem 3], the resolution of the Poincaré Conjecture, Theorem 2.5.5, and (L.3) imply that one of the following conclusions hold:
(a) N is Seifert fibered and Γ is a subgroup of the Seifert fiber subgroup, or
(b) N fibers over S^1 with surface fiber Σ and Γ is a finite-index subgroup of $\pi_1(\Sigma)$, or
(c) N is the union of two twisted I-bundles over a compact connected (necessarily non-orientable) surface Σ, which meet in the corresponding S^0-bundles, and Γ is a finite-index subgroup of $\pi_1(\Sigma)$.

In particular, if $\Gamma = \operatorname{Ker}(\phi)$ for some morphism $\phi\colon \pi_1(N) \to \mathbb{Z}$, then $\phi = p_*$ for some surface bundle $p\colon N \to S^1$. This special case was first proved by Stallings [**Sta62**] and is known as Stallings' Fibration Theorem. Generalizations to subnormal groups were formulated and proved by Elkalla [**El84**, Theorem 3.7] and Bieri–Hillman [**BiH91**].

Recall that the *normalizer* of a subgroup Γ of a group π is the subgroup

$$N_\pi(\Gamma) := \{g \in \pi : g\Gamma g^{-1} = \Gamma\}$$

of π. Now suppose N is compact, orientable, and irreducible, and let $\Sigma \subseteq N$ be a closed 2-sided incompressible orientable non-fiber surface. Heil [**Hei81**, p. 147] showed that $\pi_1(\Sigma)$ is its own normalizer in $\pi_1(N)$ unless Σ bounds in N a twisted I-bundle over a closed (necessarily non-orientable) surface. (See also [**Hei71, Swp75, Ein76a**] and [**HeR84**].) An extension to the case of properly embedded incompressible surfaces was proved by Heil–Rakovec [**HeR84**, Theorem 3.7]. Moreover, Heil [**Hei81**, p. 148] showed that if $M \subseteq \operatorname{int}(N)$ is a compact π_1-injective submanifold, then $\pi_1(M)$ is its own normalizer in $\pi_1(N)$ unless $M \cong \Sigma \times I$ for some surface Σ. This generalizes earlier work by Eisner [**Ein77a**].

(L.10) Let N be compact, and let Γ be a normal subgroup of $\pi_1(N)$ which is also a finite-index subgroup of $\pi_1(\Sigma)$, where Σ is a surface fiber of a surface bundle $N \to S^1$. We identify $\pi_1(N)$ with $\mathbb{Z} \ltimes \pi_1(\Sigma)$. Since Γ is normal, we have a subgroup $\widetilde{\pi} := \mathbb{Z} \ltimes \Gamma$ of $\mathbb{Z} \ltimes \pi_1(\Sigma) = \pi_1(N)$. Denote by $\widetilde{N}$ the finite cover of N corresponding to $\widetilde{\pi}$. It is clear that Γ is a surface fiber subgroup of $\widetilde{N}$.

(L.11) Suppose N is compact, orientable, with empty or toroidal boundary, and let Σ be a fiber of a surface bundle $N \to S^1$. Then $\pi_1(N) \cong \mathbb{Z} \ltimes \pi_1(\Sigma)$, and $\pi_1(\Sigma) \subseteq \pi_1(N)$ is therefore separable. From this it follows easily that every virtual surface fiber subgroup of a 3-manifold group is separable.

(L.12) Suppose N is compact, orientable, with empty or toroidal boundary, and let Σ be a fiber of a surface bundle $N \to S^1$. Then $\pi_1(N) \cong \mathbb{Z} \ltimes \pi_1(\Sigma)$ where $1 \in \mathbb{Z}$ acts by some $\Phi \in \operatorname{Aut}(\pi_1(\Sigma))$. Let Γ be a finite-index subgroup of $\pi_1(\Sigma)$. Because $\pi_1(\Sigma)$ is finitely generated, there are only finitely many subgroups of index $[\pi_1(\Sigma) : \Gamma]$, and so $\Phi^n(\Gamma) = \Gamma$ for some n. Now $\widetilde{\pi} := n\mathbb{Z} \ltimes \Gamma$ is a subgroup of finite index in $\pi_1(N)$ such that $\widetilde{\pi} \cap \pi_1(\Sigma) = \Gamma$. This shows that π induces the full profinite topology on the surface fiber subgroup $\pi_1(\Sigma)$. It follows easily that π also induces the full profinite topology on any virtual surface fiber subgroup.

(L.13) Suppose N is hyperbolic. The Subgroup Tameness Theorem (Theorem 4.1.2) asserts that if Γ is a finitely generated subgroup of $\pi_1(N)$, then Γ is either geometrically finite or a virtual surface fiber subgroup. See (L.14), (L.15), (L.19), (L.23) for other formulations of this fundamental dichotomy.

(L.14) Let N be a hyperbolic 3-manifold and let Γ be a geometrically finite subgroup of $\pi = \pi_1(N)$ of infinite index. Then Γ has finite index in $\operatorname{Comm}_\pi(\Gamma)$. See [**Cay08**, Theorem 8.7] for a proof (and also [**KaS96**] and [**Ar01**, Theorem 2]), and [**Ar01**, Section 5] for more results in this direction. If Γ is a virtual surface fiber subgroup of π, then $\operatorname{Comm}_\pi(\Gamma)$ is easily seen to be a finite-index subgroup of π, so Γ has infinite index in its commensurator. The commensurator thus gives another way to formulate the dichotomy of (L.13).

(L.15) Let N be a hyperbolic 3-manifold and Γ be a geometrically finite subgroup of $\pi = \pi_1(N)$. In (H.9) we saw that it follows from the Virtually Compact Special Theorem that Γ is a virtual retract of π. On the other hand, it is straightforward to see that if Γ is a virtual surface fiber subgroup of π and if the monodromy of the surface bundle does not have finite order, then Γ is not a virtual retract of π. We thus obtain one more way to formulate the dichotomy of (L.13).

(L.16) It is easy to prove that every group induces the full profinite topology on each of its virtual retracts.

(L.17) Let π be a residually finite group (e.g., a 3-manifold group, see (C.29)). Every virtual retract of π is also separable in π. See (H.10) for details.

(L.18) Suppose N is hyperbolic. As observed in Proposition 4.4.2, if N is closed then $\pi = \pi_1(N)$ is word-hyperbolic, and a subgroup of π is geometrically finite if and only if it is quasi-convex. (See [**Swp93**, Theorem 1.1 and Proposition 1.3] and also [**KaS96**, Theorem 2].) If N has toroidal boundary, then π is not word-hyperbolic, but it is hyperbolic *relative to its collection of peripheral subgroups.* By [**Hr10**, Corollary 1.3], a subgroup of π is geometrically finite if and only if it is *relatively quasi-convex.* The reader is referred to [**Hr10**] for thorough treatments of the various definitions of relative hyperbolicity and of relative quasi-convexity, as well as proofs of their equivalence.

(L.19) Suppose N is hyperbolic, and let Γ be subgroup of $\pi = \pi_1(N)$. Suppose first that Γ is a geometrically finite subgroup of π. By (L.18) this means that Γ is a relatively quasi-convex subgroup of π. The main result of [**GMRS98**] shows that the width of Γ is finite when N is closed (so π is word-hyperbolic by Proposition 4.4.2); this also holds in general, by [**HrW09**]. If, on the other hand, Γ is a virtual surface fiber subgroup of N, then the width of Γ is infinite. The width thus gives another way to formulate the dichotomy of (L.13).

(L.20) Suppose N is hyperbolic with boundary components $T_1, \dots, T_n$, and let Γ be a geometrically finite subgroup of $\pi = \pi_1(N)$. It follows from (L.18) and [**Hr10**, Theorem 9.1] that Γ is hyperbolic relative to a set $\mathcal{P}$ of Γ-conjugacy class representatives for the following collection of subgroups of π:

$$\left\{\Gamma \cap g\pi_1(T_i)g^{-1} \,:\, g \in \pi,\ i = 1, \dots, n,\ \text{and } \Gamma \cap g\pi_1(T_i)g^{-1} \neq \{1\}\right\}.$$

All elements of $\mathcal{P}$ are isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}^2$. In fact, it follows from [**Osi06**, Theorem 2.40] that Γ is hyperbolic relative to those elements of $\mathcal{P}$ isomorphic to $\mathbb{Z}^2$. In particular, if all elements of $\mathcal{P}$ are cyclic, then Γ is word-hyperbolic.

The fundamental group of a surface of negative Euler characteristic is also word-hyperbolic. It follows from (L.21), (L.13), and the above discussion that any finitely generated subgroup of the fundamental group of a closed hyperbolic 3-manifold is word-hyperbolic.

(L.21) Let Γ be a virtual surface fiber subgroup in an aspherical 3-manifold. Then Γ is torsion-free by (C.3) and contains a finite-index subgroup which is a free group or a surface group. It follows from [**Sta68b**, (0.2)] (see also [**Sta68a**, Theorem 3]) and [**EcM80**, Corollary 2] that Γ is a free group or a surface group.

Groves, Manning and Wilton [**GMW12**, Theorem 5.1] combined Geometrization, Mostow Rigidity, the Convergence Group Theorem of Casson–Jungreis–Gabai, and a theorem of Zimmerman [**Zim82**], to prove an extension of this result to 3-manifold groups: a torsion-free group is the fundamental group of a closed, irreducible 3-manifold if it is virtually the fundamental group of a closed, irreducible 3-manifold.

(L.22) Suppose N is compact, orientable, irreducible, and let Γ be a subgroup of infinite index of $\pi_1(N)$. The argument of the proof of [**How82**, Theorem 6.1] shows that $b_1(\Gamma) \geq 1$. See also (C.19).

(L.23) Each surface fiber subgroup corresponding to a surface bundle $p\colon N \to S^1$ is the kernel of the map $p_*\colon \pi_1(N) \to \pi_1(S^1) = \mathbb{Z}$. It follows immediately from the definition that a virtual surface fiber subgroup is virtually normal.

(L.24) Let Γ be a subgroup of a torsion-free group π and suppose that Γ is separable and has finite width. Let $g_1, \dots, g_n \in \pi$ be a maximal collection of essentially distinct elements of $\pi \setminus \Gamma$ such that $\Gamma \cap \Gamma^{g_i}$ is infinite for all i. Let π' be a finite-index subgroup of π that contains Γ but none of the g_i. Then Γ is easily seen to be malnormal in π'; in particular, Γ is virtually malnormal in π.

If we combine this fact with (L.15), (L.17), and (L.19) we see that any geometrically finite subgroup of the fundamental group of a hyperbolic 3-manifold is virtually malnormal. (In the closed case, this is [**Mac13**, Lemma 2.3].) Thus together with (L.22), the dichotomy for subgroups of hyperbolic 3-manifold groups can also be rephrased in terms of being virtually (mal)normal.

6.3. Additional results and implications

We conclude this chapter with a few more results and references about subgroups of 3-manifold groups.

(M.1) A group is called *locally free* if every finitely generated subgroup is free. It follows from (L.4) that if N is a compact, orientable, irreducible 3-manifold with empty or toroidal boundary, then every abelian locally free subgroup of $\pi_1(N)$

is free. On the other hand Anderson [**Ana02**, Theorem 4.1] and Kent [**Ken04**, Theorem 1] gave examples of hyperbolic 3-manifolds which contain non-abelian subgroups which are locally free but not free.

(M.2) Suppose N is compact and orientable, and let $\phi\colon \pi_1(N) \to \mathbb{Z}$ be a surjective morphism. By (L.9), ϕ is either induced by the projection of a surface bundle or $\operatorname{Ker}(\phi)$ is not finitely generated. This dichotomy can be strengthened in several ways: if ϕ is not induced by the projection of a surface bundle, then:
(a) $\operatorname{Ker}(\phi)$ admits uncountably many subgroups of finite index. (See [**FV14**, Theorem 5.2], [**SiW09a**] and [**SiW09b**, Theorem 3.4].)
(b) The pair $(\pi_1(N), \phi)$ has 'positive rank gradient.' (See [**DFV14**, Theorem 1.1] and [**DeB13**, Theorem 0.1].)
(c) $\operatorname{Ker}(\phi)$ has a finite-index subgroup which is not normally generated by finitely many elements. (See [**DFV14**, Theorem 5.1].)
(d) If N has non-empty toroidal boundary and if the restriction of ϕ to the fundamental group of each boundary component is non-trivial, then $\operatorname{Ker}(\phi)$ is not locally free. (See [**FF98**, Theorem 3].)
(e) The pair $(\pi_1(N), \phi)$ cannot be represented by an ascending HNN-extension nor a descending HNN-extension. Here, given any group π and any surjective morphism $\alpha\colon \pi \to \mathbb{Z}$ we say that the pair (π, α) is *represented by an ascending HNN-extension* if there exists an isomorphism

$$g\colon \pi \xrightarrow{\cong} \langle A, t \,|\, tAt^{-1} = \gamma(A)\rangle$$

where A is finitely generated, $\gamma\colon A \to A$ is injective and where $(\phi \circ g^{-1})(t) = 1$ and $(\phi \circ g^{-1})(a) = 0$ for all $a \in A$. We define the notion that (π, α) is *represented by a descending HNN-extension* by considering $t^{-1}At$ instead of tAt^{-1}.

Let us explain the last statement in more detail. Let π be a group. Bieri–Neumann–Strebel introduced an invariant $\Sigma(\pi)$, which by definition is a certain subset of the sphere $S(\pi) := \big(H^1(\pi;\mathbb{R}) \setminus \{0\}\big)/\mathbb{R}^{>0}\big)$. This invariant $\Sigma(\pi)$ is usually referred to as the *Bieri–Neumann–Strebel invariant* or the *geometric invariant of* π. It follows from [**BNS87**, Proposition 4.3] that if $\phi \in H^1(\pi;\mathbb{Z}) = \operatorname{Hom}(\pi, \mathbb{Z})$ is primitive, then $\phi \in \Sigma(\pi)$ if and only if (π, ϕ) is represented by an ascending HNN-extension. It follows from [**BNS87**, Theorem E] (together with the resolution of the Poincaré Conjecture) that if N is a compact 3-manifold, then $\Sigma(\pi_1(N))$ equals the image of the fibered cones in $H^1(\pi;\mathbb{R}) \setminus \{0\}$ under the natural map $H^1(\pi;\mathbb{R}) \setminus \{0\}) \to S(\pi)$. (We refer to Section 5.4.3 for background on fibered cones.) In particular $\Sigma(\pi_1(N))$ is symmetric (see [**BNS87**, Corollary F]), which in turn implies, using the aforementioned [**BNS87**, Proposition 4.3], the above statement (e).

(M.3) Let Γ be a finitely generated subgroup of a group π. We say Γ is *tight* in π if for any $g \in \pi$ there exists an n such that $g^n \in \Gamma$. Clearly a finite-index subgroup of π is tight. Let N be a hyperbolic 3-manifold. It follows from the Subgroup Tameness Theorem 4.1.2 that any tight subgroup of $\pi_1(N)$ has finite index. For N with non-trivial toroidal boundary, this was first shown by Canary [**Cay94**, Theorem 6.2].

(M.4) Suppose N is compact, orientable, with no spherical boundary components. Let Σ be an incompressible connected subsurface of ∂N. If $\pi_1(\Sigma)$ has finite index in $\pi_1(N)$, then by [**Hem76**, Theorem 10.5]:

(a) N is a solid torus, or
(b) there is a homeomorphism $f\colon N \to \Sigma \times [0,1]$ with $f(\Sigma) = \Sigma \times \{0\}$, or
(c) N is a twisted I-bundle over a (necessarily non-orientable) surface with Σ the associated S^0-bundle.

More generally, if Γ is a finite-index subgroup of $\pi_1(N)$ isomorphic to the fundamental group of a closed surface, then N is an I-bundle over a closed surface, by [**Hem76**, Theorem 10.6]. (See also [**Broa66**, Theorem 3.1], and see [**BT74**] for an extension to the case of non-compact N.)

(M.5) Suppose N is compact, and let Σ be a connected compact proper subsurface of ∂N such that $\chi(\Sigma) \geq \chi(N)$ and $\pi_1(\Sigma) \to \pi_1(N)$ is surjective. It follows from [**BrC65**, Theorem 1] that Σ and $\overline{\partial N \setminus \Sigma}$ are strong deformation retracts of N.

(M.6) Suppose N is compact, orientable, irreducible, and let $\Sigma \neq S^2$ be a closed incompressible surface in N. If Γ is a subgroup of $\pi_1(N)$ containing $\pi_1(\Sigma)$, and Γ is isomorphic to the fundamental group of a closed orientable surface, then $\pi_1(\Sigma) = \Gamma$, by [**Ja71**, Theorem 6]. (See also [**Feu70, Feu72b, Hei69b, Hei70**] and [**Sco74**, Lemma 3.5].)

(M.7) Let N be compact and Γ be a finitely generated subgroup of $\pi = \pi_1(N)$. Button [**But07**, Theorem 4.1] showed that if $t \in \pi$ with $t\Gamma t^{-1} \subseteq \Gamma$, then $t\Gamma t^{-1} = \Gamma$. If N is hyperbolic and f is an automorphism of π with $f(\Gamma) \subseteq \Gamma$, then $f(\Gamma) = \Gamma$; this follows easily from the above result of Button and the fact that $\operatorname{Out}(\pi)$ is finite (see Section 2.4).

(M.8) Moon [**Moo05**, p. 18] showed that if N is geometric and Γ a finitely generated subgroup of $\pi_1(N)$ of infinite index which has a subgroup G with $G \neq \{1\}$, $G \not\cong \mathbb{Z}$, which is normal in π, then Γ is commensurable to a virtual surface fiber group. (Recall that subgroups A, B of a group π are called *commensurable* if $A \cap B$ has finite index in both A and B.) In the hyperbolic case this can be seen as a consequence of (L.13) and (L.23). Moon also shows that this conclusion holds for certain non-geometric 3-manifolds.

(M.9) We denote by $\mathcal{K}(N)$ the set of all isomorphism classes of knot groups of N. Here a *knot group of* N is the fundamental group of $N \setminus \nu J$ where $J \subseteq N$ is a connected 1-dimensional submanifold of N. Let N_1, N_2 be orientable compact 3-manifolds whose boundaries contain no 2-spheres. Jaco–Myers and Row (see [**Row79**, Corollary 1], [**JM79**, Theorem 6.1], and [**Mye82**, Theorem 8.1]) showed that N_1 and N_2 are diffeomorphic if and only if $\mathcal{K}(N_1) = \mathcal{K}(N_2)$. Earlier work is in [**Fo52**, p. 455], [**Bry60**, p. 181] and [**Con70**].

(M.10) Soma [**Som91**] proved various results on the intersections of conjugates of virtual surface fiber subgroups.

(M.11) We refer to [**WW94, WY99**] and [**BGHM10**, Section 7] for results on finite-index subgroups of 3-manifold groups.

CHAPTER 7

Open Questions

The resolution of Thurston's questions marks a watershed in the study of 3-manifolds. In this final chapter we discuss some remaining open questions about their fundamental groups.

We have loosely organized these questions into five categories. First, we recall some well-known conjectures, such as the conjectures by Wall and Cannon (Section 7.1.1), the Simple Loop Conjecture (Section 7.1.2), and conjectures about knot groups (Section 7.1.3) and ribbon groups (Section 7.1.4). We then turn to questions motivated by the work of Agol, Kahn, Markovic, Wise, and others, discussed in the previous chapters, concerning separable subgroups in 3-manifold groups (Section 7.2.1) and properties of fundamental groups of 3-manifolds which are not non-positively curved (Section 7.2.2). Next, we ask for characterizations of those 3-manifolds with certain group-theoretic features: linearity (Section 7.3.1), residual simplicity (Section 7.3.2), potence (Section 7.3.3), left-orderability (Section 7.3.4), biautomaticity (Section 7.3.6), and the PQL property (Section 7.3.7); at that point we also discuss profinite completions (Section 7.3.5) of 3-manifold groups. Section 7.4 deals with fundamental groups of random 3-manifolds. Finally, in Sections 7.5.1, 7.5.2, and 7.5.3 we formulate some questions about finite covers of 3-manifolds.

7.1. Some classical conjectures

7.1.1. Poincaré duality groups and the Cannon Conjecture. It is natural to ask whether there is an intrinsic, group-theoretic characterization of fundamental groups of (closed) 3-manifolds. Johnson–Wall [**JW72**] introduced the notion of an *n-dimensional Poincaré duality group* (usually just referred to as a *PD_n-group*). The fundamental group of any closed, orientable, aspherical n-manifold is a PD_n-group. Now suppose that π is a PD_n-group. If $n = 1$ or $n = 2$, then π is the fundamental group of a closed, orientable, aspherical n-manifold. (The case $n = 2$ was proved by Eckmann, Linnell and Müller [**EcM80, EcL83, Ecn94, Ecn95, Ecn97**].) Davis [**Davb98**, Theorem C] showed that for any $n \geq 4$ there is a finitely generated PD_n-group which is not finitely presented and hence is not the fundamental group of an aspherical closed n-manifold. But the following conjecture of Wall is still open.

Conjecture 7.1.1 (Wall Conjecture). *Let $n \geq 3$. Every finitely presentable PD_n-group is the fundamental group of a closed, orientable, aspherical n-manifold.*

This conjecture has been studied over many years and a summary of all the results so far exceeds the scope of this book. We refer to [**Tho95, Davb00, Hil11**] for some surveys and to [**BiH91, Cas04, Cas07, Cr00, Cr07, Davb00, DuS00, Hil85, Hil87, Hil06, Hil12, Hil11, Kr90b, SSw07, Tho84, Tho95, Tur90, Wala04**] for more information on the case $n = 3$ of the Wall Conjecture and for known results.

The Geometrization Theorem implies that the fundamental group of a closed, orientable, aspherical non-hyperbolic 3-manifold contains a subgroup isomorphic to $\mathbb{Z}^2$, and so the fundamental group of a closed, aspherical 3-manifold N is word-hyperbolic if and only if N is hyperbolic. It is therefore natural to ask which word-hyperbolic groups are PD_3 groups. Bestvina [**Bea96**, Remark 2.9], extending earlier work of Bestvina–Mess [**BeM91**], characterized hyperbolic PD_3 groups in terms of their *Gromov boundaries*. (See [**BrH99**, Section III.H.PW12] for the definition of the Gromov boundary of a word-hyperbolic group.) He proved that a word-hyperbolic group π is PD_3 if and only if its Gromov boundary $\partial\pi$ is homeomorphic to S^2. Thus for word-hyperbolic groups the Wall Conjecture is equivalent to the following conjecture of Cannon.

Conjecture 7.1.2 (Cannon Conjecture). *If the boundary of a word-hyperbolic group π is homeomorphic to S^2, then π acts properly discontinuously and cocompactly on $\mathbb{H}^3$ with finite kernel.*

This conjecture, which is the 3-dimensional analogue of the 2-dimensional results by Casson–Jungreis [**CJ94**] and Gabai [**Gab92**] stated in the remark after Theorem 2.5.5, was first set out in [**CaS98**, Conjecture 5.1] and goes back to earlier work in [**Can94**] (see also [**Man07**, p. 97]). We refer to [**Bok06**, Section 5] and [**BeK02**, Section 9] for a detailed discussion of the conjecture and to [**CFP99**], [**CFP01**], [**BoK05**] and [**Rus10**] for some positive evidence. Markovic [**Mac13**, Theorem 1.1] (see also [**Hai13**, Corollary 1.5]) showed that Agol's Theorem (see [**Ag13**] and Theorem 4.6.2) gives a new approach to the Cannon Conjecture.

We finally point out that a high-dimensional analogue to the Cannon Conjecture was proved by Bartels, Lück and Weinberger: if π is a word-hyperbolic group whose boundary is homeomorphic to S^{n-1} with $n \geq 6$, then π is the fundamental group of an aspherical closed n-dimensional manifold; see [**BLW10**, Theorem A].

7.1.2. The Simple Loop Conjecture. Let $f\colon \Sigma \to N$ be an embedding of a closed, orientable surface into a compact 3-manifold. It is a consequence of the Loop Theorem that if the induced group morphism $f_*\colon \pi_1(\Sigma) \to \pi_1(N)$ is not injective, then the kernel of f_* contains an essential simple closed loop on Σ. (See Theorem 1.3.1 and [**Sco74**, Corollary 3.1].) The Simple Loop Conjecture (see, e.g., [**Kir97**, Problem 3.96]) posits that the same conclusion holds for any map of an orientable surface to a compact, orientable 3-manifold:

Conjecture 7.1.3 (Simple Loop Conjecture). *Let $f\colon \Sigma \to N$ be a map from a closed, orientable surface to a compact, orientable 3-manifold. If $f_*\colon \pi_1(\Sigma) \to \pi_1(N)$ is not injective, then the kernel of f_* contains an essential simple closed loop on Σ.*

Conjecture 7.1.3 was proved for graph manifolds by Rubinstein–Wang [**RuW98**, Theorem 3.1], extending earlier work of Gabai [**Gab85**, Theorem 2.1] and Hass [**Has87**, Theorem 2]. Usadi [**Us93**] showed that an equivariant generalization of the conjecture does not hold. Minsky [**Miy00**, Question 5.3] asked whether the conclusion of the conjecture also holds if the target is replaced by $\mathrm{SL}(2,\mathbb{C})$. This was answered in the negative by Louder [**Lou14**, Theorem 2] and Cooper–Manning [**CoM11**]; see also [**Cal13a**], [**Mnnb14**, Theorem 1.2], and [**But12**, Section 7].

7.1.3. Knot groups. An *n-knot group* is the fundamental group of a knot exterior $S^n \setminus \nu J$, where J is an $(n-2)$-sphere smoothly embedded in S^n and νJ is a tubular neighborhood of J in S^n. Every n-knot group π has the following properties:

(1) π is finitely presented,
(2) the abelianization of π is isomorphic to $\mathbb{Z}$,
(3) $H_2(\pi) = 0$,
(4) the group π has weight 1.

Here a group π is said to be of *weight* 1 if it admits a normal generator, i.e., if there exists a $g \in \pi$ such that the smallest normal subgroup containing g equals π.

The first three properties are straightforward consequences of elementary algebraic topology and Alexander duality, the fourth property follows from the fact that a meridian is a normal generator. Kervaire [**Ker65**] showed that for $n \geq 5$ these conditions in fact characterize n-knot groups. This is not true in the case that $n = 4$, see, e.g., [**Hil77, Lev78, Hil89**], and it is not true if $n = 3$. In the latter case a straightforward example is given by the Baumslag-Solitar group $BS(1)$, see Section 7.1.4. More subtle examples for $n = 3$ are given by Rosebrock; see [**Bue93, Ros94**].

Now we restrict ourselves to the case $n = 3$. In particular we henceforth refer to a 3-knot group as a knot group. The following question, which is related to the discussion in Section 7.1.1, naturally arises.

Question 7.1.4. *Is there a group-theoretic characterization of knot groups?*

Knot groups have been studied intensively since the very beginning of 3-manifold topology. They serve partly as a laboratory for the general study of 3-manifold groups, but of course there are also results and questions specific to knot groups. We refer to [**Neh65, Neh74**] for a summary of some early work, to [**GA75, Joh80, JL89**] for results on homomorphic images of knot groups, and to [**Str74, Eim00, KrM04, AL12**] for further results. Below we discuss several open questions about knot groups.

If N is obtained by Dehn surgery along a knot $J \subseteq S^3$, then the image of the meridian of J is a normal generator of $\pi_1(N)$, i.e., $\pi_1(N)$ has weight 1. The converse does not hold, i.e., there exist closed 3-manifolds N such that $\pi_1(N)$ has weight 1, but which are not obtained by Dehn surgery along a knot in S^3. For example, if $N = P_1 \# P_2$ is the connected sum of two copies of the Poincaré homology sphere P, then $\pi_1(N)$ is normally generated by $a_1 a_2$, where $a_1 \in \pi_1(P_1)$ is an element of order 3 and $a_2 \in \pi_1(P_2)$ is an element of order 5. On the other hand it follows from [**GLu89**, Corollary 3.1] that N cannot be obtained from Dehn surgery along a knot in S^3. Further reducible examples are also given by Boyer [**Boy86**, p. 104]. Boyer–Lines [**BoL90**, Theorem 5.6] respectively Doig [**Doi12**], Hoffman–Walsh [**HW13**, Theorem 4.4], Hom–Karakurt–Lidman [**HKL14**, Theorem 1.1] and Marengon [**Mar14**, Theorem 3] also provided 3-manifolds N such that

(1) N is a rational homology sphere with non-trivial cyclic first homology,
(2) $\pi_1(N)$ has weight one,
(3) N is Seifert fibered respectively hyperbolic,
(4) N is not Dehn surgery along a knot in S^3.

Auckly [**Auc93, Auc97**] gave examples of hyperbolic integral homology spheres which are not the result of Dehn surgery along a knot in S^3, but it is not known whether their fundamental groups have weight 1. The known obstructions to being Dehn surgery along a knot in S^3 mostly apply to rational homology spheres. In particular the following question is still open.

Question 7.1.5. *Let N be a closed, orientable, irreducible* 3*-manifold such that* $b_1(N) = 1$ *and* $\pi_1(N)$ *has weight* 1. *Is N the result of Dehn surgery along a knot in S^3?*

Another question concerning fundamental groups of knot complements is the following, due to Cappell–Shaneson (see [**Kir97**, Problem 1.11]).

QUESTION 7.1.6. *Let $J \subseteq S^3$ be a knot such that $\pi_1(S^3 \setminus \nu J)$ is generated by n-meridional generators. Is J an n-bridge knot?*

The case $n = 1$ is a consequence of the Loop Theorem and the case $n = 2$ follows from work of Boileau and Zimmermann [**BoZi89**, Corollary 3.3] together with the Orbifold Geometrization Theorem [**BMP03, BLP05**]. An affirmative answer is also known for torus knots [**RsZ87**], pretzel knots [**BoZ85, Cor13**], and many satellite knots [**CH14**]. Bleiler [**Kir97**, Problem 1.73] suggested a generalization to knots in general 3-manifolds and gave positive evidence for it [**BJ04**], but results of Li [**Lia13**, Theorem 1.1] can be used to show that Bleiler's conjecture is false in general.

The following question also concerns the relationship between generators of the fundamental group and the topology of a knot complement.

QUESTION 7.1.7. *Let $J \subseteq S^3$ be a knot such that $\pi_1(S^3 \setminus \nu J)$ is generated by two elements. Is J a tunnel number one knot?*

Here a knot is said to have tunnel number one if there exists a properly embedded arc A in $S^3 \setminus \nu J$ such that $S^3 \setminus \nu(J \cup A)$ is a handlebody. Some evidence towards this conjecture is given in [**Ble94, BJ04**] and [**BW05**, Corollary 7].

If $J \subseteq S^3$ is a knot, then any meridian normally generates $\pi = \pi_1(S^3 \setminus \nu J)$. An element $g \in \pi$ is called a *pseudo-meridian of J* if g normally generates π but if there is no automorphism of π which sends g to a meridian. Examples of pseudo-meridians were first given by Tsau [**Ts85**, Theorem 3.11]. Silver–Whitten–Williams [**SWW10**, Corollary 1.3] showed that every non-trivial hyperbolic 2-bridge knot, every torus knot, and every hyperbolic knot with unknotting number one admits a pseudo-meridian. The following conjecture was proposed in [**SWW10**, Conjecture 3.3].

QUESTION 7.1.8. *Does every non-trivial knot in S^3 have a pseudo-meridian?*

We also refer to work of Suzuki [**Suz13**] which suggests that knot groups have in general many pseudo-meridians.

Two knots J_1 and J_2 in S^3 are called *commensurable* if the exteriors $S^3 \setminus \nu J_1$ and $S^3 \setminus \nu J_2$ admit finite covers which are diffeomorphic. All torus knots are commensurable, but hyperbolic knots tend to be commensurable to few other knots. For example, by [**Red91, MaM08, ReW08**] many hyperbolic knots are only commensurable to itself. By [**ReW08, Hof10**] there are hyperbolic knots which are commensurable to two other knots. The following conjecture was formulated in [**ReW08**].

CONJECTURE 7.1.9. *Every hyperbolic knot is commensurable to at most two other knots.*

This conjecture has been verified by Boileau–Boyer–Cebanu–Walsh [**BBCW12**] for all hyperbolic knots with 'no hidden symmetry.' It is conjectured [**NeR92, BBCW12**] that the two dodecahedral knots of [**AiR99a**] are the only hyperbolic knots with hidden symmetries. See [**Wah11, Pao13, Fri13**] for more information on commensurability of knots.

7.1.4. Ribbon groups. A group π with $H_1(\pi;\mathbb{Z}) \cong \mathbb{Z}$ is a *ribbon group* if it has a Wirtinger presentation of deficiency 1, i.e., a presentation

$$\left\langle g_1,\dots,g_{n+1} \mid g_{\sigma(1)}^{\varepsilon_1} g_1 g_{\sigma(1)}^{-\varepsilon_1} g_2^{-1},\ \dots,\ g_{\sigma(n)}^{\varepsilon_k} g_n g_{\sigma(n)}^{-\varepsilon_n} g_{n+1}^{-1} \right\rangle$$

where $\sigma\colon \{1,\dots,n\} \to \{1,\dots,n+1\}$ is a map and $\varepsilon_i = \pm 1$ for $i = 1,\dots,n$. The name comes from the fact these groups are precisely the fundamental groups of ribbon disk complements in D^4. (See [**FTe05**, Theorem 2.1] or [**Hil02**, p. 22].)

It is well known that if π is a knot group (i.e., $\pi = \pi_1(S^3 \setminus \nu J)$ where $J \subseteq S^3$ is a knot), then π is a ribbon group. (See, e.g., [**Rol90**, p. 57].) Knot groups are fundamental groups of irreducible 3-manifolds with non-trivial toroidal boundary; in particular they are coherent by (C.5), they have a 2-dimensional Eilenberg–Mac Lane space by (C.1), and they are virtually special by Theorems 4.2.2, 4.7.2 and 4.7.3.

On the other hand not all ribbon groups are 3-manifold groups, let alone knot groups. For example, for any m the Baumslag–Solitar group

$$BS(m) = \langle a, b \mid ba^m b^{-1} = a^{m+1} \rangle = \langle a, b \mid a^m b a^{-m} = ba \rangle$$

is a ribbon group but not a knot group. Indeed, following [**Kul05**, p. 129], we see that with $x = ba$ and $\overline{g} = g^{-1}$, the group $BS(m)$ is isomorphic to

$$\begin{aligned}
&\left\langle x, b, b_1, \dots, b_{m-1} \mid (\bar{b}x)b\overline{(\bar{b}x)} = b_1, \dots, (\bar{b}x)b_{m-1}\overline{(\bar{b}x)} = x \right\rangle = \\
&\left\langle x, b, b_1, \dots, b_{m-1} \mid xb\overline{x} = bb_1\bar{b}, \dots, xb_{m-1}\overline{x} = bx\bar{b} \right\rangle = \\
&\left\langle x, b, b_1, \dots, b_{m-1}, a_1, \dots, a_m \mid xb\overline{x} = a_1 = bb_1\bar{b}, \dots, xb_{m-1}\overline{x} = a_m = bx\bar{b} \right\rangle,
\end{aligned}$$

which is a ribbon group. The group $BS(1)$ is isomorphic to the solvable group $\mathbb{Z} \ltimes \mathbb{Z}[1/2]$, which is not a 3-manifold group by Theorem 1.11.1. It is shown in [**BaS62**, Theorem 1] that $BS(m)$ is not Hopfian if $m \geq 2$, which by (C.29) and (C.30) also implies that $BS(m)$ is not a 3-manifold group. (See also [**Shn01**, Theorem 1] and [**JS79**, Theorem VI.2.1].) More examples of ribbon groups which are not knot groups are in [**Ros94**, Theorem 3].

We thus see that ribbon groups, which from the point of view of group presentations look like a mild generalization of knot groups, can exhibit very different behavior. It is an interesting question whether the 'good properties' of knot groups or the 'bad properties' of the Baumslag–Solitar groups $BS(m)$ (for $m \geq 2$) are prevalent among ribbon groups.

Very little is known about the general properties of ribbon groups. In particular, the following question is still open.

QUESTION 7.1.10. *Is the canonical 2-complex corresponding to a Wirtinger presentation of deficiency 1 of a ribbon group an Eilenberg–Mac Lane space?*

An affirmative answer to this question would be an important step towards determining which knots bound ribbon disks; see [**FTe05**, p. 2136f] for details. Howie gave a positive answer to this question for certain (e.g., locally indicable) ribbon groups. (See [**How82**, Theorem 5.2] and [**How85**, Section 10].) We refer the reader to [**IK01, HuR01, HaR03, Ivb05, Bed11, HaR12**] for further work.

We conclude this section with a conjecture due to Whitehead [**Whd41b**].

CONJECTURE 7.1.11 (Whitehead). *Each subcomplex of an aspherical 2-complex is also aspherical.*

A proof of the Whitehead Conjecture would give an affirmative answer to Question 7.1.10. Indeed, starting with the canonical 2-complex corresponding to a Wirtinger presentation of deficiency 1 for a ribbon group, the 2-complex obtained by attaching a 2-cell to any of the generators is easily seen to be aspherical. We refer to [**Bog93, Ros07**] for survey articles on the Whitehead Conjecture and to [**BeB97**, Theorem 8.7] for some negative evidence.

7.2. Questions motivated by the work of Agol, Kahn, Markovic, and Wise

7.2.1. Separable subgroups in 3-manifolds with non-trivial JSJ-decomposition. Let N be a compact, orientable, irreducible 3-manifold with empty or toroidal boundary. By the Virtually Compact Special Theorem 4.2.2, we know that if N is hyperbolic then $\pi_1(N)$ is in fact virtually *compact* special. Together with the Tameness Theorem of Agol and Calegari–Gabai and work of Haglund this implies that $\pi_1(N)$ is LERF.

The picture is considerably more complicated for non-hyperbolic 3-manifolds. Niblo–Wise [**NW01**, Theorem 4.2] showed that the fundamental group of a graph manifold N is LERF if and only if N is geometric. (See also the discussion in (H.11).) Liu [**Liu14**] gave more examples of 3-manifold groups which are not LERF. The following conjecture is a slight variation on [**Liu14**, Conjecture 1.5].

Conjecture 7.2.1. *Let N be a compact, orientable, irreducible 3-manifold with empty or toroidal boundary. If N has a non-trivial JSJ-decomposition, then $\pi_1(N)$ is not LERF.*

(In an earlier version of this survey that appeared on the arXiv we had conjectured that if N is a compact, orientable, irreducible 3-manifold with empty or toroidal boundary such that no torus of the JSJ-decomposition bounds a Seifert fibered 3-manifold on both sides, then $\pi_1(N)$ is LERF. This conjecture was disproved by Liu [**Liu14**, p. 3].)

Note that $\pi_1(N)$ being virtually compact special is in general not enough to deduce that $\pi_1(N)$ is LERF. Indeed, there exist graph manifolds with fundamental groups that are compact special but not LERF; for instance, the non-LERF link group exhibited in [**NW01**, Theorem 1.3] is a right-angled Artin group.

Despite the general failure of LERF, certain families of subgroups are known to be separable. Let N be a compact 3-manifold.

1. Suppose N is orientable, irreducible, with (not necessarily toroidal) boundary. Let X be a connected, incompressible subsurface of ∂N. Long–Niblo [**LoN91**, Theorem 1] showed that $\pi_1(X)$ is separable in $\pi_1(N)$.
2. Hamilton proved that any abelian subgroup of $\pi_1(N)$ is separable; see (C.32).
3. Hamilton [**Hamb03**] gave examples of free 2-generator subgroups in non-geometric 3-manifolds which are separable.
4. Suppose N is orientable and irreducible, with empty or toroidal boundary. Then N is efficient, by (C.35). By (C.12) and (H.11) the fundamental group of any JSJ piece is LERF, and it follows that any subgroup of $\pi_1(N)$ carried by a JSJ piece is separable.
5. For an arbitrary N, Przytycki–Wise [**PW14b**, Theorem 1.1] showed that a subgroup of $\pi_1(N)$ given by an incompressible properly embedded surface is separable in $\pi_1(N)$. (This generalizes earlier work of Long–Niblo [**LoN91**, Theorem 1].)

(6) Long–Reid [**LoR01**, Theorem 1.2] showed that the fundamental group of the double D of the exterior of the figure-8 knot is GFERF. More precisely, they found a faithful representation of $\pi_1(D)$ into the isometry group of 4-dimensional hyperbolic space, and showed that the subgroups which are geometrically finite with respect to this representation are separable.

To bring order to this menagerie of examples, it would be desirable to exhibit some large, intrinsically defined class of subgroups of general 3-manifold groups which are separable. In the remainder of this subsection, we propose the class of *fully relatively quasi-convex subgroups* (see below) as a candidate.

We work in the context of relatively hyperbolic groups. The following theorem follows quickly from [**Dah03**, Theorem 0.1]. (See also [**BiW13**, Corollary E].)

THEOREM 7.2.2. *Let N be a compact, orientable, irreducible 3-manifold with empty or toroidal boundary. Let $M_1, \dots, M_k$ be the maximal graph manifold pieces of the JSJ-decomposition of N, let $S_1, \dots, S_l$ be the tori in the boundary of N that adjoin a hyperbolic piece and let $T_1, \dots, T_m$ be the tori in the JSJ-decomposition of M that separate two (not necessarily distinct) hyperbolic components of the JSJ-decomposition. The fundamental group of N is hyperbolic relative to the set of parabolic subgroups*

$$\{H_i\} = \{\pi_1(M_p)\} \cup \{\pi_1(S_q)\} \cup \{\pi_1(T_r)\} \ .$$

In the case where N is a graph manifold, the theorem makes the vacuous assertion that $\pi_1 N$ is hyperbolic relative to itself.

There is a notion of a *relatively quasi-convex* subgroup of a relatively hyperbolic group; see (H.9) and the references mentioned there for more details. A subgroup Γ of a group π, relatively hyperbolic relative to $\{H_i\}$, is called *fully relatively quasi-convex* if it is relatively quasi-convex and, moreover, for each i, the subgroup $\Gamma \cap H_i$ of H_i is trivial or of finite index.

CONJECTURE 7.2.3. *Let N be a compact, orientable, non-positively curved 3-manifold with empty or toroidal boundary. If Γ is a subgroup of $\pi = \pi_1(N)$ that is fully relatively quasi-convex with respect to the natural relatively hyperbolic structure on π, then Γ is a virtual retract of π; in particular, Γ is separable.*

An earlier version of this book that appeared on the arXiv noted that Conjecture 7.2.3 would follow if the fundamental group of every compact, non-positively curved 3-manifold with empty or toroidal boundary were virtually *compact* special, and asked if this holds. Hagen–Przytycki answered this question completely by showing that, on the contrary, the fundamental group of such a manifold is usually not virtually compact special [**HaP13**]; see also (H.4) above.

7.2.2. Non-non-positively curved 3-manifolds. The preceding chapters show that a clear picture of the properties of aspherical non-positively curved 3-manifolds is emerging. The 'last frontier,' oddly enough, seems to be the study of 3-manifolds which are not non-positively curved.

It is interesting to note that solvable fundamental groups of 3-manifolds in some sense have 'worse' properties than fundamental groups of hyperbolic 3-manifolds. In fact, in contrast to the picture we developed in Flowchart 4 for hyperbolic 3-manifold groups, we have the following lemma:

LEMMA 7.2.4. *Let N be a Sol-manifold and $\pi = \pi_1(N)$. Then π*

(1) *is not virtually RFRS,*
(2) *is not virtually special,*
(3) *has no finite-index subgroup which is residually p for all primes p, and*
(4) *does not virtually retract onto all its cyclic subgroups.*

The first statement was shown by Agol [**Ag08**, p. 271], the second is an immediate consequence of the first statement and (H.17), the third is proved in [**AF11**, Proposition 1.3], and the fourth statement follows easily from the fact that any finite cover N' of a Sol-manifold is a Sol-manifold again and so $b_1(N') \le 1$, contradicting (H.19).

We summarize some known properties in the following theorem.

THEOREM 7.2.5. *Let N be a compact, orientable, aspherical 3-manifold with empty or toroidal boundary which does not admit a non-positively curved metric. Then*

(1) *N is a closed graph manifold;*
(2) *$\pi_1(N)$ is conjugacy separable; and*
(3) *for any prime p, the group $\pi_1(N)$ is virtually residually p.*

The first statement was proved by Leeb [**Leb95**, Theorems 3.2 and 3.3] and the other two statements are known to hold for fundamental groups of all graph manifolds by [**WZ10**, Theorem D] and [**AF13**], respectively.

We saw in Lemma 7.2.4 and Proposition 5.4.7 that there are many desirable properties which are not shared by fundamental groups of Sol-manifolds and some graph manifolds. Also, recall that there are graph manifolds which are not virtually fibered (cf. (H.4)). We can nonetheless pose the following question.

QUESTIONS 7.2.6. *Let N be a compact, orientable, aspherical 3-manifold with empty or toroidal boundary which doesn't admit a non-positively curved metric.*

(1) *Is $\pi := \pi_1(N)$ linear over $\mathbb{C}$?*
(2) *Is π linear over $\mathbb{Z}$?*
(3) *If π is not solvable, does it have a finite-index subgroup which is residually p for any prime p?*
(4) *Is π virtually bi-orderable?*
(5) *Does π satisfy the Atiyah Conjecture?*
(6) *Is the group ring $\mathbb{Z}[\pi]$ a domain?*

7.3. Further group-theoretic properties

Throughout this section we let N be a compact 3-manifold.

7.3.1. Linear representations of 3-manifold groups. As we explained in Section 5, we now know that the fundamental groups of most compact 3-manifolds are linear. It is natural to ask what is the minimal dimension of a faithful representation for a given 3-manifold group. For example, Thurston [**Kir97**, Problem 3.33] asked whether every finitely generated 3-manifold group has a faithful representation in $\mathrm{GL}(4,\mathbb{R})$. This question was partly motivated by the study of projective structures on 3-manifolds, since a projective structure on a 3-manifold naturally gives rise to a (not necessarily faithful) representation of its fundamental group in $\mathrm{PGL}(4,\mathbb{R})$. See [**CLT06, CLT07, HeP11**] for more about projective structures on 3-manifolds, and [**CoG12**] for a proof that $\mathbb{R}P^3\#\mathbb{R}P^3$ does not admit a projective structure.

Thurston's question was answered in the negative by Button [**But14**, Corollary 5.2]. More precisely, Button found a closed graph manifold whose fundamental group admits no faithful representation in $\mathrm{GL}(4, K)$ for any field K.

One of the main themes which emerges from this book is that fundamental groups of closed graph manifolds are at times less well behaved than fundamental groups of compact, orientable, irreducible 3-manifolds which are not closed graph manifolds (e.g., which have a hyperbolic JSJ-component). We can therefore ask:

QUESTION 7.3.1.

(1) *Suppose that N is orientable, irreducible, and not a closed graph manifold. Does $\pi_1(N)$ admit a faithful representation in* $\mathrm{GL}(4, \mathbb{R})$*?*
(2) *Does there exist an n such that the fundamental group of any compact* 3-*manifold group has a faithful representation in* $\mathrm{GL}(n, \mathbb{R})$*?*

Note, however, that we do not even know whether there is an n such that the fundamental group of any Seifert fibered manifold embeds in $\mathrm{GL}(n, \mathbb{R})$; see (C.11).

We remark that Lubotzky [**Lu88**] characterized group-theoretically the finitely generated virtually residually p groups which embed into $\mathrm{GL}(n, K)$ for some n and some field K of characteristic 0. On general grounds (Theorem of Łos-Tarski [**Ho93**, Corollary 6.5.3]), for each n there also exists an intrinsic characterization of those groups which embed into $\mathrm{GL}(n, K)$ for some field K of characteristic 0. Since the Compactness Theorem of first-order logic is invoked in the proof, we don't have much information about the nature of this characterization; for $n = 1$ it can be worked out explicitly, see, e.g., [**Weh73**, Theorem 2.2 (i)]. Unfortunately, these abstract results seem of little help in tackling part (2) of the question above.

The following was conjectured by Luo [**Luo12**, Conjecture 1].

CONJECTURE 7.3.2. *Given any $g \in \pi_1(N)$, $g \neq 1$, there exists a finite commutative ring R and a morphism $\alpha\colon \pi_1(N) \to \mathrm{SL}(2, R)$ such that $\alpha(g) \neq 1$.*

So far we only discussed faithful representations over $\mathbb{Z}$ or over fields of characteristic 0. Now we turn to representations over fields of non-zero characteristic. As we had mentioned in (C.11), it follows from the Auslander–Swan Theorem [**Weh73**, Theorem 2.5] that fundamental groups of Sol-manifolds are linear over $\mathbb{Z}$. On the other hand, by [**Weh73**, p. 21] the fundamental groups of Sol-manifolds do not admit faithful representations over fields of non-zero characteristic.

QUESTION 7.3.3. *Does the fundamental group of any hyperbolic* 3-*manifold admit a faithful representation over a field of non-zero characteristic?*

7.3.2. 3-manifold groups which are residually simple. Long–Reid [**LoR98**, Corollary 1.3] showed that the fundamental group of any hyperbolic 3-manifold is residually simple. On the other hand, there are examples of 3-manifold groups which are not residually simple:

(1) certain finite fundamental groups like $\mathbb{Z}/4\mathbb{Z}$,
(2) non-abelian solvable groups, like fundamental groups of non-trivial torus bundles, and
(3) non-abelian groups with non-trivial center, i.e., infinite non-abelian fundamental groups of Seifert fibered manifolds.

We are not aware of any other examples of 3-manifold groups which are not residually finite simple. We therefore pose the following question:

QUESTION 7.3.4. *Suppose N is orientable, irreducible, with empty or toroidal boundary. If N is not geometric, is $\pi_1(N)$ residually finite simple?*

7.3.3. Potence. A group π is called *potent* if for any $g \in \pi \setminus \{1\}$ and any n there exists a morphism α from π onto a finite group such that $\alpha(g)$ has order n. As we saw above, many 3-manifold groups are virtually potent. It is also straightforward to see that fundamental groups of fibered 3-manifolds are potent. Also, Shalen [**Shn12**] proved that if π be the fundamental group of a hyperbolic 3-manifold and $n \geq 3$, then there are finitely many conjugacy classes $C_1, \dots, C_m$ in π such that for any $g \notin C_1 \cup \dots \cup C_m$ there exists a morphism α from $\pi_1(N)$ onto a finite group such that $\alpha(g)$ has order n.

The following question naturally arises:

QUESTION 7.3.5. *Suppose N is orientable and aspherical, with empty or toroidal boundary. Is $\pi_1(N)$ potent?*

7.3.4. Left-orderability. Suppose N is orientable, irreducible, with empty or toroidal boundary. By (C.19) and (C.20) above, if $b_1(N) \geq 1$, then $\pi_1(N)$ is left-orderable. (See also [**BRW05**, Theorem 1.1] for a different approach.) On the other hand there is presently no good criterion for determining whether $\pi_1(N)$ is left-orderable if $b_1(N) = 0$, i.e., if N is a rational homology sphere. Before we formulate the subsequent conjecture we recall that a rational homology sphere N is called an *L-space* if the total rank of its Heegaard Floer homology $\widehat{HF}(N)$ equals $|H_1(N;\mathbb{Z})|$. We refer to the foundational papers of Ozsváth–Szabó [**OzS04a, OzS04b**] for details on Heegaard Floer homology and to [**OzS05**] for the definition of L-spaces.

The following was formulated by Boyer–Gordon–Watson [**BGW13**, Conjecture 3]:

CONJECTURE 7.3.6. *Suppose that N is an irreducible rational homology sphere. Then $\pi_1(N)$ is left-orderable if and only if N is not an L-space.*

See [**BGW13**] for some background. We refer to the papers [**Pet09, BGW13, CyW13, CyW11, CLW13, LiW14, ClT13, LeL12, Ter13, Gre11, HaTe14b, HaTe12, HaTe14a, Tra13a, MTe13, BoB13, It13, DJRZ13, Nak13, BoC14, IT14, GLid14, CGHV14**] for evidence towards an affirmative answer, for examples of non left-orderable fundamental groups, and for relations of these notions to the existence of taut foliations. We also refer to [**DPT05, Hu13, Tra13b**] for more results on the left-orderability of fundamental groups of rational homology spheres.

A link between left-orderability and L-spaces is given by the (non-) existence of certain foliations on 3-manifolds. See [**CD03**, Section 7] and [**RSS03, RoS10**] for the interaction between left-orderability and foliations. Ozsváth–Szabó [**OzS04c**, Theorem 1.4], on the other hand, proved that no L-space has a co-orientable taut foliation. An affirmative answer to Conjecture 7.3.6 would thus imply that the fundamental group of a rational homology sphere admitting a co-orientable taut foliation is left-orderable. The following theorem can be seen as positive evidence for the conjecture.

THEOREM 7.3.7. *The fundamental group of each irreducible $\mathbb{Z}$-homology sphere which admits a co-orientable taut foliation is left-orderable.*

This theorem follows from [**BoB13**, Lemma 0.4] together with the proof that the second statement of [**BoB13**, Corollary 0.3] implies the third statement. This theorem was first proved in the Seifert fibered case by [**BRW05**, Corollary 3.12], the hyperbolic case also follows from [**CD03**, Theorems 6.3 and 7.2], and the graph manifold case was first proved in [**CLW13**, Theorem 1].

7.3.5. Finite quotients of 3-manifold groups. As before, we denote by $\hat{\pi}$ the profinite completion of a group π. It is natural to ask to what degree a residually finite group is determined by its profinite completion. This question goes back to Grothendieck [**Grk70**] and it is studied in the general group-theoretic context in detail in [**Pil74, GPS80, GZ11**]. We start out the discussion of 3-manifold groups with the following theorem.

THEOREM 7.3.8. *Let π and Γ be finitely generated groups. The following are equivalent:*

(1) *there exists an isomorphism $\widehat{\pi} \to \widehat{\Gamma}$ of topological groups;*
(2) *there exists an isomorphism $\widehat{\pi} \to \widehat{\Gamma}$ of groups;*
(3) *the set of isomorphism classes of finite quotients of π is the same as the set of isomorphism classes of finite quotients of Γ.*

The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are obvious. Nikolov–Segal [**NiS03, NiS07**] showed (2) $\Rightarrow$ (1). See [**DFPR82**] and [**RiZ10**, Corollary 3.2.8] for a proof of (3) $\Rightarrow$ (1).

We denote by $\mathcal{G}_3$ the set of isomorphism classes $[\Gamma]$ of fundamental groups Γ of compact, orientable 3-manifolds. Inspired by [**GZ11**, p. 132], given a group π, we call

$$g_3(\pi) := \left\{[\Gamma] \in \mathcal{G}_3 \,:\, \widehat{\pi} \cong \widehat{\Gamma}\right\}$$

the 3-*manifold genus of* π. We say that the 3-manifold genus of π is *trivial* if $g_3(\pi)$ consists of one element; that is, if Γ is the fundamental group of a compact 3-manifold with $\widehat{\pi} \cong \widehat{\Gamma}$, then $\pi \cong \Gamma$. Which 3-manifold groups π have non-trivial 3-manifold genus? Funar [**Fun13**, Corollary 1.4], building on [**Ste72**, p. 3], and also Hempel [**Hem14**] gave examples of fundamental groups of Sol-manifolds and of Seifert fibered manifolds that have non-trivial 3-manifold genus. As of now these are the only known examples of 3-manifold groups with non-trivial 3-manifold genus.

We propose the following conjecture, which is a variation on [**LoR11**, p. 481] and [**CFW10**, Remark 3.7].

CONJECTURE 7.3.9. *The fundamental group of a hyperbolic* 3*-manifold with empty or toroidal boundary has trivial* 3*-manifold genus.*

The results and calculations in [**Eim07**, Section 2D] and [**FV07, FV13, Lin01, BF15**] give some positive evidence for the conjecture. Further evidence is given by Wilton-Zalesskii [**WZ14**, Theorem A] who showed that the 3-manifold genus of a closed hyperbolic 3-manifold contains only fundamental groups of hyperbolic 3-manifolds. A proof of the conjecture, together with the equivalence of (1) and (3) in Theorem 7.3.8, would give an alternative solution to the isomorphism problem for hyperbolic 3-manifold groups. We also expect many other 3-manifold groups to have trivial 3-manifold genus.

Note that there are infinite classes of finitely presented groups which have the same profinite completion; see, e.g., [**Pil74**]. We nonetheless propose that this does not hold for 3-manifold groups. More precisely, we propose the following conjecture.

Conjecture 7.3.10. *The* 3*-manifold genus of a fundamental group of a compact* 3*-manifold is finite.*

Fundamental groups of Sol-manifolds are virtually polycyclic. So from [**GPS80**, p. 155] it follows that the conjecture holds for fundamental groups of Sol-manifolds. Also, Hempel's examples [**Hem14**] seem to be consistent with the conjecture.

As mentioned in (J.4), Cavendish used the fact that 3-manifold groups are good in the sense of Serre (see (H.26)) to show that fundamental groups of compact 3-manifolds are Grothendieck rigid. In fact, one can deduce more.

Proposition 7.3.11. *Let N_1, N_2 be compact, aspherical* 3*-manifolds, where N_1 is closed and N_2 has non-empty boundary. Then*

$$\widehat{\pi_1(N_1)} \ncong \widehat{\pi_1(N_2)}\ .$$

Proof. Because $\pi_1(N_1)$ and $\pi_1(N_2)$ are both good,

$$H^3(\widehat{\pi_1(N_i)};\mathbb{Z}_2) \cong H^3(\pi_1(N_i);\mathbb{Z}_2) \quad \text{for } i = 1, 2.$$

But

$$H^3(\pi_1(N_1);\mathbb{Z}_2) \cong H^3(N_1;\mathbb{Z}_2) \cong \mathbb{Z}_2$$

whereas

$$H^3(\pi_1(N_2);\mathbb{Z}_2) \cong H^3(N_2;\mathbb{Z}_2) \cong 0,$$

so the profinite completions of $\pi_1(N_1)$ and of $\pi_1(N_2)$ cannot be isomorphic. □

7.3.6. Biautomaticity. The reader is referred to [**ECHLPT92**] for the definitions of *automatic* and *biautomatic* structures on groups. The original formulations of these definitions, by Thurston and others, were strongly motivated by attempts to understand the fundamental groups of 3-manifolds using computers. Automatic groups have quadratic Dehn function, and as a result it follows that fundamental groups of Nil- and Sol-manifolds are not automatic. However, Epstein–Cannon–Holt–Levy–Paterson–Thurston [**ECHLPT92**, Theorem 12.4.6] proved the following theorem.

Theorem 7.3.12. *If N is orientable, irreducible, with empty or toroidal boundary, then $\pi_1(N)$ has an automatic structure.*

However, the following question remains open.

Question 7.3.13. *Suppose N is orientable, irreducible, with empty or toroidal boundary, and does not admit* Nil *or* Sol *geometry. Is $\pi_1(N)$ biautomatic?*

7.3.7. Stable commutator length. Let π be a group. The commutator subgroup $[\pi,\pi]$ of π is the subgroup of π that is generated by commutators $[g,h] = ghg^{-1}h^{-1}$ $(g,h \in \pi)$. Each element g of $[\pi,\pi]$ is a product $[g_1,h_1]\cdots[g_n,h_n]$ $(g_i,h_i \in \pi)$ of commutators; the *commutator length* $\mathrm{cl}(g)$ is the minimal number n of commutators which are needed to express g in this way. The *stable commutator length* of an element $g \in [\pi,\pi]$ is then defined as

$$\mathrm{scl}(g) := \lim_{n\to\infty} \frac{1}{n}\,\mathrm{cl}(g^n).$$

The function scl yields a seminorm $B_1^H \to \mathbb{R}^{\geq 0}$ on a certain real vector space B_1^H. This seminorm, also denoted by scl, is a norm if π is word-hyperbolic; see [**CK10, Cal09a**].

The vector space B_1^H is defined as the tensor product of a certain rational vector space with $\mathbb{R}$. It thus makes sense to ask whether scl is a rational seminorm. A group is said to have the *PQL property* if scl is a piecewise rational linear seminorm on B_1^H. Not all groups have the PQL property: Zhuang [**Zhu08**] exhibited finitely presented groups such that scl is not rational. On the other hand, Calegari [**Cal09b**, Theorem 3.11] showed that fundamental groups of Seifert fibered 3-manifolds with non-empty boundary have the PQL property. (See also [**Sus13**].) Calegari [**Cal09b**, Question 3.13] poses the following question.

QUESTION 7.3.14. *Does every* 3*-manifold group have the PQL property?*

In fact, it is not even known whether fundamental groups of closed surfaces have the PQL property.

7.4. Random 3-manifolds

It is natural to ask, what properties does a 'random' (or 'generic') 3-manifold have? For example, is a 'random' closed 3-manifold N a rational homology sphere (i.e., satisfies $b_1(N) = 0$)? Is it hyperbolic? Is it Haken?

There are several reasonable interpretations of the vague phrase 'a random 3-manifold.' The basic idea is always a variation on the following: First one describes the set of homeomorphism classes of 3-manifolds as an exhaustion by a family $\{M_n\}$ of finite sets; then, given a property of 3-manifolds $\mathcal{P}$ one says that a *random* 3*-manifold has Property* $\mathcal{P}$ if in the limit the ratio of manifolds in M_n satisfying $\mathcal{P}$ goes to one. As we will see, the different ways of constructing 3-manifolds give rise to different implementations of the above strategy.

7.4.1. The Heegaard splitting model. From (C.4) recall that every closed 3-manifold N has a Heegaard splitting, i.e., a closed surface in N which splits N into two handlebodies. Let Σ be a surface. We let $M(\Sigma)$ be the mapping class group of Σ, that is, the group of isotopy classes of orientation preserving self-diffeomorphisms of Σ. Given $\varphi \in M(\Sigma)$ we denote by $H_g \cup_\varphi H_g$ the result of gluing two copies of the standard genus g handlebody along the boundary via a representative of φ. The homeomorphism type of $H_g \cup_\varphi H_g$ does not depend on the choice of the representative.

The group $M(\Sigma)$ is finitely generated; see [**FaM12**, Theorem 4.1]. Given a finite generating set S for $M(\Sigma)$ and an element $\varphi \in M(\Sigma)$, we denote by $l_S(\varphi)$ the minimal length of φ as a word in $S \cup S^{-1}$. We denote by $B_S(n) := \#\{\varphi \in M(\Sigma) : l_S(\varphi) \leq n\}$ the n-ball in $M(\Sigma)$ with respect to l_S. Given a property $\mathcal{P}$ of 3-manifolds we say that *in the Heegaard splitting model a random* 3*-manifold has property* $\mathcal{P}$ if for any closed surface Σ of genus $g \geq 2$ and for any finite generating set S of $M(\Sigma)$ we have

$$\lim_{n\to\infty} \frac{\#\{\varphi \in B_S(n) \,:\, H_g \cup_\varphi H_g \text{ has property } \mathcal{P}\}}{\#B_S(n)} = 1.$$

The work of Maher [**Mah10a**, Theorem 1.1] together with that of Hempel [**Hem01**] and Kobayashi [**Koi88**] and the Geometrization Theorem implies that in the Heegaard splitting model a random 3-manifold is hyperbolic. (See also [**LMW14**, Theorem 2].) Furthermore, Dunfield–Thurston [**DnTb06**, Corollary 8.5] showed that in the Heegaard splitting model a random 3-manifold is a rational homology sphere. (See also Kowalski [**Kow08**, Section 6.2], [**Kow**], Sarnak [**Sar12**, p. 4], and Ma [**Ma12**, Corollary 1.2] for extensions and related results.)

The following conjecture is due to Rivin [**Riv14**, Conjecture 11.9].

QUESTION 7.4.1. *In the Heegaard splitting model a random* 3*-manifold is not Haken.*

The following conjecture is due to Dunfield [**Dun14**].

CONJECTURE 7.4.2. *In the Heegaard splitting model a random 3-manifold*

(1) *is not an L-space,*
(2) *has left-orderable fundamental group,*
(3) *admits a taut foliation, and*
(4) *admits a tight contact structure.*

(See [**CdC00**] for the definition of a *taut foliation* and [**Gei08**] for the definition of a *tight contact structure.*)

7.4.2. Dehn surgery on links. Another approach to constructing 3-manifold is to start out from knots and links in S^3. Links can for example be described by closures of braids. In such a model Ma [**Ma14**] showed that a random link is hyperbolic. (On the other hand, in other models a random link turns out to be a satellite link [**Jun94**].) Now let $L = L_1 \cup \cdots \cup L_m$ be an m-component link and $r_1, \ldots, r_m \in \mathbb{Q}$. Denote by μ_i, λ_i the meridian respectively longitude of L_i. Let $S^3_{r_1,\ldots,r_m}(L)$ be the result of $(r_1, \ldots, r_m)$-Dehn surgery on L. More precisely, for $r_i = \frac{p_i}{q_i}$ with coprime $p_i, q_i \in \mathbb{Z}$, $q_i \neq 0$ consider

$$S^3_{r_1,\ldots,r_m}(L) = (S^3 \setminus \nu L) \cup \bigcup_{i=1}^{m} S^1 \times D^2_i \quad \text{where } \partial D^2_i \text{ gets glued to } p_i\mu_i + q_i\lambda_i.$$

Lickorish [**Lic62**] and Wallace [**Wac60**] showed that any closed 3-manifold is obtained by Dehn surgery on a link. There are various ways for giving an exhaustion of the set of isotopy classes of links by finite sets, e.g., by considering isotopy classes of links with an upper bound on the crossing number. Together with bounds on the Dehn surgery coefficients one obtains an exhaustion of all closed 3-manifolds and thus a concept of random 3-manifold. In this model, a random 3-manifold clearly is a rational homology sphere, but it is much harder to answer other questions about random 3-manifolds. The difficulty here is that 'crossing number' is a combinatorial notion and arguably not very natural in this context.

Nonetheless there are again good reasons to believe that Dehn surgeries 'generically' produce hyperbolic 3-manifolds. For example, Lackenby–Meyerhoff [**LaM13**] showed that if N is a hyperbolic 3-manifold with one boundary component (e.g., the exterior of a knot), then there exist at most 10 Dehn fillings of N that are not hyperbolic. (The bound of 10 is attained by the exterior of the figure-8 knot.) This result comes at the end of a long list of results starting with Thurston's Hyperbolic Dehn Surgery Theorem [**Thu79**] and work of many others [**Ag00, Ag10a, BGZ01, BCSZ08, FP07, BlH96, Lac00, Ter06, HoK05, Tay13**]; see also the surveys [**Boy02, Gon98**].

It is much less clear whether Dehn surgeries generically give rise to Haken 3-manifolds. On the one hand, if a knot exterior contains a closed incompressible surface, then for almost all Dehn fillings the resulting manifold is Haken [**Wu92**]. There exist many knots whose exterior contains a closed incompressible surface. (See, e.g., [**FiM00, LMo99**] and [**Thp97**, Corollary 3].) However, work of Hatcher [**Hat82**] together with [**CJR82, Men84**], [**HaTh85**, Theorem 2(b)], [**FlH82**, Theorem 1.1], [**Lop92**, Theorem A], and [**Lop93**, Theorem A] shows that almost all Dehn fillings of many knot exteriors and 3-manifolds with toroidal boundary are non-Haken.

7.4.3. Branched covers along links. Knots and links also give rise to closed 3-manifolds via branched covers. In fact, Alexander [**Ale20, Fei86**] showed that any closed 3-manifold is the branched cover of S^3 along a link. (See also [**Hin74**], [**Mon74,**

Mon83] and [**Rol90**, Theorem G.1] for more precise statements.) But as for Dehn surgery along links, a corresponding notion of a 'random 3-manifold' is not very natural. We do point out though that there is strong evidence that 'most' closed 3-manifolds arising that way are again hyperbolic. For example, if J is a hyperbolic knot, then any n-fold branched cover of S^3 along J with $n \geq 3$ is hyperbolic again. This is a consequence of the Orbifold Geometrization Theorem [**Thu82a, BP01, BLP01, BLP05, BMP03, CHK00**] and work of Dunbar [**Dub88a, Dub88b**]; see [**CHK00**, Corollary 1.26].

7.4.4. Betti number vs. fiberedness for random 3-manifolds. We have just seen that in reasonable models a random 3-manifold is a rational homology sphere, hence it is in particular not fibered. The following question asks whether the Betti number is the only reason which prevents a random 3-manifold from being fibered.

QUESTION 7.4.3. *Given an appropriate model of random 3-manifolds (with empty or with toroidal boundary), is a 3-manifold with $b_1 \geq 1$ fibered?*

The work of Dunfield–D. Thurston [**DnTa06**] gives strong evidence that the answer is no for 'random tunnel number one manifolds.'

7.4.5. Random fibered 3-manifolds. Now we turn to the study of random fibered 3-manifolds. Recall that a surface Σ and an element $\varphi \in M(\Sigma)$ gives rise to a corresponding mapping torus. In the above model of a 'random 3-manifold in the Heegaard splitting model' we now replace 'gluings of handlebodies' by mapping tori, and we obtain the notion of a 'random fibered 3-manifold.'

It follows from the work of Maher [**Mah11**, Theorem 1.1] and Rivin [**Riv08**, Theorem 8.2], and the geometrization of fibered 3-manifolds (Section 1.9), that a random fibered 3-manifold is hyperbolic. We refer to the work of Maher [**Mah10b, Mah12**], Rivin [**Riv12**, Section 8] and [**Riv08, Riv09, Riv10**], Lubotzky–Meiri [**LMe11**], Atalan–Korkmaz [**AK10**], as well as Malestein–Souto [**MlS13**], [**Cau13, CWi13**], and [**Sis11b**] for alternative proofs, more precise statements, and related results. Moreover, we refer to [**Riv14**] for a wealth of further results on random fibered 3-manifolds.

Above we have seen that in the Heegaard splitting model a random 3-manifold is a rational homology sphere, that is, its first Betti number is as small as possible. The first Betti number of a fibered 3-manifold is ≥ 1, and Rivin [**Riv14**, Theorem 4.1] showed that a random fibered 3-manifold N in fact satisfies $b_1(N) = 1$.

The last question of the question can be viewed as a variation on Conjecture 7.4.1.

QUESTION 7.4.4. *Does a random fibered 3-manifold have an incompressible surface which is not the fiber of a fiber bundle?*

By [**FlH82**] the answer is 'yes' if we restrict ourselves to fibered 3-manifolds where the fiber is a once-punctured torus.

7.5. Finite covers of 3-manifolds

7.5.1. Homology of finite regular covers and the volume of 3-manifolds. Let N be a compact, orientable, aspherical 3-manifold with empty or toroidal boundary. We saw in (C.37) that for any cofinal regular tower $\{\tilde{N}_n\}$ of N we have

$$\lim_{n\to\infty} \frac{b_1(\tilde{N}_n;\mathbb{Z})}{[\tilde{N}_n : N]} = 0.$$

It is natural to ask about the limit behavior of other 'measures of complexity' of groups and spaces for cofinal regular towers of N. In particular we ask the following question:

QUESTION 7.5.1. *Let N and $\{\tilde{N}_n\}$ be as above.*

(1) *Does the equality*

$$\lim_{n\to\infty} \frac{b_1(\tilde{N}_n;\mathbb{F}_p)}{[\tilde{N}_n : N]} = 0$$

hold for any prime p?

(2) *If* (1) *is answered affirmatively, then does the following hold?*

$$\liminf_{n\to\infty} \frac{\operatorname{rank}(\pi_1(\tilde{N}_n))}{[\tilde{N}_n : N]} = 0.$$

Note that it is not even clear that the first limit exists. The second limit is called the *rank gradient* and was first studied by Lackenby [**Lac05**]. The first question is a particular case of [**EL14**, Question 1.5] and the second question is asked in [**KaN12**]. It follows from [**KaN12**, Proposition 2.1] and [**AJZN11**, Theorem 4] together with [**Pas13**, Theorems 1.4 and 1.5] that (1) and (2) hold for graph manifolds, and that to answer the general case it is enough to answer (1) and (2) for hyperbolic 3-manifolds. It follows from [**BGLS10**, Theorem 1.5] that there exists a $C > 0$ such that the rank gradient is $\leq C$ for *any* hyperbolic 3-manifold.

Arguably the most interesting question is about the growth rate of the size of the torsion homology of finite covers. The following question has been raised by several authors (see, e.g., [**BV13**], [**Lü02**, Question 13.73], [**Lü13**, Conjecture 1.12], [**Lü15**, Conjecture 10.1], and [**Le14b**, Conjecture 1]).

QUESTION 7.5.2. *Let N be a compact, orientable, irreducible* 3*-manifold with empty or toroidal boundary, and denote by* $\operatorname{vol}(N)$ *the sum of the volumes of the hyperbolic JSJ-components of N. Does there exist a cofinal regular tower $\{\tilde{N}_n\}$ of N such that*

$$\lim_{n\to\infty} \frac{1}{[\tilde{N}_n : N]} \ln|\operatorname{Tor} H_1(\tilde{N}_n;\mathbb{Z})| = \frac{1}{6\pi}\operatorname{vol}(N)\ ?$$

More optimistically, does this equality hold for any *cofinal regular tower $\{\tilde{N}_n\}$ of N?*

In the following remarks we let N be as in the preceding question.

(1) A good motivation to this question is given in the introduction of [**BD13**]. There the authors summarize and add to the evidence towards an affirmative answer for arithmetic hyperbolic 3-manifolds (see also [**BSV14**] for further evidence), and they also give some evidence that the answer might be negative for general hyperbolic 3-manifolds.

(2) Le [**Le14b**, Theorem 1] showed that given any cofinal regular tower $\{\tilde{N}_n\}$ of N we have the inequality

$$\lim_{n\to\infty} \frac{1}{[\tilde{N}_n : N]} \ln|\operatorname{Tor} H_1(\tilde{N}_n;\mathbb{Z})| \leq \frac{1}{6\pi}\operatorname{vol}(N),$$

which would in particular imply that the left hand side is zero for graph manifolds. The details of the proof have not appeared yet.

(3) We refer to [**ACCS96**], [**ACS06**, Theorem 1.1], [**Shn07**],[**CuS08**, Proposition 10.1], [**CDS09**, Theorem 6.7], [**DeS09**, Theorem 1.2], [**ACS10**, Theorem 9.6], [**CuS11**, Theorem 1.2], and [**Riv14**, Section 9] for results relating the homology of a hyperbolic 3-manifold to its hyperbolic volume.

(4) An attractive approach to this question is the result of Lück–Schick [**LüS99**, Theorem 0.7] that vol(N) can be expressed in terms of a certain L^2–torsion of N. By [**LiZ06**, Equation 8.2] and [**Lü02**, Lemma 13.53] the L^2–torsion corresponding to the abelianization corresponds to the Mahler measure of the Alexander polynomial. The relationship between the Mahler measure of the Alexander polynomial and the growth of torsion homology in finite abelian covers is explored by Silver–Williams [**SiW02a**, Theorem 2.1], [**SiW02b**] (extending earlier work in [**Gon72, CwS78, Ril90, GoS91**]), Kitano–Morifuji–Takasawa [**KMT03**], Le [**Le14a**], and Raimbault [**Rai12a**, Theorem 0.2].

(5) It follows from Gabai–Meyerhoff–Milley [**GMM09**, Corollary 1.3], [**Mie09**, Theorem 1.3] that for hyperbolic N we have $\operatorname{vol}(N) > 0.942$. (See also [**Ada87**, Theorem 3], [**Ada88**], [**CaM01**], [**GMM10**], and [**Ag10b**, Theorem 3.6].)

An affirmative answer to the question above would imply that the order of torsion in the homology of a hyperbolic 3-manifold grows exponentially by going to finite covers. A first step towards this conjecture is to show that any hyperbolic 3-manifold admits a finite cover with non-trivial torsion in its homology. This was shown by Sun [**Sun13**, Theorem 1.5] for closed hyperbolic 3-manifolds. (See also (H.14).) The following question is still open. (See [**Sun13**, Question 1.7].)

QUESTION 7.5.3. *Let N be a hyperbolic 3-manifold with non-trivial toroidal boundary. Does N have a finite cover $\tilde{N}$ with* $\operatorname{Tor} H_1(\tilde{N};\mathbb{Z}) \neq 0$*?*

It is also interesting to study the behavior of the $\mathbb{F}_p$-Betti numbers in finite covers and the number of generators of the first homology group in finite covers. Little seems to be known about these two problems; but see [**LLS11**] for some partial results regarding the former problem. One intriguing question is whether, for any compact 3-manifold N,

$$\lim_{\tilde{N}} \frac{b_1(\tilde{N};\mathbb{F}_p)}{[\tilde{N}:N]} = \lim_{\tilde{N}} \frac{b_1(\tilde{N};\mathbb{Z})}{[\tilde{N}:N]}.$$

We also refer to [**Lac11**] for further questions on $\mathbb{F}_p$-Betti numbers in finite covers.

Given a compact 3-manifold N, the behavior of the homology in a cofinal regular tower $\{\tilde{N}_n\}$ can depend on the particular choice of $\{\tilde{N}_n\}$. For example, F. Calegari–Dunfield [**CD06**, Theorem 1] together with Boston–Ellenberg [**BE06**] showed that there exists a closed hyperbolic 3-manifold N and a cofinal regular tower $\{\tilde{N}_n\}$ of N such that $b_1(\tilde{N}_n) = 0$ for any n; on the other hand we know by (H.14) that $vb_1(N) > 0$. Another instance of this phenomenon can be seen in [**LLuR08**, Theorem 1.2].

7.5.2. Ranks of finite-index subgroups. The rank $\operatorname{rk}(\pi)$ of a finitely generated group π is defined as the minimal number of generators of π. Reid [**Red92**, p. 212] exhibited a closed hyperbolic 3-manifold N with a finite cover $\tilde{N}$ such that $\operatorname{rk}(\pi_1(\tilde{N})) = \operatorname{rk}(\pi_1(N)) - 1$. Examples of Seifert fibered manifolds with this property are in [**NuR11**].

It is still unknown whether the rank can drop by more than 1 while going to a finite cover. The following was formulated by Shalen [**Shn07**, Conjecture 4.2].

CONJECTURE 7.5.4. *If N is a hyperbolic 3–manifold, then for any finite cover $\tilde{N}$ of N we have* $\operatorname{rk}(\pi_1(\tilde{N})) \geq \operatorname{rk}(\pi_1(N)) - 1$.

It follows from a transfer argument that if Γ is a finite-index subgroup of a finitely generated group π, then $b_1(\Gamma) \geq b_1(\pi)$. More subtle evidence towards the conjecture is given by [**ACS06**, Corollary 7.3] which states that the rank of $\mathbb{F}_p$-homology of a closed orientable 3-manifold can decrease by at most 1 by going to a finite cover.

7.5.3. (Non-) Fibered faces in finite covers of 3-manifolds. By Proposition 5.4.12, every irreducible 3-manifold which is not a graph manifold has finite covers with an arbitrarily large number of fibered faces (in the Thurston norm ball). We conclude this book with the following two questions on virtual (non-) fiberedness:

QUESTION 7.5.5. *Does every compact, irreducible non-positively curved 3-manifold with empty or toroidal boundary have a finite cover such that all faces of the Thurston norm ball are fibered?*

Let N be a 3-manifold with $vb_1(N) \geq 2$ which is not finitely covered by a torus bundle. Then N has a finite cover N' with non-vanishing Thurston norm and $b_1(N') \geq 2$, and $H^1(N';\mathbb{R})$ contains a non-zero class which is not fibered, by [**Thu86a**, Theorem 5].

Surprisingly, though, the following question is still open.

QUESTION 7.5.6. *Does every compact, orientable, irreducible 3-manifold which is not a graph manifold admit a finite cover such that at least one top-dimensional face of the Thurston norm ball is not fibered?*

Bibliography

[ABBGNRS11] M. Abért, N. Bergeron, I. Biringer, T. Gelander, N. Nikolov, J. Raimbault, I. Samet, *On the growth of Betti numbers of locally symmetric spaces*, C. R. Math. Acad. Sci. Paris **349**, no. 15-16 (2011), 831–835.

[ABBGNRS12] ———, *On the growth of L^2-invariants for sequences of lattices in Lie groups*, preprint, 2012.

[AJZN11] M. Abért, A. Jaikin-Zapirain, N. Nikolov, *The rank gradient from combinatorial viewpoint,* Groups Geom. Dyn. **5** (2011), 213–230.

[AN12] M. Abért, N. Nikolov, *Rank gradient, cost of groups and the rank versus Heegaard genus problem*, J. Eur. Math. Soc. **14** (2012), no. 5, 1657–1677.

[Ada87] C. Adams, *The noncompact hyperbolic* 3*-manifold of minimal volume*, Proc. Amer. Math. Soc. **100** (1987), no. 4, 601–606.

[Ada88] ———, *Volumes of N-cusped hyperbolic* 3*-manifolds*, J. London Math. Soc. (2) **38** (1988), no. 3, 555–565.

[Ady55] S. I. Adyan, *Algorithmic unsolvability of problems of recognition of certain properties of groups*, Dokl. Akad. Nauk SSSR (N. S.) **103** (1955), 533–535.

[Ag00] I. Agol, *Bounds on exceptional Dehn filling*, Geom. Topol. **4** (2000), 431–449.

[Ag03] ———, *Small* 3*-manifolds of large genus*, Geom. Dedicata **102** (2003), 53–64.

[Ag04] ———, *Tameness of hyperbolic* 3*-manifolds*, manuscript, `math/0405568`, 2004.

[Ag06] ———, *Virtual Betti numbers of symmetric spaces*, manuscript, `math/0611828`, 2006.

[Ag08] ———, *Criteria for virtual fibering*, J. Topol. **1** (2008), no. 2, 269–284.

[Ag10a] ———, *Bounds on exceptional Dehn filling*, II, Geom. Topol. **14** (2010), no. 4, 1921–1940.

[Ag10b] ———, *The minimal volume orientable hyperbolic* 2*-cusped* 3*-manifolds*, Proc. Amer. Math. Soc. **138** (2010), no. 10, 3723–3732.

[Ag11] ———, *Finite index subgroups of free groups and torsion-free amenable quotients of free groups*, MathOverflow.
`http://mathoverflow.net/questions/80657`

[Ag13] ———, *The virtual Haken conjecture*, with an appendix by I. Agol, D. Groves and J. Manning, Doc. Math. 18 (2013), 1045–1087.

[Ag14] ———, *Virtual properties of* 3*-manifolds*, preprint, 2014.

[ABZ08] I. Agol, S. Boyer, X. Zhang, *Virtually fibered Montesinos links*, J. Topol. **1** (2008), no. 4, 993–1018.

[ACS06] I. Agol, M. Culler, P. Shalen, *Dehn surgery, homology and hyperbolic volume*, Algebr. Geom. Topol. **6** (2006), 2297–2312.

[ACS10] ———, *Singular surfaces, mod 2 homology, and hyperbolic volume,* I, Trans. Amer. Math. Soc. **362** (2010), no. 7, 3463–3498.

[AGM09] I. Agol, D. Groves, J. Manning, *Residual finiteness, QCERF and fillings of hyperbolic groups*, Geom. Topol. **13** (2009), no. 2, 1043–1073.

[AGM14] ———, *An alternate proof of Wise's Malnormal Special Quotient Theorem*, preprint, 2014.

[AL12] I. Agol, Y. Liu, *Presentation length and Simon's conjecture*, J. Amer. Math. Soc. **25** (2012), 151–187.

[ALR01] I. Agol, D. Long, A. Reid, *The Bianchi groups are separable on geometrically finite subgroups*, Ann. of Math. (2) **153** (2001), no. 3, 599–621.

[AS60] L. V. Ahlfors, L. Sario, *Riemann Surfaces*, Princeton Mathematical Series, vol. 26, Princeton University Press, Princeton, N. J., 1960.

[AMR97] I. Aitchison, S. Matsumoto, J. Rubinstein, *Immersed surfaces in cubed manifolds*, Asian J. Math. **1** (1997), no. 1, 85–95.

[AMR99] ———, *Dehn surgery on the figure* 8 *knot: immersed surfaces*, Proc. Amer. Math. Soc. **127** (1999), no. 8, 2437–2442.

[AiR89] I. Aitchison, J. Rubinstein, *An introduction to polyhedral metrics of nonpositive curvature on* 3*-manifolds*, in: *Geometry of Low-dimensional Manifolds,* 2, pp. 127–161, London Math. Soc. Lecture Note Series, vol. 151, Cambridge University Press, Cambridge, 1990.

[AiR92] ———, *Canonical surgery on alternating link diagrams*, in: *Knots 90,* pp. 543–558, Walter de Gruyter & Co., Berlin, 1992.

[AiR99a] ———, *Polyhedral metrics and* 3*-manifolds which are virtual bundles*, Bull. London Math. Soc. **31** (1999), no. 1, 90–96.

[AiR99b] ———, *Combinatorial Dehn surgery on cubed and Haken* 3*-manifolds*, in: *Proceedings of the Kirbyfest,* pp. 1–21 (electronic), Geometry & Topology Monographs, vol. 2, Geometry & Topology Publications, Coventry, 1999.

[AiR04] ———, *Localising Dehn's lemma and the loop theorem in* 3*-manifolds*, Math. Proc. Cambridge Philos. Soc. **137** (2004), no. 2, 281–292.

[Ale19] J. W. Alexander, *Note on two three-dimensional manifolds with the same group*, Trans. Amer. Math. Soc. **20** (1919), 339–342.

[Ale20] ———, *Note on Riemann spaces*, Bull. Amer. Math. Soc. **26** (1920), 370–372.

[Ale24a] ———, *New topological invariants expressible as tensors*, Proc. Nat. Acad. Sci. U.S.A. **10** (1924), no. 3, 99–101.

[Ale24b] ———, *An example of a simply connected surface bounding a region which is not simply connected*, Proc. Nat. Acad. Sci. U.S.A. **10** (1924), no. 1, 8–10.

[Alf70] W. R. Alford, *Complements of minimal spanning surfaces of knots are not unique*, Ann. of Math. (2) **91** (1970), 419–424.

[AS70] W. R. Alford, C. B. Schaufele, *Complements of minimal spanning surfaces of knots are not unique,* II, in: *Topology of Manifolds,* pp. 87–96, Markham, Chicago, IL, 1970.

[AHR13] Y. Algom-Kfir, E. Hironaka, K. Rafi, *Digraphs and cycle polynomials for free-by-cyclic groups*, preprint, 2013.

[AlR12] Y. Algom-Kfir, K. Rafi, *Mapping tori of small dilatation irreducible train-track maps*, preprint, 2012.

[ABEMT79] R. Allenby, J. Boler, B. Evans, L. Moser, C. Y. Tang, *Frattini subgroups of* 3*-manifold groups*, Trans. Amer. Math. Soc. **247** (1979), 275–300.

[AKT05] R. Allenby, G. Kim, C. Y. Tang, *Conjugacy separability of certain Seifert* 3*-manifold groups*, J. Algebra **285** (2005), 481–507.

[AKT06] ———, *Outer automorphism groups of certain orientable Seifert* 3*-manifold groups*, in: *Combinatorial Group Theory, Discrete Groups, and Number Theory,* pp. 15–22, Contemporary Mathematics, vol. 421, Amer. Math. Soc., Providence, RI, 2006.

[AKT09] ———, *Outer automorphism groups of Seifert* 3*-manifold groups over non-orientable surfaces*, J. Algebra **322** (2009), no. 4, 957–968.

[AKT10] ———, *Conjugacy separability of Seifert* 3*-manifold groups over non-orientable surfaces*, J. Algebra **323** (2010), no. 1, 1–9.

[AH99] E. Allman, E. Hamilton, *Abelian subgroups of finitely generated Kleinian groups are separable*, Bull. London Math. Soc. **31** (1999), no. 2, 163–172.

[Alt12] I. Altman, *Sutured Floer homology distinguishes between Seifert surfaces*, Topology Appl. **159** (2012), no. 14, 3143–3155.

[ABCKT96] J. Amorós, M. Burger, A. Corlette, D. Kotschick, D. Toledo, *Fundamental Groups of Compact Kähler Manifolds*, Mathematical Surveys and Monographs, vol. 44, Amer. Math. Soc., Providence, RI, 1996.

[Ana02] J. W. Anderson, *Finite volume hyperbolic* 3*-manifolds whose fundamental group contains a subgroup that is locally free but not free*, Sci. Ser. A Math. Sci. (N.S.) **8** (2002), 13–20.

[ACCS96] J. W. Anderson, R. Canary, M. Culler, P. Shalen, *Free Kleinian groups and volumes of hyperbolic* 3*-manifolds*, J. Differential Geom. **43** (1996), no. 4, 738–782.

[Anb04] M. T. Anderson, *Geometrization of* 3*-manifolds via the Ricci flow*, Notices Amer. Math. Soc. **51** (2004), 184–193.

[ADL11] Y. Antolín, W. Dicks, P. Linnell, *Non-orientable surface-plus-one-relation groups*, J. Algebra **326** (2011), 4–33.

[AM11] Y. Antolín, A. Minasyan, *Tits alternatives for graph products*, J. Reine Angew. Math., to appear.

[AMS13] Y. Antolín, A. Minasyan, A. Sisto, *Commensurating endomorphisms of acylindrically hyperbolic groups and applications*, preprint, 2013.

[Ao11] R. Aoun, *Random subgroups of linear groups are free*, Duke Math. J. **160** (2011), no. 1, 117–173.

[Ar01] G. N. Arzhantseva, *On quasi-convex subgroups of word-hyperbolic groups*, Geom. Dedicata **87** (2001), no. 1–3, 191–208.

[AF11] M. Aschenbrenner, S. Friedl, *Residual properties of graph manifold groups*, Topology Appl. **158** (2011), 1179–1191.

[AF13] ———, *3-manifold groups are virtually residually p*, Mem. Amer. Math. Soc. **225** (2013), no. 1058.

[AFW13] M. Aschenbrenner, S. Friedl, H. Wilton, *Decision problems for 3-manifolds and their fundamental groups*, preprint, 2013.

[AK10] F. Atalan, M. Korkmaz, *Number of pseudo-Anosov elements in the mapping class group of a four-holed sphere*, Turkish J. Math. **34** (2010), 585–592.

[At76] M. Atiyah, *Elliptic operators, discrete groups, and von Neumann algebras*, Astérisque **32** (1976), 43–72.

[APS75a] M. F. Atiyah, V. K. Patodi, I. M. Singer, *Spectral asymmetry and Riemannian geometry,* I, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69.

[APS75b] ———, *Spectral asymmetry and Riemannian ge- ometry,* II, Math. Proc. Cambridge Philos. Soc. **78** (1975), 405–432.

[Auc93] D. Auckly, *An irreducible homology sphere which is not Dehn surgery on any knot*, manuscript, 1993.
`http://www.math.ksu.edu/~dav/irr.pdf`

[Auc97] ———, *Surgery numbers of 3-manifolds: a hyperbolic example*, in: *Geometric Topology*, pp. 21–34, AMS/IP Studies in Advanced Mathematics, vol. 2.1, Amer. Math. Soc., Providence, RI; International Press, Cambridge, MA, 1997.

[Aum56] R. Aumann, *Asphericity of alternating knots*, Ann. of Math. (2) **64** (1956), 374–392.

[Aus67] L. Auslander, *On a problem of Philip Hall*, Ann. of Math. (2) **86** (1967), 112–116.

[Av70] A. Avez, *Variétés riemanniennes sans points focaux*, C. R. Acad. Sci. Paris Sér. A-B **270** (1970), A188–A191.

[ANW13] G. Avramidi, T. Tam Nguyen-Phan, Y. Wu, *Fundamental groups of finite volume, bounded negatively curved 4-manifolds are not 3-manifold groups*, preprint, 2013.

[BcS05] D. Bachman, S. Schleimer, *Surface bundles versus Heegaard splittings*, Comm. Anal. Geom. **13** (2005), 903–928.

[Bak88] M. Baker, *The virtual $\mathbb{Z}$-representability of certain 3-manifold groups*, Proc. Amer. Math. Soc. **103** (1988), no. 3, 996–998.

[Bak89] ———, *Covers of Dehn fillings on once-punctured torus bundles*, Proc. Amer. Math. Soc. **105** (1989), no. 3, 747–754.

[Bak90] ———, *Covers of Dehn fillings on once-punctured torus bundles,* II, Proc. Amer. Math. Soc. **110** (1990), no. 4, 1099–1108.

[Bak91] ———, *On coverings of figure eight knot surgeries*, Pacific J. Math. **150** (1991), no. 2, 215–228.

[BBW02] M. Baker, M. Boileau, S. Wang, *Towers of covers of hyperbolic 3-manifolds*, Rend. Istit. Mat. Univ. Trieste **32** (2001), suppl. 1, 35–43 (2002).

[BaC13] M. Baker, D. Cooper, *Conservative subgroup separability for surfaces with boundary*, Algebr. Geom. Topol. **13** (2013), no. 1, 115–125.

[BaC14] ———, *Finite-volume hyperbolic 3-manifolds contain immersed quasi-Fuchsian surfaces*, preprint, 2014.

[BGS85] W. Ballmann, M. Gromov, V. Schroeder, *Manifolds of nonpositive curvature*, Progress in Mathematics, vol. 61, Birkhäuser Boston Inc., Boston, MA, 1985.

[Ban11a] J. E. Banks, *On links with locally infinite Kakimizu complexes*, Algebr. Geom. Topol. **11** (2011), no. 3, 1445–1454.

[Ban11b] ———, *Minimal genus Seifert surfaces for alternating links*, preprint, 2011.

[Ban13] ———, *Knots with many minimal genus Seifert surfaces*, preprint, 2013.
[Bar01] A. Bart, *Surface groups in some surgered manifolds*, Topology **40** (2001), 197–211.
[Bar04] ———, *Immersed and virtually embedded boundary slopes in arithmetic manifolds*, J. Knot Theory Ramifications **13** (2004), 587–596.
[BFL14] A. Bartels, F. T. Farrell, W. Lück, *The Farrell-Jones Conjecture for cocompact lattices in virtually connected Lie groups*, J. Amer. Math. Soc. **27** (2014), no. 2, 339–388.
[BLW10] A. Bartels, W. Lück, S. Weinberger, *On hyperbolic groups with spheres as boundary*, J. Differential Geom. **86** (2010), no. 1, 1–16.
[BaL12] A. Bartels, W. Lück, *The Borel Conjecture for hyperbolic and CAT(0)-groups*, Ann. of Math. (2) **175** (2012), 631–689.
[Bas93] H. Bass, *Covering theory for graphs of groups,* J. Pure Appl. Algebra **89** (1993), no. 1–2, 3–47.
[Bat71] J. Batude, *Singularité générique des applications différentiables de la* 2*-sphère dans une* 3*-variété différentiable*, Ann. Inst. Fourier (Grenoble) **21** (1971), no. 3, 155–172.
[Bah81] A. Baudisch, *Subgroups of semifree groups*, Acta Math. Acad. Sci. Hungar. **38** (1981), no. 1–4, 19–28.
[Bag62] G. Baumslag, *On generalized free products*, Math. Z. **78** (1962), 423–438.
[BaS62] G. Baumslag, D. Solitar, *Some two-generator one-relator non-Hopfian groups*, Bull. Amer. Math. Soc. **68** (1962), 199–201.
[BL14] B. Beeker, N. Lazarovich, *Resolutions of CAT(0) cube complexes and accessibility properties*, preprint, 2014.
[Bed11] T. Bedenikovic, *Asphericity results for ribbon disk complements via alternate descriptions*, Osaka J. Math. **48** (2011), no. 1, 99–125.
[BN08] J. A. Behrstock, W. D. Neumann, *Quasi-isometric classification of graph manifold groups*, Duke Math. J. **141** (2008), no. 2, 217–240.
[Bn12] ———, *Quasi-isometric classification of non-geometric* 3*-manifold groups*, J. Reine Angew. Math. **669** (2012), 101–120.
[BdlHV08] B. Bekka, P. de la Harpe, A. Valette, *Kazhdan's property* (T), New Mathematical Monographs, vol. 11, Cambridge University Press, Cambridge, 2008.
[Bek13] I. Belegradek, *Topology of open nonpositively curved manifolds*, preprint, 2013. To appear in the proceedings of the ICM satellite conference (Bangalore, 2010) Geometry, Topology and Dynamics in Negative Curvature, London Math Society Lecture Notes series.
[Bel12] M. Belolipetsky, *On* 2*-systoles of hyperbolic* 3*-manifolds*, preprint, 2012.
[BGLS10] M. Belolipetsky, T. Gelander, A. Lubotzky, A. Shalev, *Counting arithmetic lattices and surfaces*, Ann. Math. (2) **172** (2010), no. 3, 2197–2221.
[BeL05] M. Belolipetsky, A. Lubotzky, *Finite groups and hyperbolic manifolds*, Invent. Math. **162** (2005), no. 3, 459–472.
[BeK02] N. Benakli, I. Kapovich, *Boundaries of hyperbolic groups*, in: *Combinatorial and Geometric Group Theory,* pp. 39–93, Contemporary Mathematics, vol. 296, Amer. Math. Soc., Providence, RI, 2002.
[BeP92] R. Benedetti, C. Petronio, *Lectures on Hyperbolic Geometry*, Universitext, Springer-Verlag, Berlin, 1992.
[Bei07] V. N. Berestovskii, *Poincaré Conjecture and Related Statements*, Izv. Vyssh. Uchebn. Zaved. Mat. **2007**, no. 9, 3–41.
[Ber08] N. Bergeron, *Virtual fibering of certain cover of* $\mathbb{S}^3$*, branched over the figure eight knot*, C. R. Math. Acad. Sci. Paris **346** (2008), no. 19-20, 1073–1078.
[Ber12] ———, *La conjecture des sous-groupes de surfaces* (*d'après Jeremy Kahn et Vladimir Markovic*), Séminaire Bourbaki, vol. 2011/2012, exposés 1043–1058. Astérisque No. **352** (2013), exp. no. 1055, x, 429–458.
[Ber14] ———, *Toute variété de dimension 3 compacte et asphérique est virtuellement de Haken* (*d'après Ian Agol et Daniel T. Wise*), Séminaire Bourbaki, vol. 2013-14, exp. no. 1078.
[BeG04] N. Bergeron, D. Gaboriau, *Asymptotique des nombres de Betti, invariants* l^2 *et laminations*, Comment. Math. Helv. **79** (2004), no. 2, 362–395.
[BHW11] N. Bergeron, F. Haglund, D. Wise, *Hyperplane sections in arithmetic hyperbolic manifolds*, J. London Math. Soc. (2) **83** (2011), no. 2, 431–448.

[BSV14] N. Bergeron, M. H. Sengun, A. Venkatesh, *Torsion homology growth and cycle complexity of arithmetic manifolds*, preprint, 2014.

[BV13] N. Bergeron, A. Venkatesh, *The asymptotic growth of torsion homology for arithmetic groups*, J. Inst. Math. Jussieu **12** (2013), no. 2, 391–447.

[BeW12] N. Bergeron, D. Wise, *A boundary criterion for cubulation*, Amer. J. Math. **134** (2012), 843–859.

[Bem91] G. M. Bergman, *Right orderable groups that are not locally indicable*, Pacific J. Math. **147** (1991), 243–248.

[BBBMP10] L. Bessières, G. Besson, M. Boileau, S. Maillot, J. Porti, *Geometrisation of* 3*-Manifolds*, EMS Tracts in Mathematics, vol. 13, European Mathematical Society, Zürich, 2010.

[Ben06] G. Besson, *Preuve de la conjecture de Poincaré en déformant la métrique par la courbure de Ricci* (*d'après G. Perelman*), Séminaire Bourbaki, vol. 2004/2005, Astérisque **307** (2006), exp. no. 947, ix, 309–347.

[BCG11] G. Besson, G. Courtois, S. Gallot, *Uniform growth of groups acting on Cartan-Hadamard spaces*, J. Eur. Math. Soc. **13** (2011), no. 5, 1343–1371.

[Bet71] L. A. Best, *On torsion-free discrete subgroups of* $PSL(2,\mathbb{C})$ *with compact orbit space*, Canad. J. Math. **23** (1971), 451–460.

[Bea96] M. Bestvina, *Local homology properties of boundaries of groups*, Michigan Math. J. **43** (1996), no. 1, 123–139.

[Bea14] ———, *Geometric group theory and* 3*-manifolds hand in hand: the fulfillment of Thurston's vision*, Bull. Amer. Math. Soc. **51** (2014), 53–70.

[BeB97] M. Bestvina, N. Brady, *Morse theory and finiteness properties of groups*, Invent. Math. **129** (1997), no. 3, 445–470.

[BF92] M. Bestvina, M. Feighn, *A combination theorem for negatively curved groups*, J. Differential Geom. **35** (1992), no. 1, 85–101.

[BeH92] M. Bestvina, M. Handel, *Train tracks and automorphisms of free groups*, Ann. of Math. (2) **135** (1992), no. 1, 1–51.

[BeM91] M. Bestvina, G. Mess, *The boundary of negatively curved groups*, J. Amer. Math. Soc. **4** (1991), no. 3, 469–481.

[BiH91] R. Bieri, J. A. Hillman, *Subnormal subgroups in* 3*-dimensional Poincaré duality groups*, Math. Z. **206** (1991), 67–69.

[BNS87] R. Bieri, W. Neumann, R. Strebel, *A geometric invariant of discrete groups*, Invent. Math. **90** (1987), 451–477.

[BR88] R. Bieri, B. Renz, *Valuations on free resolutions and higher geometric invariants of groups*, Comment. Math. Helv. **63** (1988), 464–497.

[BiW13] H. Bigdely, D. Wise, *Quasiconvexity and relatively hyperbolic groups that split*, Michigan Math. J. **62** (2013), no. 2, 387–406.

[Bie07] R. Bieri, *Deficiency and the geometric invariants of a group*, J. Pure Appl. Algebra **208** (2007), 951–959.

[Bin52] R. H. Bing, *A homeomorphism between the* 3*-sphere and the sum of two solid horned spheres*, Ann. of Math. (2) **56** (1952), 354–362.

[Bin59] ———, *An alternative proof that* 3*-manifolds can be triangulated*, Ann. of Math. (2) **69** (1959), 37–65.

[Bin83] ———, *The Geometric Topology of* 3*-Manifolds*, Amer. Math. Soc. Colloquium Publications, vol. 40, Amer. Math. Soc., Providence, RI, 1983.

[Bir09] I. Biringer, *Geometry and rank of fibered hyperbolic* 3*-manifolds*, Algebr. Geom. Topol. **9** (2009), no. 1, 277–292.

[BiS14] I. Biringer, J. Souto, *Ranks of mapping tori via the curve complex*, preprint, 2014.

[BiM12] I. Biswas, M. Mj, *Low dimensional projective groups*, preprint, 2012.

[BiM14] ———, *Quasiprojective* 3*-manifold groups and complexification of* 3*-manifolds*, preprint, 2014.

[BMS12] I. Biswas, M. Mj, H. Seshadri, *Three manifold groups, Kähler groups and complex surfaces*, Commun. Contemp. Math. **14** (2012), no. 6.

[Bla57] R. Blanchfield, *Intersection theory of manifolds with operators with applications to knot theory*, Ann. of Math. (2) **65** (1957), 340–356.

[Ble94] S. Bleiler, *Two generator cable knots are tunnel one*, Proc. Am. Math. Soc. **122** (1994), 1285–1287.

[BlC88] S. Bleiler, A. Casson, *Automorphisms of Surfaces after Nielsen and Thurston*, London Math. Soc. Student Texts, vol. 9, Cambridge University Press, Cambridge, 1988.

[BlH96] S. Bleiler, C. Hodgson, *Spherical space forms and Dehn filling*, Topology **35** (1996), 809–833.

[BJ04] S. Bleiler, A. Jones, *On two generator satellite knots*, Geom. Dedicata **104** (2004), 1–14.

[Boe72] H. Boehme, *Fast genügend grosse irreduzible* 3*-dimensionale Mannigfaltigkeiten*, Invent. Math. **17** (1972), 303–316.

[Bog93] W. Bogley, *J. H. C. Whitehead's asphericity question*, in: *Two-dimensional Homotopy and Combinatorial Group Theory,* pp. 309–334, London Math. Soc. Lecture Note Series, vol. 197, Cambridge University Press, Cambridge, 1993.

[BoB13] M. Boileau, S. Boyer, *Graph manifolds* $\mathbb{Z}$*-homology* 3*-spheres and taut foliations*, preprint, 2013.

[BBCW12] M. Boileau, S. Boyer, R. Cebanu, G. S. Walsh, *Knot commensurability and the Berge conjecture*, Geom. Topol. **16** (2012), no. 2, 625–664.

[BF15] M. Boileau, S. Friedl, *The profinite completion of* 3*-manifold groups, fiberedness and the Thurston norm*, Preprint (2015)

[BLP01] M. Boileau, B. Leeb, J. Porti, *Uniformization of small* 3*-orbifolds*, C. R. Acad. Sci. Paris. Sér I Math. **332** (2001), no. 1, 57–62.

[BLP05] ———, *Geometrization of* 3*-dimensional orbifolds*, Ann. of Math. (2) **162** (2005), no. 1, 195–290.

[BMP03] M. Boileau, S. Maillot, J. Porti, *Three-dimensional Orbifolds and their Geometric Structures*, Panoramas et Synthèses, vol. 15, Société Mathématique de France, Paris, 2003.

[BO86] M. Boileau, J.-P. Otal, *Groupe des difféotopies de certaines variétés de Seifert*, C. R. Acad. Sci. Paris Sér. I Math. **303** (1986), no. 1, 19–22.

[BO91] ———, *Heegaard splittings and homeotopy groups of small Seifert fibre spaces*, Invent. Math. **106** (1991), no. 1, 85–107.

[BP01] M. Boileau, J. Porti, *Geometrization of* 3*-orbifolds of cyclic type*, Astérisque **272** (2001).

[BW05] M. Boileau, R. Weidmann, *The structure of* 3*-manifolds with two-generated fundamental group*, Topology **44** (2005), no. 2, 283–320.

[BoZ83] M. Boileau, H. Zieschang, *Genre de Heegaard d'une variété de dimension 3 et générateurs de son groupe fondamental*, C. R. Acad. Sci. Paris Sér. I Math. **296** (1983), no. 22, 925–928.

[BoZ84] ———, *Heegaard genus of closed orientable Seifert* 3*-manifolds*, Invent. Math. **76** (1984), 455–468.

[BoZ85] ———, *Nombre de ponts et générateurs méridiens des entrelacs de Montesinos*, Comm. Math. Helv. **60** (1985), 270–279.

[BoZi89] M. Boileau, B. Zimmermann, *On the p-orbifold group of a link*, Math. Z. **200** (1989), no. 2, 187–208.

[Bon83] F. Bonahon, *Difféotopies des espaces lenticulaires*, Topology **22** (1983), no. 3, 305–314.

[Bon86] ———, *Bouts des variétés hyperboliques de dimension* 3, Ann. of Math. (2) **124** (1986), 71–158.

[Bon02] ———, *Geometric structures on* 3*-manifolds*, in: *Handbook of Geometric Topology,* pp. 93–164, North-Holland, Amsterdam, 2002.

[Bon09] ———, *Low-dimensional geometry. From Euclidean surfaces to hyperbolic knots*, Student Mathematical Library, vol. 49, IAS/Park City Mathematical Subseries, Amer. Math. Soc., Providence, RI; Institute for Advanced Study, Princeton, NJ, 2009.

[BoS87] F. Bonahon, L. C. Siebenmann, *The characteristic toric splitting of irreducible compact* 3*-orbifolds*, Math. Ann. **278** (1987), no. 1-4, 441–479.

[Bok06] M. Bonk, *Quasiconformal geometry of fractals*, in: *International Congress of Mathematicians,* vol. II, pp. 1349–1373, European Mathematical Society, Zürich, 2006.

[BoK05] ———, *Conformal dimension and Gromov hyperbolic groups with* 2*-sphere boundary*, Geom. Topol. **9** (2005), 219–246.

[BHP68] W. Boone, W. Haken, V. Poenaru, *On recursively unsolvable problems in topology and their classification*, in: *Contributions to Mathematical Logic,* pp. 37–74, North-Holland, Amsterdam, 1968.

[Bor81] A. Borel, *Commensurability classes and volumes of hyperbolic* 3*-manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **8** (1981), no. 1, 1–33.

[BoP89] A. Borel, G. Prasad, *Finiteness theorems for discrete subgroups of bounded covolume in semi-simple groups*, Inst. Hautes Études Sci. Publ. Math. **69** (1989), 119–171.

[BE06] N. Boston, J. Ellenberg, *Pro-p groups and towers of rational homology spheres*, Geom. Topol. **10** (2006), 331–334 (electronic).

[BHPa14] K. Bou-Rabee, M. F. Hagen, P. Patel, *Residual finiteness growths of virtually special groups*, preprint, 2014.

[Bou81] N. Bourbaki, *Éléments de Mathématique. Groupes et Algèbres de Lie. Chapitres 4, 5 et 6,* Masson, Paris, 1981.

[Bowe04] L. Bowen, *Weak forms of the Ehrenpreis Conjecture and the Surface Subgroup Conjecture*, manuscript, `math/0411662`, 2004.

[Bow93] B. Bowditch, *Geometrical finiteness for hyperbolic groups*, J. Funct. Anal. **113** (1993), no. 2, 245–317.

[Bow00] ———, *A variation on the unique product property*, J. London Math. Soc. (2) **62** (2000), no. 3, 813–826.

[Bow04] ———, *Planar groups and the Seifert conjecture*, J. Reine Angew. Math. **576** (2004), 11–62.

[Bow06] ———, *End invariants of hyperbolic 3-manifolds*, preprint, 2011.
`http://www.warwick.ac.uk/~masgak/papers/bhb-endinvariants.pdf`

[Bow05] ———, *Geometric models for hyperbolic 3-manifolds*, preprint, 2011.
`http://www.warwick.ac.uk/~masgak/papers/bhb-models.pdf`

[Bow10] ———, *Notes on tameness*, Enseign. Math. (2) **56** (2010), no. 3–4, 229–285.

[Bow11] ———, *The ending lamination theorem*, preprint, 2011.
`http://www.warwick.ac.uk/~masgak/papers/elt.pdf`

[Boy86] S. Boyer, *On the non-realizability for certain 3-manifolds by Dehn surgery*, Math. Proc. Camb. Phil. Soc. **99** (1986), 103–106.

[Boy02] ———, *Dehn surgery on knots*, in: *Handbook of Geometric Topology,* pp. 165–218, North-Holland, Amsterdam, 2002.

[Boy] ———, *Linearisibility of the fundamental groups of 3-manifolds*, manuscript.

[BoC14] S. Boyer, A. Clay, *Foliations, orders, representations, L-spaces and graph manifolds*, preprint, 2014.

[BCSZ08] S. Boyer, M. Culler, P. Shalen, X. Zhang, *Characteristic subsurfaces, character varieties and Dehn fillings*, Geom. Topol. **12** (2008), no. 1, 233–297.

[BGW13] S. Boyer, C. McA. Gordon, L. Watson, *On L–spaces and left-orderable fundamental groups*, Math. Ann. **356** (2013), 1213–1245.

[BGZ01] S. Boyer, C. McA. Gordon, X. Zhang, *Dehn fillings of large hyperbolic 3-manifolds*, J. Differential Geom. **58** (2001), no. 2, 263–308.

[BoL90] S. Boyer, D. Lines, *Surgery formulae for Casson's invariant and extensions to homology lens spaces*, J. Reine Angew. Math. **405** (1990), 181–220.

[BRW05] S. Boyer, D. Rolfsen, B. Wiest, *Orderable 3-manifold groups*, Ann. Inst. Fourier (Grenoble) **55** (2005), no. 1, 243–288.

[BrZ00] S. Boyer, X. Zhang, *Virtual Haken 3-manifolds and Dehn filling*, Topology **39** (2000), no. 1, 103–114.

[Bra95] T. Brady, *Complexes of nonpositive curvature for extensions of F_2 by $\mathbb{Z}$*, Topology Appl. **63** (1995), no. 3, 267–275.

[Brh21] H. R. Brahana, *Systems of circuits on two-dimensional manifolds*, Annals of Math. (2)**23** (1921), 144–168.

[Brd93] M. Bridson, *Combings of semidirect products and 3-manifold groups*, Geom. Funct. Anal. **3** (1993), no. 3, 263–278.

[Brd99] ———, *Non-positive curvature in group theory*, in: *Groups St. Andrews 1997 in Bath,* I, pp. 124–175, London Math. Soc. Lecture Note Series, vol. 260, Cambridge University Press, Cambridge, 1999.

[Brd01] ———, *On the subgroups of semihyperbolic groups*, in: *Essays on Geometry and Related Topics*, Vol. 1, pp. 85–111, Monographies de L'Enseignement Mathématique, vol. 38, L'Enseignement Mathématique, Geneva, 2001.

[Brd13] ———, *On the subgroups of right angled Artin groups and mapping class groups*, Math. Res. Lett. **20** (2013), no. 2, 203–212.

[BrGi96] M. Bridson, R. Gilman, *Formal language theory and the geometry of* 3*-manifolds*, Comment. Math. Helv. **71** (1996), no. 4, 525–555.

[BGHM10] M. Bridson, D. Groves, J. Hillman, G. Martin, *Cofinitely Hopfian groups, open mappings and knot complements*, Groups Geom. Dyn. **4** (2010), no. 4, 693–707.

[BrGd04] M. Bridson, F. Grunewald, *Grothendieck's problems concerning profinite completions and representations of groups*, Ann. of Math. (2) **160** (2004), no. 1, 359–373.

[BrH99] M. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften, vol. 319, Springer-Verlag, New York, 1999.

[Brn93] M. G. Brin, *Seifert fibered* 3*-manifolds*, lecture notes, Binghamton University, `arXiv:0711.1346`, 1993.

[BJS85] M. Brin, K. Johannson, P. Scott, *Totally peripheral* 3*-manifolds*, Pacific J. Math. **118** (1985), 37–51.

[Brm00] P. Brinkmann, *Hyperbolic automorphisms of free groups*, Geom. Funct. Anal. **10** (2000), no. 5, 1071–1089.

[Brt08] M. Brittenham, *Knots with unique minimal genus Seifert surface and depth of knots*, J. Knot Theory Ramifications **17** (2008), no. 3, 315–335.

[Brs05] N. Broaddus, *Noncyclic covers of knot complements*, Geom. Dedicata **111** (2005), 211–239.

[BB11] J. Brock, K. Bromberg, *Geometric inflexibility and 3-manifolds that fiber over the circle*, J. Top. **4** (2011), 1–38.

[BCM12] J. Brock, R. Canary, Y. Minsky, *The classification of Kleinian surface groups,* II. *The ending lamination conjecture*, Ann. of Math. (2) **176** (2012), no. 1, 1–149.

[BD13] J. Brock, N. Dunfield, *Injectivity radii of hyperbolic integer homology* 3*-spheres*, Geom. Topol., to appear.

[Bry60] E. J. Brody, *The topological classification of the lens spaces*, Ann. of Math. (2) **71** (1960), 163–184.

[Brk85] R. Brooks, *On branched coverings of* 3*-manifolds which fiber over the circle*, J. Reine Angew. Math. **362** (1985), 87–101.

[Brk86] ———, *Circle packings and co-compact extensions of Kleinian groups*, Invent. Math. **86** (1986), no. 3, 461–469.

[Broa66] E. M. Brown, *Unknotting in* $M^2 \times I$, Trans. Amer. Math. Soc. **123** (1966), 480–505.

[BrC65] E. M. Brown, R. H. Crowell, *Deformation retractions of* 3*-manifolds into their boundaries*, Ann. of Math. (2) **82** (1965), 445–458.

[BT74] E. M. Brown, T. W. Tucker, *On proper homotopy theory for noncompact* 3*-manifolds*, Trans. Amer. Math. Soc. **188** (1974), 105–126.

[Brob87] K. Brown, *Trees, valuations and the Bieri–Neumann–Strebel invariant*, Invent. Math. **90** (1987), no. 3, 479–504.

[Broc60] M. Brown, *A proof of the generalized Schoenflies theorem*, Bull. Amer. Math. Soc. **66** (1960), 74–76.

[Bru92] A. M. Brunner, *Geometric quotients of link groups*, Topology Appl. **48** (1992), no. 3, 245–262.

[BBS84] A. M. Brunner, R. G. Burns, D. Solitar, *The subgroup separability of free products of two free groups with cyclic amalgamation*, in: *Contributions to Group Theory,* pp. 90–115, Contemporary Mathematics, vol. 33, Amer. Math. Soc., Providence, RI, 1984.

[BBI13] M. Bucher, M. Burger, A. Iozzi, *A dual interpretation of the Gromov-Thurston proof of Mostow Rigidity and Volume Rigidity*, in: *Trends in Harmonic Analysis,* pp. 47–76, Springer INdAM Series, vol. 3, Springer, Milan, 2013.

[BdlH00] M. Bucher, P. de la Harpe, *Free products with amalgamation, and HNN-extensions of uniformly exponential growth*, Math. Notes **67** (2000), no. 5-6, 686–689.

[Bue93] G. Burde, *Knot groups*, in: *Topics in Knot Theory,* pp. 25–31, NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences, vol. 399, Kluwer Academic Publishers, Dordrecht, 1993.

[BdM70] G. Burde, K. Murasugi, *Links and Seifert fiber spaces*, Duke Math. J. **37** (1970), 89–93.

[BZ66] G. Burde, H. Zieschang, *Eine Kennzeichnung der Torusknoten*, Math. Ann. **167** (1966), 169–176

[BZH14] G. Burde, H. Zieschang, M. Heusener, *Knots*, 3rd ed., De Gruyter Studies in Mathematics, vol. 5, De Gruyter, Berlin, 2014.

[BuM06] J. Burillo, A. Martino, *Quasi-potency and cyclic subgroup separability*, J. Algebra **298** (2006), no. 1, 188–207.

[Bus69] R. G. Burns, *A note on free groups*, Proc. Amer. Math. Soc. **23** (1969), 14–17.

[Bus71] ______, *On finitely generated subgroups of free products*, J. Austral. Math. Soc. **12** (1971), 358–364.

[BHa72] R. G. Burns, V. Hale, *A note on group rings of certain torsion-free groups*, Canad. Math. Bull. **15** (1972), 441–445.

[BKS87] R. G. Burns, A. Karrass, D. Solitar, *A note on groups with separable finitely generated subgroups*, Bull. Austral. Math. Soc. **36** (1987), no. 1, 153–160.

[BCT14] B. Burton, A. Coward, S. Tillmann, *Computing closed essential surfaces in knot complements*, in: *SoCG '13: Proceedings of the Twenty-Ninth Annual Symposium on Computational Geometry,* pp. 405–414, Association for Computing Machinery (ACM), New York, 2013.

[BRT12] B. Burton, H. Rubinstein, S. Tillmann, *The Weber-Seifert dodecahedral space is non-Haken*, Trans. Amer. Math. Soc. **364** (2012), no. 2, 911–932.

[But04] J. O. Button, *Strong Tits alternatives for compact* 3*-manifolds with boundary*, J. Pure Appl. Algebra **191** (2004), no. 1-2, 89–98.

[But05] ______, *Fibred and virtually fibred hyperbolic* 3*-manifolds in the censuses*, Experiment. Math. **14** (2005), no. 2, 231–255.

[But07] ______, *Mapping tori with first Betti number at least two*, J. Math. Soc. Japan **59** (2007), no. 2, 351–370

[But08] ______, *Large groups of deficiency* 1, Israel J. Math. **167** (2008), 111–140.

[But10] ______, *Largeness of LERF and* 1*-relator groups*, Groups Geom. Dyn. **4** (2010), no. 4, 709–738.

[But11a] ______, *Proving finitely presented groups are large by computer*, Exp. Math. **20** (2011), no. 2, 153–168.

[But11b] ______, *Virtual finite quotients of finitely generated groups*, New Zealand J. Math. **41** (2011), 1–15.1179–4984

[But12] ______, *Groups possessing only indiscrete embeddings in* $\mathrm{SL}(2,\mathbb{C})$, preprint, 2012.

[But13] ______, *Free by cyclic groups are large*, preprint, 2013.

[But14] ______, *A* 3*-manifold group which is not four dimensional linear*, J. Pure Appl. Algebra **218** (2014), 1604–1619.

[BuK96a] S. Buyalo, V. Kobelskii, *Geometrization of graph-manifolds.* I. *Conformal geometrization*, St. Petersburg Math. J. **7** (1996), no. 2, 185–216.

[BuK96b] ______, *Geometrization of graph-manifolds.* II. *Isometric geometrization*, St. Petersburg Math. J. **7** (1996), no. 3, 387–404.

[BuS05] S. Buyalo, P. Svetlov, *Topological and geometric properties of graph-manifolds*, St. Petersburg Math. J. **16** (2005), no. 2, 297–340.

[Cal06] D. Calegari, *Real places and torus bundles*, Geom. Dedicata **118** (2006), no. 1, 209–227.

[Cal07] ______, *Hyperbolic* 3*-manifolds, tameness, and Ahlfors' measure conjecture*, lecture notes from a minicourse at the IUM during February 19–23, 2007
`http://math.uchicago.edu/~dannyc/papers/moscow.pdf.gz`

[Cal09a] ______, *scl*, MSJ Memoirs, vol. 20, Mathematical Society of Japan, Tokyo, 2009.

[Cal09b] ______, *Stable commutator length is rational in free groups*, J. Amer. Math. Soc. **22** (2009), no. 4, 941–961.

[Cal13a] ______, *Certifying incompressibility of non-injective surfaces with scl*, Pacific J. Math. **262** (2013), no. 3, 257–262.

[Cal13b] ______, *Notes on Agol's virtual Haken theorem*, lecture notes, 2013.
`http://math.uchicago.edu/~dannyc/courses/agol_virtual_haken/agol_notes.pdf`

[Cal14a] ______, *Notes on* 3*-manifolds*, lectures notes, 2014.
`http://math.uchicago.edu/~dannyc/courses/3manifolds_2014/3_manifolds_notes.pdf`

[Cal14b] ______, *Groups quasi-isometric to planes*
`https://lamington.wordpress.com/2014/08/22/groups-quasi-isometric-to-planes/`

[CD03] D. Calegari, N. Dunfield, *Laminations and groups of homeomorphisms of the circle*, Invent. Math. **152** (2003), no. 1, 149–204.

[CaG06] D. Calegari, D. Gabai, *Shrinkwrapping and the taming of hyperbolic* 3*-manifolds*, J. Amer. Math. Soc. **19** (2006), no. 2, 385–446.

[CFW10] D. Calegari, M. H. Freedman, K. Walker, *Positivity of the universal pairing in* 3 *dimensions*, J. Amer. Math. Soc. **23** (2010), no. 1, 107–188.

[CK10] D. Calegari, K. Fujiwara, *Stable commutator length in word hyperbolic groups*, Groups Geom. Dyn. **4** (2010), no. 1, 59–90.

[CSW11] D. Calegari, H. Sun, S. Wang, *On fibered commensurability*, Pacific J. Math. **250** (2011), no. 2, 287–317.

[CD06] F. Calegari, N. Dunfield, *Automorphic forms and rational homology* 3*-spheres*, Geom. Topol. **10** (2006), 295–330.

[CaE11] F. Calegari, M. Emerton, *Mod-p cohomology growth in p-adic analytic towers of* 3*-manifolds*, Groups Geom. Dyn. **5** (2011), no. 2, 355–366.

[Cay94] R. Canary, *Covering theorems for hyperbolic* 3*-manifolds*, in: *Low-dimensional Topology,* pp. 21–30, Conference Proceedings and Lecture Notes in Geometry and Topology, III, International Press, Cambridge, MA, 1994.

[Cay96] ———, *A covering theorem for hyperbolic* 3*-manifolds and its applications*, Topology **35** (1996), no. 3, 751–778.

[Cay08] ———, *Marden's Tameness Conjecture: history and applications*, in: *Geometry, Analysis and Topology of Discrete Groups,* pp. 137–162, Advanced Lectures in Mathematics (ALM), vol. 6, International Press, Somerville, MA; Higher Education Press, Beijing, 2008.

[CEG87] R. Canary, D. B. A. Epstein, P. Green, *Notes on notes of Thurston*, in: *Analytical and Geometric Aspects of Hyperbolic Space*, pp. 3–92, London Math. Soc. Lecture Note Series, vol. 111, Cambridge University Press, Cambridge, 1987.

[CEG06] ———, *Notes on notes of Thurston. With a new foreword by Canary*, in: *Fundamentals of Hyperbolic Geometry: Selected Expositions,* pp. 1–115, London Math. Soc. Lecture Note Series, vol. 328, Cambridge University Press, Cambridge, 2006.

[CyM04] R. Canary, D. McCullough, *Homotopy equivalences of* 3*-manifolds and deformation theory of Kleinian groups*, Mem. Amer. Math. Soc. **172** (2004), no. 812.

[CdC00] A. Candel, L. Conlon, *Foliations*, I, Graduate Studies in Mathematics, vol. 23, Amer. Math. Soc., Providence, RI, 2000.

[CdC03] ———, *Foliations*, II, Graduate Studies in Mathematics, vol. 60, Amer. Math. Soc., Providence, RI, 2003.

[CtC93] J. Cantwell, L. Conlon, *Foliations of* $E(5_2)$ *and related knot complements*, Proc. Amer. Math. Soc. **118** (1993), no. 3, 953–962.

[Can94] J. Cannon, *The combinatorial Riemann mapping theorem*, Acta Math. **173** (1994), no. 2, 155–234.

[CF76] J. Cannon, C. Feustel, *Essential embeddings of annuli and Möbius bands in* 3*-manifolds*, Trans. Amer. Math. Soc. **215** (1976), 219–239.

[CFP99] J. W. Cannon, W. J. Floyd, W. R. Parry, *Sufficiently rich families of planar rings*, Ann. Acad. Sci. Fenn. Math. **24** (1999), no. 2, 265–304.

[CFP01] ———, *Finite subdivision rules*, Conform. Geom. Dyn. **5** (2001), 153–196 (electronic).

[CaS98] J. W. Cannon, E. L. Swenson, *Recognizing constant curvature discrete groups in dimension* 3, Trans. Amer. Math. Soc. **350** (1998), no. 2, 809–849.

[CaM01] C. Cao, R. Meyerhoff, *The orientable cusped hyperbolic* 3*-manifolds of minimum volume*, Invent. Math. **146** (2001), no. 3, 451–478.

[CZ06a] H.-D. Cao, X.-P. Zhu, *A complete proof of the Poincaré and geometrization conjectures–application of the Hamilton-Perelman theory of the Ricci flow*, Asian J. Math. **10** (2006), no. 2, 165–492.

[CZ06b] ———, *Erratum to: "A complete proof of the Poincaré and geometrization conjectures—application of the Hamilton-Perelman theory of the Ricci flow"*, Asian J. Math. **10** (2006), no. 4, 663.

[CaT89] J. A. Carlson, D. Toledo, *Harmonic mapping of Kähler manifolds to locally symmetric spaces*, Inst. Hautes Études Sci. Publ. Math. **69** (1989), 173–201.

[Car14] M. Carr, *Two generator subgroups of right-angled Artin groups are quasi-isometrically embedded,* preprint, `arXiv:1412.0642`, 2014.

[Cau13] S. Caruso, *On the genericity of pseudo-Anosov braids* I*: rigid braids*, preprint, 2013.

[CWi13] S. Caruso, B. Wiest, *On the genericity of pseudo-Anosov braids* II*: conjugations to rigid braids*, preprint, 2013.

[CJ94] A. Casson, D. Jungreis, *Convergence groups and Seifert fibered* 3*-manifolds*, Invent. Math. **118** (1994), no. 3, 441–456.

[Cas04] F. Castel, *Centralisateurs dans les groupes à dualité de Poincaré de dimension* 3, C. R. Math. Acad. Sci. Paris **338** (2004), no. 12, 935–940.

[Cas07] ———, *Centralisateurs d'éléments dans les* PD(3)*-paires*, Comment. Math. Helv. **82** (2007), no. 3, 499–517.

[Cav12] W. Cavendish, *Finite-sheeted covering spaces and solenoids over* 3*-manifolds*, PhD thesis, Princeton University, 2012.

[Cav14] ———, *On finite derived quotients of* 3*-manifold groups*, preprint, 2014.

[CSC14] T. Ceccherini–Silberstein, M. Coornaert, *Cellular Automata and Groups*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2010.

[dCe09] L. Di Cerbo, *A gap property for the growth of closed* 3*-manifold groups*, Geom. Dedicata **143** (2009), 193–199.

[Ce68] J. Cerf, *Sur les Difféomorphismes de la Sphère de Dimension Trois* ($\Gamma_4 = 0$), Lecture Notes in Mathematics, vol. 53, Springer-Verlag, Berlin-New York, 1968.

[CdSR79] E. César de Sá, C. Rourke, *The homotopy type of homeomorphisms of* 3*-manifolds*, Bull. Amer. Math. Soc. (N.S.) **1** (1979), no. 1, 251–254.

[ChO13] J. Cha, K. Orr, *Hidden torsion,* 3*-manifolds, and homology cobordism*, J. Topology **6** (2013), no. 2, 490–512.

[ChZ10] S. C. Chagas, P. A. Zalesskii, *Bianchi groups are conjugacy separable*, J. Pure and Applied Algebra **214** (2010), no. 9, 1696–1700.

[Cha07] R. Charney, *An introduction to right-angled Artin groups*, Geom. Dedicata **125** (2007), 141–158.

[Chp74] T. A. Chapman, *Topological invariance of Whitehead torsion*, Amer. J. Math. **96** (1974), 488–497.

[ChS74] S. Chern, J. Simons, *Characteristic forms and geometric invariants*, Ann. of Math. (2) **99** (1974), 48–69.

[CDW12] E. Chesebro, J. DeBlois, H. Wilton, *Some virtually special hyperbolic* 3*-manifold groups*, Comment. Math. Helv. **87** (2012), no. 3, 727–787.

[ChT07] E. Chesebro, S. Tillmann, *Not all boundary slopes are strongly detected by the character variety*, Comm. Anal. Geom. **15** (2007), no. 4, 695–723.

[Chi83] T. Chinburg, *Volumes of hyperbolic manifolds*, J. Differential Geom. **18** (1983), no. 4, 783–789.

[CGW14] I. M. Chiswell, A. M. W. Glass, J. S. Wilson, *Residual nilpotence and ordering in one-relator groups and knot groups*, preprint, 2014.

[Cho06] S. Choi, *The PL-methods for hyperbolic* 3*-manifolds to prove tameness*, preprint, `math/0602520`, 2006.

[ChE80] S. Chen, P. Eberlein, *Isometry groups of simply connected manifolds of nonpositive curvature*, Illinois J. Math. **24** (1980), no. 1, 73–103.

[CV77] C. Christenson, W. Voxman, *Aspects of Topology*, Pure and Applied Mathematics, vol. 39, Marcel Dekker, Inc., New York-Basel, 1977.

[CGHV14] K. Christianson, J. Goluboff, L. Hamann, S. Varadaraj, *Non-left-orderable surgeries on twisted torus knots*, preprint, 2014.

[ClW03] B. Clair, K. Whyte, *Growth of Betti numbers*, Topology **42** (2003), no. 5, 1125–1142.

[ClS84] B. Clark, V. Schneider, *All knot groups are metric*, Math. Z. **187** (1984), no. 2, 269–271.

[CDN14] A. Clay, C. Desmarais, P. Naylor, *Testing bi-orderability of knot groups*, preprint, 2014.

[CLW13] A. Clay, T. Lidman, L. Watson, *Graph manifolds, left-orderability and amalgamation*, Algebr. Geom. Topol. **13** (2013), no. 4, 2347–2368.

[CR12] A. Clay, D. Rolfsen, *Ordered groups, eigenvalues, knots, surgery and L-spaces*, Math. Proc. Cambridge Philos. Soc. **152** (2012), no. 1, 115–129.

[ClT13] A. Clay, M. Teragaito, *Left-orderability and exceptional Dehn surgery on two-bridge knots*, in: *Geometry and Topology Down Under,* pp. 225–233, Contemporary Mathematics, vol. 597, Amer. Math. Soc., Providence, RI, 2013.

[CyW11] A. Clay, L. Watson, *On cabled knots, Dehn surgery, and left-orderable fundamental groups*, Math. Res. Lett. **18** (2011), no. 6, 1085–1095.

[CyW13] ———, *Left-orderable fundamental groups and Dehn surgery*, Int. Math. Res. Not. IMRN **2013**, no. 12, 2862–2890.

[Cl87] L. Clozel, *On the cuspidal cohomology of arithmetic subgroups of $SL(2n)$ and the first Betti number of arithmetic 3-manifolds*, Duke Math. J. **55** (1987), no. 2, 475–486.

[Coc04] T. Cochran, *Noncommutative knot theory*, Algebr. Geom. Topol. **4** (2004), 347–398.

[CMa06] T. Cochran, J. Masters, *The growth rate of the first Betti number in abelian covers of 3-manifolds*, Math. Proc. Cambridge Philos. Soc. **141** (2006), no. 3, 465–476.

[CoO98] T. Cochran, K. Orr, *Stability of lower central series of compact 3-manifold groups*, Topology **37** (1998), no. 3, 497–526.

[COT03] T. Cochran, K. Orr, P. Teichner, *Knot concordance, Whitney towers and L^2-signatures*, Ann. of Math. (2) **157** (2003), no. 2, 433–519.

[Coh73] M. Cohen, *A course in simple-homotopy theory*, Graduate Texts in Mathematics, vol. 10, Springer-Verlag, New York-Berlin, 1973.

[CMi77] D. J. Collins, C. F. Miller, *The conjugacy problem and subgroups of finite index*, Proc. London Math. Soc. (3) **34** (1977), no. 3, 535–556.

[CZi93] D. J. Collins, H. Zieschang, *Combinatorial group theory and fundamental groups*, in: *Algebra,* VII, pp. 1–166, 233–240, Encyclopaedia of Mathematical Sciences, vol. 58, Springer-Verlag, Berlin, 1993.

[Con70] A. C. Conner, *An algebraic characterization of 3-manifolds*, Notices Amer. Math. Soc. **17** (1970), 266, abstract # 672–635.

[CoS03] J. Conway, D. Smith, *On Quaternions and Octonions: their Geometry, Arithmetic, and Symmetry*, A K Peters, Ltd., Natick, MA, 2003.

[CoG12] D. Cooper, W. Goldman, *A 3-manifold with no real projective structure*, preprint, 2012.

[CHK00] D. Cooper, C. Hodgson, S. Kerckkoff, *Three-dimensional Orbifolds and Cone-Manifolds*, MSJ Memoirs, vol. 5, Mathematical Society of Japan, Tokyo, 2000.

[CoL99] ———, *Virtually Haken Dehn-filling*, J. Differential Geom. **52** (1999), no. 1, 173–187.

[CoL00] ———, *Free actions of finite groups on rational homology 3-spheres*, Topology Appl. **101** (2000), no. 2, 143–148.

[CoL01] ———, *Some surface subgroups survive surgery*, Geom. Topol. **5** (2001), 347–367.

[CLR94] D. Cooper, D. Long, A. W. Reid, *Bundles and finite foliations*, Invent. Math. **118** (1994), 253–288.

[CLR97] ———, *Essential surfaces in bounded 3-manifolds*, J. Amer. Math. Soc. **10** (1997), 553–564.

[CLR07] ———, *On the virtual Betti numbers of arithmetic hyperbolic 3-manifolds*, Geom. Topol. **11** (2007), 2265–2276.

[CLT06] D. Cooper, D. Long, M. Thistlethwaite, *Computing varieties of representations of hyperbolic 3-manifolds into* $\mathrm{SL}(4,\mathbb{R})$, Experiment. Math. **15** (2006), no. 3, 291–305.

[CLT07] ———, *Flexing closed hyperbolic manifolds*, Geom. Topol. **11** (2007), 2413–2440.

[CLT09] ———, *Constructing non-congruence subgroups of flexible hyperbolic 3-manifold groups*, Proc. Amer. Math. Soc. **137** (2009), no. 11, 3943–3949.

[CoM11] D. Cooper, J. Manning, *Non-faithful representations of surface groups into $SL(2,\mathbb{C})$ which kill no simple closed curve*, preprint, 2011.

[CoW06a] D. Cooper, G. Walsh, *Three-manifolds, virtual homology, and group determinants*, Geom. Topol. **10** (2006), 2247–2269.

[CoW06b] ———, *Virtually Haken fillings and semi-bundles*, Geom. Topol. **10** (2006), 2237–2245.

[Cor13] C. Cornwell, *Knot contact homology and representations of knot groups*, J. Topology, to appear.

[CH14] C. Cornwell, D. Hemminger, *Augmentation rank of satellites with braid pattern*, preprint, 2014.

[CwS78] R. C. Cowsik, G. A. Swarup, *A remark on infinite cyclic covers*, J. Pure Appl. Algebra **11** (1977/78), no. 1–3, 131–138.

[Cr00] J. Crisp, *The decomposition of 3-dimensional Poincaré complexes*, Comment. Math. Helv. **75** (2000), no. 2, 232–246.

[Cr07] ———, *An algebraic loop theorem and the decomposition of PD^3-pairs*, Bull. London Math. Soc. **39** (2007), no. 1, 46–52.

[CrW04] J. Crisp, B. Wiest, *Embeddings of graph braid and surface groups in right-angled Artin groups and braid groups*, Alg. Geom. Top. **4** (2004), 439–472.

[Cu86] M. Culler, *Lifting representations to covering groups*, Adv. in Math. **59** (1986), no. 1, 64–70.

[CDS09] M. Culler, J. Deblois, P. Shalen, *Incompressible surfaces, hyperbolic volume, Heegaard genus and homology*, Comm. Anal. Geom. **17** (2009), no. 2, 155–184.

[CGLS85] M. Culler, C. Gordon, J. Luecke, P. Shalen, *Dehn surgery on knots*, Bull. Amer. Math. Soc. (N.S.) **13** (1985), no. 1, 43–45.

[CGLS87] ———, *Dehn surgery on knots*, Ann. of Math. (2) **125** (1987), 237–300.

[CJR82] M. Culler, W. Jaco, H. Rubinstein, *Incompressible surfaces in once-punctured torus bundles*, Proc. London Math. Soc. (3) **45** (1982), no. 3, 385–419.

[CuS83] M. Culler, P. Shalen, *Varieties of group representations and splittings of* 3*-manifolds*, Ann. of Math. (2) **117** (1983), no. 1, 109–146.

[CuS84] ———, *Bounded, separating, incompressible surfaces in knot manifolds*, Invent. Math. **75** (1984), no. 3, 537–545.

[CuS08] ———, *Volume and homology of one-cusped hyperbolic* 3*-manifolds*, Algebr. Geom. Topol. **8** (2008), no. 1, 343–379.

[CuS11] ———, *Singular surfaces, mod* 2 *homology, and hyperbolic volume*, II, Topology Appl. **158** (2011), no. 1, 118–131.

[CuS12] ———, 4*-free groups and hyperbolic geometry*, J. Topol. **5** (2012), no. 1, 81–136.

[DPT05] M. Dabkowski, J. Przytycki, A. Togha, *Non-left-orderable* 3*-manifold groups*, Canad. Math. Bull. **48** (2005), no. 1, 32–40.

[Dah03] F. Dahmani, *Combination of convergence groups*, Geom. Topol. **7** (2003), 933–963.

[Dam91] R. Daverman, 3*-manifolds with geometric structure and approximate fibrations*, Indiana Univ. Math. J. **40** (1991), no. 4, 1451–1469.

[Dan96] K. Davidson, *C^*-algebras by Example*, Fields Institute Monographs, vol. 6, Amer. Math. Soc., Providence, RI, 1996.

[Dava83] J. Davis, *The surgery semicharacteristic*, Proc. London Math. Soc. (3) **47** (1983), no. 3, 411–428.

[DJ00] M. W. Davis, T. Januszkiewicz, *Right-angled Artin groups are commensurable with right-angled Coxeter groups*, J. Pure Appl. Algebra **153** (2000), 229–235.

[Davb98] M. Davis, *The cohomology of a Coxeter group with group ring coefficients*, Duke Math. J. **91** (1998), 297–313.

[Davb00] ———, *Poincaré duality groups*, in: *Surveys on Surgery Theory,* vol. 1, pp. 167–193, Annals of Mathematics Studies, vol. 145, Princeton University Press, Princeton, NJ, 2000.

[DeB10] J. DeBlois, *On the doubled tetrus*, Geom. Dedicata **144** (2010), 1–23.

[DeB13] ———, *Explicit rank bounds for cyclic covers*, preprint, 2013.

[DFV14] J. Deblois, S. Friedl, S. Vidussi, *The rank gradient for infinite cyclic covers of* 3*-manifolds*, Michigan Math. J. **63** (2014), 65–81.

[DeS09] J. DeBlois, P. Shalen, *Volume and topology of bounded and closed hyperbolic* 3*-manifolds*, Comm. Anal. Geom. **17** (2009), no. 5, 797–849.

[De10] M. Dehn, *Über die Topologie des dreidimensionalen Raumes*, Math. Ann. **69** (1910), no. 1, 137–168.

[De11] ———, *Über unendliche diskontinuierliche Gruppen*, Math. Ann. **71** (1911), 116–144.

[De87] ———, *Papers on Group Theory and Topology*, Springer-Verlag, New York, 1987.

[DS14] W. Dicks, Z. Sunić, *Orders on trees and free products of left-ordered groups*, preprint, 2014.

[DPS11] A. Dimca, S. Papadima, A. Suciu, *Quasi-Kähler groups,* 3*-manifold groups, and formality*, Math. Z. **268** (2011), no. 1-2, 169–186.

[DiS09] A. Dimca, A. Suciu, *Which* 3*-manifold groups are Kähler groups?*, J. Eur. Math. Soc. **11** (2009), no. 3, 521–528.

[DL09] J. Dinkelbach, B. Leeb, *Equivariant Ricci flow with surgery and applications to finite group actions on geometric* 3*-manifolds*, Geom. Topol. **13** (2009), no. 2, 1129–1173.

[Di77] J. Dieudonné, *Treatise on Analysis,* Volume V, Pure and Applied Mathematics, vol. 10, Academic Press, New York-London, 1977.

[DFPR82] J. Dixon, E. Formanek, J. Poland, L. Ribes, *Profinite completions and finite quotients*, J. Pure Appl. Algebra **23** (1982), 22–231.

[DLMSY03] J. Dodziuk, P. Linnell, V. Mathai, T. Schick, S. Yates, *Approximating L^2-invariants and the Atiyah conjecture*, Comm. Pure Appl. Math. **56** (2003), no. 7, 839–873.

[Doi12] M. Doig, *Finite knot surgeries and Heegaard Floer homology*, preprint, 2013.

[Don83] S. K. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Differential Geom. **18** (1983), 279–315.

[DJRZ13] F. Doria Medina, M. Jackson, J. Ruales, H. Zeilberger, *On left-orderability and double branched covers of Kanenobu's knots*, preprint, 2013.

[DKL12] S. Dowdall, I. Kapovich, C. Leininger, *Dynamics on free-by-cyclic groups*, preprint, 2012.

[DKL13] ———, *McMullen polynomials and Lipschitz flows for free-by-cyclic groups*, preprint, 2013.

[Dr83] C. Droms, *Graph Groups*, Ph.D. thesis, Syracuse University, 1983.

[Dr87] ———, *Graph groups, coherence, and three-manifolds*, J. Algebra **106** (1987), no. 2, 484–489.

[DSS89] C. Droms, B. Servatius, H. Servatius, *Surface subgroups of graph groups*, Proc. Amer. Math. Soc. **106** (1989), 573–578.

[DuV64] P. Du Val, *Homographies, Quaternions and Rotations*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.

[DK92] G. Duchamp, D. Krob, *The lower central series of the free partially commutative group*, Semigroup Forum **45** (1992), no. 3, 385–394.

[DpT92] G. Duchamp, J.-Y. Thibon, *Simple orderings for free partially commutative groups*, Internat. J. Algebra Comput. **2** (1992), no. 3, 351–355.

[Duf12] G. Dufour, *Cubulations de variétés hyperboliques compactes*, thèse de doctorat, 2012.

[Dub88a] W. Dunbar, *Classification of solvorbifolds in dimension three*, I, in: *Braids,* pp. 207–216, Contemporary Mathematics, vol. 78, Amer. Math. Soc., Providence, RI, 1988.

[Dub88b] ———, *Geometric orbifolds*, Rev. Mat. Univ. Complut. Madrid **1** (1988), no. 1-3, 67–99.

[Dun01] N. Dunfield, *Alexander and Thurston norms of fibered 3-manifolds*, Pacific J. Math. **200** (2001), no. 1, 43–58.

[Dun14] ———, *What does a random 3-manifold look like?*, talk at Combinatorial Link Homology Theories, Braids, and Contact Geometry, ICERM, August 2014. `http://www.math.uiuc.edu/~nmd/preprints/slides/icerm-prob.pdf`

[DFJ12] N. Dunfield, S. Friedl, N. Jackson, *Twisted Alexander polynomials of hyperbolic knots*, Exp. Math. **21** (2012), 329–352.

[DG12] N. Dunfield, S. Garoufalidis, *Incompressibility criteria for spun-normal surfaces*, Trans. Amer. Math. Soc. **364** (2012), no. 11, 6109–6137.

[DKR14] . N. Dunfield, S. Kionke, J. Raimbault, *On geometric aspects of diffuse groups*, preprint, 2014.

[DR10] N. Dunfield, D. Ramakrishnan, *Increasing the number of fibered faces of arithmetic hyperbolic 3-manifolds*, Amer. J. Math. **132** (2010), no. 1, 53–97.

[DnTa06] N. Dunfield, D. Thurston, *A random tunnel number one 3-manifold does not fiber over the circle*, Geom. Topol. **10** (2006), 2431–2499.

[DnTb03] N. Dunfield, W. Thurston, *The virtual Haken conjecture: experiments and examples*, Geom. Topol. **7** (2003), 399–441.

[DnTb06] ———, *Finite covers of random 3-manifolds*, Invent. Math. **166** (2006), no. 3, 457–521.

[Duw85] M. J. Dunwoody, *An equivariant sphere theorem*, Bull. London Math. Soc. **17** (1985), no. 5, 437–448.

[DuS00] M. J. Dunwoody, E. L. Swenson, *The algebraic torus theorem*, Invent. Math. **140** (2000), 605–637.

[ELL13] Q. E, F. Lei, F. Li, *A proof of Lickorish and Wallace's theorem*, preprint, 2013.

[Ea06] V. Easson, *Surface subgroups and handlebody attachment*, Geom. Topol. **10** (2006), 557–591.

[Eb82] P. Eberlein, *A canonical form for compact nonpositively curved manifolds whose fundamental groups have nontrivial center*, Math. Ann. **260** (1982), no. 1, 23–29.

[Ecr08] K. Ecker, *Heat equations in geometry and topology*, Jahresber. Deutsch. Math.-Verein. **110** (2008), no. 3, 117–141.

[Ecn94] B. Eckmann, *Sur les groupes fondamentaux des surfaces closes*, Riv. Mat. Univ. Parma (4) **10** (1984), special vol. 10*, 41–46.

[Ecn95] ———, *Surface groups and Poincaré duality*, in: *Conference on Algebraic Topology in Honor of Peter Hilton,* pp. 51–59, Contemporary Mathematics, vol. 37, Amer. Math. Soc., Providence, RI, 1985.

[Ecn97] ———, *Poincaré duality groups of dimension* 2 *are surface groups*, in: *Combinatorial Group Theory and Topology,* pp. 35–52, Annals of Mathematics Studies, vol. 111, Princeton University Press, Princeton, NJ, 1987.

[Ecn00] ———, *Introduction to l_2-methods in topology: Reduced l_2-homology, harmonic chains, l_2-Betti numbers*, Isr. J. Math. **117** (2000), 183–219.

[EcL83] B. Eckmann, P. Linnell, *Poincaré duality groups of dimension two*, II, Comment. Math. Helv. **58** (1983), no. 1, 111–114.

[EcM80] B. Eckmann, H. Müller, *Poincaré duality groups of dimension two*, Comm. Math. Helv. **55** (1980), 510–520.

[Ed86] A. Edmonds, *A topological proof of the equivariant Dehn lemma*, Trans. Amer. Math. Soc. **297** (1986), no. 2, 605–615.

[EdL83] A. Edmonds, C. Livingston, *Group actions on fibered three-manifolds*, Comment. Math. Helv. **58** (1983), no. 4, 529–542.

[EN85] D. Eisenbud, W. Neumann, *Three-dimensional Link Theory and Invariants of Plane Curve Singularities*, Annals of Mathematics Studies, vol. 110, Princeton University Press, Princeton, NJ, 1985.

[Eim00] M. Eisermann, *The number of knot group representations is not a Vassiliev invariant*, Proc. Amer. Math. Soc. **128** (2000), no. 5, 1555–1561.

[Eim07] ———, *Knot colouring polynomials*, Pacific J. Math. **231** (2007), 305–336.

[Ein76a] J. Eisner, *A characterisation of nonfibered knots*, Notices Amer. Math. Soc. **23** (2) (1976).

[Ein76b] ———, *Notions of spanning surface equivalence*, Proc. Amer. Math. Soc. **56** (1976), 345–348.

[Ein77a] ———, *A characterisation of non-fibered knots*, Michigan Math. J. **24** (1977), 41–44.

[Ein77b] ———, *Knots with infinitely many minimal spanning surfaces*, Trans. Amer. Math. Soc. **229** (1977), 329–349.

[El84] H. Elkalla, *Subnormal subgroups in* 3*-manifold groups*, J. London Math. Soc. (2) **30** (1984), no. 2, 342–360.

[Epp99] M. Epple, *Geometric aspects in the development of knot theory*, in: *History of Topology,* pp. 301–357, North-Holland, Amsterdam, 1999.

[Eps61a] D. B. A. Epstein, *Finite presentations of groups and* 3*-manifolds*, Quart. J. Math. Oxford Ser. (2) **12** (1961), 205–212.

[Eps61b] ———, *Projective planes in* 3*-manifolds*, Proc. London Math. Soc. (3) **11** (1961), 469–484.

[Eps61c] ———, *Free products with amalgamation and* 3*-manifolds*, Proc. Amer. Math. Soc. **12** (1961), 669–670.

[Eps61d] ———, *Factorization of* 3*-manifolds*, Comment. Math. Helv. **36** (1961), 91–102.

[Eps62] ———, *Ends*, in: *Topology of* 3*-Manifolds and Related Topics,*, pp. 110–117, Prentice-Hall, Englewood Cliffs, NJ, 1962.

[Eps72] ———, *Periodic flows on* 3*-manifolds*, Ann. of Math. (2) **95** (1972), 66–82.

[ECHLPT92] D B. A. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson, W. P. Thurston, *Word Processing in Groups*, Jones and Bartlett Publishers, Boston, MA, 1992.

[EL14] M. Ershov, W. Lück, *The first L^2-Betti number and approximation in arbitrary characteristic*, Doc. Math. **19** (2014), 313–332.

[EJ73] B. Evans, W. Jaco, *Varieties of groups and three-manifolds,* Topology **12** (1973), 83–97.

[EvM72] B. Evans, L. Moser, *Solvable fundamental groups of compact* 3*-manifolds*, Trans. Amer. Math. Soc. **168** (1972), 189–210.

[FaM12] B. Farb, D. Margalit, *A Primer on Mapping Class Groups*, Princeton Mathematical Series, vol. 49, Princeton University Press, Princeton, NJ, 2012.

[Far98] M. Farber, *Geometry of growth: approximation theorems for L^2 invariants*, Math. Ann. **311** (1998), no. 2, 335–375.

[Fak75] D. Farkas, *Miscellany on Bieberbach group algebras*, Pac. J. Math. **59** (1975), 427–435.

[FaH81] F. Farrell, W.-C. Hsiang, *The Whitehead group of poly-(finite or cyclic) groups*, J. London Math. Soc. **24** (1981), 308–324.

[FJ86] F. Farrell, L. Jones, *K-theory and dynamics*, I, Ann. of Math. (2) **124** (1986), 531–569.

[FJ87] ———, *Implication of the geometrization conjecture for the algebraic K-theory of 3-manifolds*, in: *Geometry and Topology*, pp. 109–113, Lecture Notes in Pure and Applied Mathematics, vol. 105, Marcel Dekker, Inc., New York, 1987.

[FLP79a] A. Fathi, F. Laudenbach, V. Poénaru, *Travaux de Thurston sur les surfaces*, Astérisque, 66-67, Soc. Math. France, Paris, 1979.

[FLP79b] ———, *Thurston's work on surfaces*, `https://wikis.uit.tufts.edu/confluence/display/~dmarga01/FLP`

[Fei86] M. Feighn, *Branched covers according to J. W. Alexander*, Collect. Math. **37** (1986), no. 1, 55-60.

[Feu70] C. D. Feustel, *Some applications of Waldhausen's results on irreducible surfaces*, Trans. Amer. Math. Soc. **149** (1970), 575–583.

[Feu72a] ———, *A splitting theorem for closed orientable* 3*-manifolds*, Topology **11** (1972), 151–158.

[Feu72b] ———, *S-maximal subgroups of* $\pi_1(M^3)$, Canad. J. Math. **24** (1972), 439–449.

[Feu73] ———, *A generalization of Kneser's conjecture*, Pacific J. Math. **46** (1973), 123–130.

[Feu76a] ———, *On the Torus Theorem and its applications*, Trans. Amer. Math. Soc. **217** (1976), 1–43.

[Feu76b] ———, *On the Torus Theorem for closed* 3*-manifolds*, Trans. Amer. Math. Soc. **217** (1976), 45–57.

[Feu76c] ———, *On realizing centralizers of certain elements in the fundamental group of a* 3*-manifold*, Proc. Amer. Math. Soc. **55** (1976), no. 1, 213–216.

[FeG73] C. D. Feustel, R. J. Gregorac, *On realizing HNN groups in* 3*-manifolds*, Pacific J. Math. **46** (1973), 381–387.

[FeW78] C. D. Feustel, W. Whitten, *Groups and complements of knots*, Canad. J. Math. **30** (1978), no. 6, 1284–1295.

[FiM99] E. Finkelstein, Y. Moriah, *Tubed incompressible surfaces in knot and link complements*, Topology Appl. **96** (1999), no. 2, 153–170.

[FiM00] ———, *Closed incompressible surfaces in knot complements*, Trans. Amer. Math. Soc. **352** (2000), no. 2, 655–677.

[FlH82] W. Floyd, A. Hatcher, *Incompressible surfaces in punctured-torus bundles*, Topology Appl. **13**, no. 3, 263–282.

[FoM10] F. Fong, J. Morgan, *Ricci Flow and Geometrization of* 3*-Manifolds*, University Lecture Series, vol. 53, Amer. Math. Soc., Providence, RI, 2010.

[FR12] B. Foozwell, H. Rubinstein, *Four-dimensional Haken cobordism theory*, preprint, 2012.

[Fo52] R. H. Fox, *Recent development of knot theory at Princeton*, in: *Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950,* vol. 2, pp. 453–457, Amer. Math. Soc., Providence, RI, 1952.

[FW99] G. Francis, J. Weeks, *Conway's ZIP proof*, Amer. Math. Monthly **106** (1999), no. 5, 393–399.

[FF98] B. Freedman, M. Freedman, *Kneser-Haken finiteness for bounded* 3*-manifolds locally free groups, and cyclic covers*, Topology **37** (1998), no. 1, 133–147.

[Fre82] M. H. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom. **17** (1982), no. 3, 357–453.

[Fre84] ———, *The disk theorem for four-dimensional manifolds*, in: *Proceedings of the International Congress of Mathematicians,* vol. 1, 2, pp. 647–663, PWN—Polish Scientific Publishers, Warsaw; North-Holland Publishing Co., Amsterdam, 1984.

[FrG07] M. H. Freedman, D. Gabai, *Covering a nontaming knot by the unlink*, Algebr. Geom. Topol. **7** (2007), 1561–1578.

[FHT97] M. Freedman, R. Hain, P. Teichner, *Betti number estimates for nilpotent groups*, in: *Fields Medallists' Lectures,* pp. 413–434, World Scientific Series in 20th Century Mathematics, vol. 5, World Scientific Publishing Co., Inc., River Edge, NJ; Singapore University Press, Singapore, 1997.

[Fri07] S. Friedl, *Reidemeister torsion, the Thurston norm and Harvey's invariants*, Pac. J. Math. **230** (2007), 271–296.

[Fri11] ———, *Centralizers in 3-manifold groups*, RIMS Kôkyûroku **1747** (2011), 23–34.

[Fri13] ———, *Commensurability of knots and L^2 invariants*, in: *Geometry and Topology Down Under,* pp. 263–279, Contemporary Mathematics, vol. 597, American Mathematical Society, Providence, RI, 2013.

[Fri14] ———, *A note on the growth of Betti numbers and ranks of 3-manifold groups*, Bull. Sci. Math. **138** (2014), no. 1, 63–70.

[FJR11] S. Friedl, A. Juhász, J. Rasmussen, *The decategorification of sutured Floer homology*, J. Topology **4** (2011), 431–478.

[FKm06] S. Friedl, T. Kim, *The Thurston norm, fibered manifolds and twisted Alexander polynomials*, Topology **45** (2006), 929–953.

[FKt14] S. Friedl, T. Kitayama, *The virtual fibering theorem for 3-manifolds*, Enseign. Math. **60** (2014), no. 1, 79–107.

[FN14] S. Friedl, M. Nagel, *3-manifolds that can be made acyclic*, preprint, 2014.

[FST15] S. Friedl, K. Schreve, S. Tillmann, *Thurston norm via Fox calculus*, preprint, 2015.

[FSW13] S. Friedl, D. Silver, S. Williams, *Splittings of knot groups*, Math. Ann., to appear.

[FS12] S. Friedl, A. Suciu, *Kähler groups, quasi-projective groups, and 3-manifold groups*, J. London Math. Soc. (2) **89** (2014), no. 1, 151–168.

[FTe05] S. Friedl, P. Teichner, *New topologically slice knots*, Geom. Topol. **9** (2005), 2129–2158.

[FTi15] S. Friedl, S. Tillmann, *Two-generator one-relator groups and marked polytopes*, preprint, 2015.

[FV07] S. Friedl, S. Vidussi, *Nontrivial Alexander polynomials of knots and links*, Bull. Lond. Math. Soc. **39** (2007), 614–622.

[FV12] ———, *The Thurston norm and twisted Alexander polynomials*, J. Reine Angew. Math., to appear.

[FV13] ———, *Twisted Alexander invariants detect trivial links*, preprint, 2013.

[FV14] ———, *A vanishing theorem for twisted Alexander polynomials with applications to symplectic 4-manifolds*, J. Eur. Math. Soc. **15** (2013), no. 6, 2127–2041.

[FrW13] S. Friedl, H. Wilton, *The membership problem for 3-manifold groups*, preprint, 2013.

[Frg04] R. Frigerio, *Hyperbolic manifolds with geodesic boundary which are determined by their fundamental group*, Topology Appl. **145** (2004), no. 1–3, 69–81.

[Fuj99] K. Fujiwara, *3-manifold groups and property T of Kazhdan*, Proc. Japan Acad. Ser. A Math. Sci. **75** (1999), no. 7, 103–104.

[Fun13] L. Funar, *Torus bundles not distinguished by TQFT invariants*, Geom. Top. **17** (2013), 2289–2344.

[FP07] D. Futer, J. Purcell, *Links with no exceptional surgeries*, Comment. Math. Helv. **82** (2007), no. 3, 629–664.

[Gab83a] D. Gabai, *Foliations and the topology of 3-manifolds*, J. Differential Geometry **18** (1983), no. 3, 445–503.

[Gab83b] ———, *Foliations and the topology of 3-manifolds*, Bull. Amer. Math. Soc. **8** (1983), 77–80.

[Gab85] ———, *The simple loop conjecture*, J. Differential Geom. **21** (1985), no. 1, 143–149.

[Gab86] ———, *On 3-manifolds finitely covered by surface bundles*, in: *Low-dimensional Topology and Kleinian Groups*, pp. 145–155, London Mathematical Society Lecture Note Series, vol. 112. Cambridge University Press, Cambridge, 1986.

[Gab87] ———, *Foliations and the topology of 3-manifolds*, III, J. Differential Geom. **26** (1987), no. 3, 479–536.

[Gab92] ———, *Convergence groups are Fuchsian groups*, Ann. of Math. (2) **136** (1992), no. 3, 447–510.

[Gab94a] ———, *Homotopy hyperbolic 3-manifolds are virtually hyperbolic*, J. Amer. Math. Soc. **7** (1994), no. 1, 193–198.

[Gab94b] ———, *On the geometric and topological rigidity of hyperbolic 3-manifolds*, Bull. Amer. Math. Soc. (N.S.) **31** (1994), no. 2, 228–232.

[Gab97] ———, *On the geometric and topological rigidity of hyperbolic 3-manifolds*, J. Amer. Math. Soc. **10** (1997), no. 1, 37–74.

[Gab01] ———, *The Smale conjecture for hyperbolic 3-manifolds:* $\mathrm{Isom}(M^3) \simeq \mathrm{Diff}(M^3)$, J. Differential Geom. **58** (2001), no. 1, 113–149.

[Gab09] ———, *Hyperbolic geometry and* 3*-manifold topology*, in: *Low Dimensional Topology,* pp. 73–103, IAS/Park City Mathematics Series, vol. 15, Amer. Math. Soc., Providence, RI, 2009.

[GMM09] D. Gabai, R. Meyerhoff, P. Milley, *Minimum volume cusped hyperbolic three-manifolds*, J. Amer. Math. Soc. **22** (2009), no. 4, 1157–1215.

[GMM10] ———, *Mom technology and hyperbolic* 3*-manifolds,* in: *In the Tradition of Ahlfors-Bers,* V, pp. 84–107, Contemporary Mathematics, vol. 510, American Mathematical Society, Providence, RI, 2010.

[GMT03] D. Gabai, G. Meyerhoff, N. Thurston, *Homotopy hyperbolic* 3*-manifolds are hyperbolic*, Ann. of Math. (2) **157** (2003), no. 2, 335–431.

[Gar11] S. Garoufalidis, *The Jones slopes of a knot*, Quantum Topol. **2** (2011), no. 1, 43–69.

[Gei08] H. Geiges, *An Introduction to Contact Topology*, Cambridge Studies in Advanced Mathematics, vol. 109, Cambridge University Press, Cambridge, 2008.

[Ger83] S. M. Gersten, *Geometric automorphisms of a free group of rank at least three are rare*, Proc. Amer. Math. Soc. **89** (1983), no. 1, 27–31.

[Ger94] ———, *Divergence in* 3*-manifold groups*, Geom. Funct. Anal. **4** (1994), no. 6, 633–647.

[Gie12] H. Gieseking, *Analytische Untersuchungen über topologische Gruppen*, Ph.D. Thesis, Münster, 1912.

[Gll99] C. P. Gill, *Growth series of stem products of cyclic groups*, Int. J. Algebra Comput. **9** (1999), no. 1, 1–30.

[Gin81] J. Gilman, *On the Nielsen type and the classification for the mapping class group*, Adv. Math. **40** (1981), no. 1, 68–96.

[Gil09] P. Gilmer, *Heegaard genus, cut number, weak p-congruence, and quantum invariants*, J. Knot Theory Ramifications **18** (2009), no. 10, 1359–1368.

[GiM07] P. Gilmer, G. Masbaum, *Integral lattices in TQFT*, Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 5, 815–844.

[Gir13] D. Girão, *Rank gradient of small covers*, Pacific J. Math. **266** (2013), no. 1, 23–29.

[Gir14] ———, *Rank gradient in co-final towers of certain Kleinian groups*, Groups Geom. Dyn. **8** (2014), no. 1, 143—155.

[Git97] R. Gitik, *Graphs and separability properties of groups*, J. Algebra **188** (1997), no. 1, 125–143.

[Git99a] ———, *Ping-pong on negatively curved groups*, J. Algebra **217** (1999), no. 1, 65–72.

[Git99b] ———, *Doubles of groups and hyperbolic LERF* 3*-manifolds*, Ann. of Math. (2) **150** (1999), 775–806.

[GMRS98] R. Gitik, M. Mitra, E. Rips, M. Sageev, *Widths of subgroups*, Trans. Amer. Math. Soc. **350** (1998), no. 1, 321–329.

[GR95] R. Gitik, E. Rips, *On separability properties of groups*, Internat. J. Algebra Comput. **5** (1995), no. 6, 703–717.

[GR13] ———, *On double cosets in free groups*, manuscript, `arXiv:1306.0033,` 2013.

[GSS10] Y. Glasner, J. Souto, P. Storm, *Finitely generated subgroups of lattices in* $\mathrm{PSL}(2,\mathbb{C})$, Proc. Amer. Math. Soc. **138** (2010), no. 8, 2667–2676.

[GI06] H. Goda, M. Ishiwata, *A classification of Seifert surfaces for some pretzel links*, Kobe J. Math. **23** (2006), no. 1-2, 11–28.

[Gom95] R. Gompf, *A new construction of symplectic manifolds*, Ann. of Math. (2) **142** (1995), 527–595.

[GfS99] R. Gompf, A. Stipsicz, 4*-manifolds and Kirby calculus*, Graduate Studies in Mathematics, vol. 20, American Mathematical Society, Providence, RI, 1999.

[GA75] F. González-Acuña, *Homomorphs of knot groups*, Ann. of Math. (2) **102** (1975), no. 2, 373–377.

[GLW94] F. González-Acuña, R. Litherland, W. Whitten, *Cohopficity of Seifert-bundle groups*, Trans. Amer. Math. Soc. **341** (1994), no. 1, 143–155.

[GoS91] F. González-Acuña, H. Short, *Cyclic branched coverings of knots and homology spheres*, Rev. Mat. Univ. Complut. Madrid **4** (1991), no. 1, 97–120.

[GW87] F. González-Acuña, W. Whitten, *Imbeddings of knot groups in knot groups*, in: *Geometry and Topology*, pp. 147–156, Lecture Notes in Pure and Applied Mathematics, vol. 105, Marcel Dekker, Inc., New York, 1987.

[GW92] F. González-Acuña, W. Whitten, *Imbeddings of Three-Manifold Groups*, Mem. Amer. Math. Soc. **99** (1992), no. 474.

[GW94] ———, *Cohopficity of* 3*-manifold groups*, Topology Appl. **56** (1994), no. 1, 87–97.

[Gon72] C. McA. Gordon, *Knots whose branched covers have periodic homology*, Trans. Amer. Math. Soc. **168** (1972), 357–370.

[Gon98] ———, *Dehn filling: a survey,* in: *Knot Theory,* pp. 129–144, Banach Center Publications, vol. 42, Polish Academy of Sciences, Institute of Mathematics, Warsaw, 1998.

[Gon99] ———, 3*-dimensional topology up to 1960*, in: *History of Topology,* pp. 449–489, North-Holland, Amsterdam, 1999.

[Gon04] ———, *Artin groups,* 3*-manifolds and coherence*, Bol. Soc. Mat. Mex. III. Ser. 10, Spec. Iss. (2004), 193–198.

[GoH75] C. McA. Gordon, W. Heil, *Cyclic normal subgroups of fundamental groups of* 3*-manifolds*, Topology **14** (1975), 305–309.

[GLid14] C. McA. Gordon, T. Lidman, *Taut foliations, left-orderability, and cyclic branched covers*, preprint, 2014.

[GLit84] C. McA. Gordon, R. Litherland, *Incompressible surfaces in branched coverings*, in: *The Smith Conjecture,* pp. 139–152, Pure and Applied Mathematics, vol. 112, Academic Press, Inc., Orlando, FL, 1984.

[GLu89] C. McA. Gordon, J. Luecke, *Knots are determined by their complements*, J. Amer. Math. Soc. **2** (1989), no. 2, 371–415.

[Goa86] A. V. Goryaga, *Example of a finite extension of an FAC-group that is not an FAC-group*, Sibirsk. Mat. Zh. **27** (1986), no. 3, 203–205, 225.

[Gra71] A. Gramain, *Topologie des Surfaces*, Collection SUP: “Le Mathématicien”, vol. 7, Presses Universitaires de France, Paris, 1971.

[Gra77] ———, *Rapport sur la théorie classique des nœuds,* I, in: *Séminaire Bourbaki,* Volume 1975/1976, 28e année: Exposés Nos. 471–488, pp. 222–237, Lecture Notes in Mathematics, vol. 567, Springer-Verlag, Berlin-New York, 1977.

[Gra84] ———, *Topology of Surfaces*, BCS Associates, Moscow, ID, 1984.

[Gra92] ———, *Rapport sur la théorie classique des nœuds,* II, Séminaire Bourbaki, Vol. 1990/91, Astérisque no. 201-203 (1991), exp. no. 732, 89–113 (1992).

[Gre11] J. Greene, *Alternating links and left-orderability*, preprint, 2011.

[Gre13] ———, *Lattices, graphs and Conway mutation*, Invent. Math. **192** (2013), 717–750.

[Grv81a] M. Gromov, *Hyperbolic manifolds according to Thurston and Jørgensen*, in: *Séminaire Bourbaki,* vol. 1979/80, Exposés 543–560, pp. 40–53, Lecture Notes in Mathematics, vol. 842, Springer, Berlin, 1981.

[Grv81b] ———, *Hyperbolic manifolds, groups and actions*, in: *Riemann Surfaces and Related Topics,* pp. 183–213, Annals of Mathematics Studies, vol. 97, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1981.

[Grv82] ———, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. **56** (1982), 5–99.

[Grv87] ———, *Hyperbolic groups*, in: *Essays in Group Theory,* Mathematical Sciences Research Institute Publications, vol. 8, Springer-Verlag, New York, 1987.

[Grv89] ———, *Sur le groupe fondamental d'une variété kählérienne*, C. R. Acad. Sci. Paris Sér. I Math. **308** (1989), 67–70.

[Grv93] ———, *Asymptotic invariants of infinite groups*, in: *Geometric Group Theory,* vol. 2, pp. 1–295, London Mathematical Society Lecture Note Series, vol. 182, Cambridge University Press, Cambridge, 1993.

[GrL12] M. Gromov, L. Guth, *Generalizations of the Kolmogorov-Barzdin embedding estimates*, Duke Math. J. **161** (2012), no. 13, 2549–2603.

[Grs69] J. Gross, *A unique decomposition theorem for* 3*-manifolds with connected boundary*, Trans. Amer. Math. Soc. **142** (1969), 191–199.

[Grs70] ———, *The decomposition of* 3*-manifolds with several boundary components*, Trans. Amer. Math. Soc. **147** (1970), 561–572.

[Grk70] A. Grothendieck, *Représentations linéaires et compactification profinie des groupes discrets*, Manuscripta Math. **2** (1970), 375–396.

[GrM08] D. Groves, J. F. Manning, *Dehn filling in relatively hyperbolic groups*, Israel J. Math. **168** (2008), 317–429.

[GMW12] D. Groves, J. F. Manning, H. Wilton, *Recognizing geometric 3-manifold groups using the word problem*, preprint, 2012.

[Gru57] K. Gruenberg, *Residual properties of infinite soluble groups*, Proc. London Math. Soc. (3) **7** (1957), 29–62.

[GJZ08] F. Grunewald, A. Jaikin-Zapirain, P. Zalesskii, *Cohomological goodness and the profinite completion of Bianchi groups*, Duke Math. J. **144** (2008), no. 1, 53–72.

[GPS80] F. Grunewald, P. F. Pickel, D. Segal, *Polycyclic groups with isomorphic finite quotients*, Ann. of Math. (2) **111** (1980), no. 1, 155–195.

[GZ11] F. Grunewald, P. Zalesskii, *Genus for groups*, J. Algebra **326** (2011), 130–168.

[GMZ11] A. Guazzi, M. Mecchia, B. Zimmermann, *On finite groups acting on acyclic low-dimensional manifolds*, Fund. Math. **215** (2011), no. 3, 203–217.

[GZ13] A. Guazzi, B. Zimmermann, *On finite simple groups acting on homology spheres*, Monatsh. Math. **169** (2013), no. 3-4, 371–381.

[GuH10] E. Guentner, N. Higson, *Weak amenability of CAT(0)–cubical groups*, Geom. Dedicata **148** (2010), 137–156.

[GZ09] X. Guo, Y. Zhang, *Virtually fibred Montesinos links of type* $\widetilde{SL(2)}$, Topology Appl. **156** (2009), no. 8, 1510–1533.

[Gus81] R. Gustafson, *A simple genus one knot with incompressible spanning surfaces of arbitrarily high genus*, Pacific J. Math. **96** (1981), 81–98.

[Gus94] ———, *Closed incompressible surfaces of arbitrarily high genus in complements of certain star knots*, Rocky Mountain J. Math. **24** (1994), no. 2, 539–547.

[Guz12] R. Guzman, *Hyperbolic 3-manifolds with k-free fundamental group*, Topology Appl. **173** (2014), 142–156.

[HaP13] M. Hagen, P. Przytycki, *Cocompactly cubulated graph manifolds*, Israel J. Math., to appear.

[HnW13] M. Hagen, D. Wise, *Cubulating hyperbolic free-by-cyclic groups: the irreducible case*, preprint, 2013.

[HnW14] ———, *Cubulating hyperbolic free-by-cyclic groups: the general case*, Geom. Funct. Anal., to appear.

[Hag08] F. Haglund, *Finite index subgroups of graph products*, Geom. Dedicata **135** (2008), 167–209.

[HaW08] F. Haglund, D. Wise, *Special cube complexes*, Geom. Funct. Anal. **17** (2008), no. 5, 1551–1620.

[HaW12] ———, *A combination theorem for special cube complexes*, Annals of Math. (2) **176** (2012), 1427–1482.

[Hai13] P. Haissinsky, *Hyperbolic groups with planar boundaries*, preprint, 2013.

[HaTe12] R. Hakamata, M. Teragaito, *Left-orderable fundamental group and Dehn surgery on twist knots*, preprint, 2012.

[HaTe14a] ———, *Left-orderable fundamental group and Dehn surgery on genus one two-bridge knots*, Alg. Geom. Topology **14** (2014), 2125–2148.

[HaTe14b] ———, *Left-orderable fundamental group and Dehn surgery on the knot* 5_2, Canad. Math. Bull. **57** (2014), no. 2, 310–317.

[Hak61a] W. Haken, *Theorie der Normalflächen,* Acta Math. **105** (1961), 245–375.

[Hak61b] W. Haken, *Ein Verfahren zur Aufspaltung einer 3-Mannigfaltigkeit in irreduzible 3-Mannigfaltigkeiten*, Math. Z. **76** (1961), 427–467.

[Hak70] ———, *Various aspects of the three-dimensional Poincaré problem*, in: *Topology of Manifolds,* pp. 140–152, Markham, Chicago, IL, 1970.

[Hal49] M. Hall, *Coset representations in free groups*, Trans. Amer. Math. Soc. **67** (1949), 421–432.

[Hab14] I. Hambleton, *Topological spherical space forms*, Handbook of Group Actions (Vol. II), ALM 32, 151–172. International Press, Beijing-Boston (2014)

[Han14] U. Hamenstädt, *Incompressible surfaces in rank one locally symmetric spaces*, preprint, 2014.

[Hama76] A. J. S. Hamilton, *The triangulation of 3-manifolds*, Quart. J. Math. Oxford Ser. (2) **27** (1976), no. 105, 63–70.

[Hamb01] E. Hamilton, *Abelian subgroup separability of Haken* 3*-manifolds and closed hyperbolic* n*-orbifolds*, Proc. London Math. Soc. **83** (2001), no. 3, 626–646.

[Hamb03] ———, *Classes of separable two-generator free subgroups of* 3*-manifold groups*, Topology Appl. **131** (2003), no. 3, 239–254.

[HWZ13] E. Hamilton, H. Wilton, P. Zalesskii *Separability of double cosets and conjugacy classes in* 3*-manifold groups*, J. London Math. Soc. **87** (2013), 269–288.

[Hamc82] R. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), no. 2, 255–306.

[Hamc95] R. Hamilton, *The formation of singularities in the Ricci flow*, in: *Surveys in Differential Geometry,* Vol. II, pp. 7–136, International Press, Cambridge, MA, 1995.

[Hamc99] ———, *Non-singular solutions of the Ricci flow on three-manifolds*, Comm. Anal. Geom. **7** (1999), no. 4, 695–729.

[HnTh85] M. Handel, W. Thurston, *New proofs of some results of Nielsen*, Adv. in Math. **56** (1985), no. 2, 173–191.

[HzW35] W. Hantzsche, H. Wendt, *Dreidimensionale euklidische Raumformen*, Math. Ann. **110** (1935), 593–611.

[HaR03] J. Harlander, S. Rosebrock, *Generalized knot complements and some aspherical ribbon disc complements*, J. Knot Theory Ramifications **12** (2003), no. 7, 947–962.

[HaR12] ———, *Injective labeled oriented trees are aspherical*, preprint, 2012.

[dlHP07] P. de la Harpe, J.-P. Préaux, *Groupes fondamentaux des variétés de dimension* 3 *et algébres d'opérateurs*, Ann. Fac. Sci. Toulouse Math. (6) **16** (2007), no. 3, 561–589.

[dlHP11] ———, C^**-simple groups: amalgamated free products, HNN-extensions, and fundamental groups of* 3*-manifolds*, J. Topol. Anal. **3** (2011), no. 4, 451–489.

[dlHW11] P. de la Harpe, C. Weber, *On malnormal peripheral subgroups in fundamental groups of* 3*-manifolds*, preprint, 2011.

[Har02] S. Harvey, *On the cut number of a* 3*-manifold*, Geom. Top. **6** (2002), 409–424.

[Har05] ———, *Higher-order polynomial invariants of* 3*-manifolds giving lower bounds for the Thurston norm*, Topology **44** (2005), 895–945.

[Has87] J. Hass, *Minimal surfaces in manifolds with* S^1 *actions and the simple loop conjecture for Seifert fiber spaces*, Proc. Amer. Math. Soc. **99** (1987), 383–388.

[Hat] A. Hatcher, *Notes on Basic* 3*-Manifold Topology,* `http://www.math.cornell.edu/~hatcher/3M/3M.pdf`

[Hat76] ———, *Homeomorphisms of sufficiently large* P^2*-irreducible* 3*-manifolds*, Topology **15** (1976), 343–347.

[Hat82] ———, *On the boundary curves of incompressible surfaces*, Pacific J. Math. **99**, no. 2, 373–377.

[Hat83] ———, *A proof of the Smale conjecture,* $\mathrm{Diff}(S^3) \cong O(4)$, Ann. of Math. (2) **117** (1983), no. 3, 553–607.

[Hat99] ———, *Spaces of incompressible surfaces*, manuscript, `arXiv: math/9906074`, 1999.

[Hat02] ———, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002.

[Hat13] ———, *The Kirby Torus Trick for surfaces*, preprint, 2013.

[HO89] A. Hatcher, U. Oertel, *Boundary slopes for Montesinos knots*, Topology **28** (1989), no. 4, 453–480.

[HaTh85] A. Hatcher, W. Thurston, *Incompressible surfaces in* 2*-bridge knot complements*, Invent. Math. **79**, no. 2, 225–246.

[Hau81] J.-C. Hausmann, *On the homotopy of nonnilpotent spaces*, Math. Z. **178** (1981), 115–123.

[HJS13] M. Hedden, A. Juhász, S. Sarkar, *On Sutured Floer homology and the equivalence of Seifert surfaces*, Alg. Geom. Topology **13** (2013), 505–548.

[Hee1898] P. Heegaard, *Forstudier til en topologisk Teori for de algebraiske Fladers Sammenhang*, Ph.D. thesis, Copenhagen University, 1898. `http://www.maths.ed.ac.uk/~aar/papers/heegaardthesis.pdf`

[Hee16] ———, *Sur l'analysis situs*, Bull. Soc. Math. France **44** (1916), 161–142.

[Hei69a] W. Heil, *On* P^2*-irreducible* 3*-manifolds*, Bull. Amer. Math. Soc. **75** (1969), 772–775.

[Hei69b] ———, *On the existence of incompressible surfaces in certain* 3*-manifolds*, Proc. Amer. Math. Soc. **23** (1969), 704–707.

[Hei70] ———, *On the existence of incompressible surfaces in certain* 3*-manifolds,* II, Proc. Amer. Math. Soc. **25** (1970), 429–432.

[Hei71] ———, *On subnormal subgroups of fundamental groups of certain* 3*-manifolds*, Michigan Math. J. **18** (1971), 393–399.

[Hei72] ———, *On Kneser's conjecture for bounded* 3*-manifolds*, Proc. Cambridge Philos. Soc. **71** (1972), 243–246.

[Hei81] ———, *Normalizers of incompressible surfaces in* 3*-manifolds*, Glas. Mat. Ser. III **16(36)** (1981), no. 1, 145–150.

[HeR84] W. Heil, J. Rakovec, *Surface groups in* 3*-manifold groups*, in: *Algebraic and Differential Topology–Global Differential Geometry,* pp. 101–133, BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1984.

[HeT78] W. Heil, J. L. Tollefson, *Deforming homotopy involutions of* 3*-manifolds to involutions*, Topology **17** (1978), no. 4, 349–365.

[HeT83] ———, *Deforming homotopy involutions of* 3*-manifolds to involutions,* II, Topology **22** (1983), no. 2, 169–172.

[HeT87] ———, *On Nielsen's theorem for* 3*-manifolds*, Yokohama Math. J. **35** (1987), no. 1–2, 1–20.

[HeW94] W. Heil, W. Whitten, *The Seifert fibre space conjecture and torus theorem for nonorientable* 3*-manifolds*, Canad. Math. Bull. **37** (4) (1994), 482–489.

[Hen79] G. Hemion, *On the classification of homeomorphisms of* 2*-manifolds and the classification of* 3*-manifolds*, Acta Math. **142** (1979), no. 1-2, 123–155.

[Hem76] J. Hempel, 3*-Manifolds*, Ann. of Math. Studies, vol. 86, Princeton University Press, Princeton, NJ, 1976.

[Hem82] ———, *Orientation reversing involutions and the first Betti number for finite coverings of* 3*-manifolds*, Invent. Math. **67** (1982), no. 1, 133–142.

[Hem84] ———, *Homology of coverings*, Pacific J. Math. **112** (1984), no. 1, 83–113.

[Hem85a] ———, *Virtually Haken manifolds*, in: *Combinatorial Methods in Topology and Algebraic geometry,* pp. 149–155, Contemporary Mathematics, vol. 44, American Mathematical Society, Providence, RI, 1985.

[Hem85b] ———, *The finitely generated intersection property for Kleinian groups*, in: *Knot Theory and Manifolds,* pp. 18–24, Lecture Notes in Mathematics, vol. 1144, Springer-Verlag, Berlin, 1985.

[Hem87] ———, *Residual finiteness for* 3*-manifolds*, in: *Combinatorial Group Theory and Topology,* pp. 379–396, Annals of Mathematics Studies, vol. 111, Princeton University Press, Princeton, NJ, 1987.

[Hem90] ———, *Branched covers over strongly amphicheiral links*, Topology **29** (1990), no. 2, 247–255.

[Hem01] ———, 3*-manifolds as viewed from the curve complex*, Topology **40** (2001), 631–657.

[Hem14] ———, *Some* 3*-manifold groups with the same finite quotients*, preprint, 2014.

[HJ72] J. Hempel, W. Jaco, *Fundamental groups of* 3*-manifolds which are extensions*, Ann. of Math. (2) **95** (1972) 86–98.

[HMPR91] K. Henckell, S.T.Margolis, J. E. Pin, J. Rhodes, *Ash's Type II Theorem, profinite topology and Malcev products*, I, Int J. of Algebra and Computation **1** (1991), 411–436.

[HL84] H. Hendriks, F. Laudenbach, *Difféomorphismes des sommes connexes en dimension trois*, Topology **23** (1984), no. 4, 423–443.

[HM99] S. Hermiller, J. Meier, *Artin groups, rewriting systems and three-manifolds*, J. Pure Appl. Algebra **136** (1999), no. 2, 141–156.

[HeS07] S. Hermiller, Z. Šunić, *Poly-free constructions for right-angled Artin groups*, J. Group Theory **10** (2007), 117–138.

[HeP11] M. Heusener, J. Porti, *Infinitesimal projective rigidity under Dehn filling*, Geom. Topol. **15** (2011), no. 4, 2017–2071.

[Hig40] G. Higman, *The units of group-rings*, Proc. London Math. Soc. **46** (1940), 231–248.

[HNN49] G. Higman, B. H. Neumann, H. Neumann, *Embedding theorems for groups*, J. London Math. Soc. **24** (1949), 247–254.

[Hin74] H. Hilden, *Every closed orientable* 3*-manifold is a* 3*-fold branched covering space of* S^3, Bull. Amer. Math. Soc. **80** (1974), 1243–1244.

[HLMA06] H. Hilden, M. Lozano, J. Montesinos-Amilibia, *On hyperbolic 3-manifolds with an infinite number of fibrations over S^1*, Math. Proc. Cambridge Philos. Soc. **140** (2006), no. 1, 79–93.

[Hil77] J. Hillman, *High dimensional knot groups which are not two-knot groups*, Bull. Austral. Math. Soc. **16** (1977), no. 3, 449–462.

[Hil85] ———, *Seifert fibre spaces and Poincaré duality groups*, Math. Z. **190** (1985), no. 3, 365–369.

[Hil87] ———, *Three-dimensional Poincaré duality groups which are extensions*, Math. Z. **195** (1987), 89–92.

[Hil89] ———, *2-Knots and their Groups*, Australian Mathematical Society Lecture Series, vol. 5, Cambridge University Press, Cambridge, 1989.

[Hil02] ———, *Algebraic Invariants of Links*, Series on Knots and Everything, vol. 32, World Scientific Publishing Co., Inc., River Edge, NJ, 2002.

[Hil06] ———, *Centralizers and normalizers of subgroups of PD_3-groups and open PD_3-groups*, J. Pure Appl. Algebra **204** (2006), no. 2, 244–257.

[Hil11] ———, *Some questions on subgroups of 3-dimensional Poincaré duality groups*, manuscript, 2011.
`http://www.maths.usyd.edu.au:8000/u/jonh/pdq.pdf`

[Hil12] ———, *Indecomposable PD_3-complexes*, Alg. Geom. Top. **12** (2012), 131–153.

[HiS97] M. Hirasawa, M. Sakuma, *Minimal genus Seifert surfaces for alternating links*, in: *KNOTS '96,* pp. 383–394, World Scientific Publishing Co., Inc., River Edge, NJ, 1997.

[HiK14] S. Hirose, Y. Kasahara, *A uniqueness of periodic maps on surfaces*, preprint, 2014.

[Ho93] W. Hodges, *Model Theory*, Encyclopedia of Mathematics and its Applications, vol. 42, Cambridge University Press, Cambridge, 1993.

[HoK05] C. Hodgson, S. Kerckhoff, *Universal bounds for hyperbolic Dehn surgery*, Ann. of Math. (2) **162** (2005), 367–421.

[HoR85] C. Hodgson, J. H. Rubinstein, *Involutions and isotopies of lens spaces,* in: *Knot Theory and Manifolds,* pp. 60–96, Lecture Notes in Mathematics, vol. 1144, Springer-Verlag, Berlin, 1985.

[Hof10] N. Hoffman, *Commensurability classes containing three knot complements*, Algebr. Geom. Topol. **10** (2010), 663–677.

[HW13] N. Hoffman, G. Walsh, *The Big Dehn Surgery Graph and the link of S^3*, preprint, 2013.

[Hog00] C. Hog-Angeloni, *Detecting 3-manifold presentations*, in: *Computational and Geometric Aspects of Modern Algebra,* pp. 106–119, London Mathematical Society Lecture Note Series, vol. 275, Cambridge University Press, Cambridge, 2000.

[HAM08] C. Hog-Angeloni, S. Matveev, *Roots in 3-manifold topology*, in: *The Zieschang Gedenkschrift,* pp. 295–319, Geometry & Topology Monographs, vol. 14, Geometry & Topology Publications, Coventry, 2008.

[HKL14] J. Hom, C. Karakurt, T. Lidman, *Surgery obstructions and Heegaard Floer homology*, preprint, 2014.

[Hom57] T. Homma, *On Dehn's lemma for S^3*, Yokohama Math. J. **5** (1957), 223–244.

[Hop26] H. Hopf, *Zum Clifford-Kleinschen Raumproblem*, Math. Ann. **95** (1926), 313–339.

[HoSh07] J. Hoste, P. Shanahan, *Computing boundary slopes of 2-bridge links*, Math. Comp. **76** (2007), no. 259, 1521–1545.

[How82] J. Howie, *On locally indicable groups*, Math. Z. **180** (1982), 445–461.

[How85] ———, *On the asphericity of ribbon disc complements*, Trans. Amer. Math. Soc. **289** (1985), no. 1, 281–302.

[HoS85] J. Howie, H. Short, *The band-sum problem*, J. London Math. Soc. **31** (1985), 571–576.

[Hr10] C. Hruska, *Relative hyperbolicity and relative quasi-convexity for countable groups*, Alg. Geom. Topology **10** (2010), 1807–1856.

[HrW09] C. Hruska, D. Wise, *Packing subgroups in relatively hyperbolic groups*, Geom. Topol. **13** (2009), no. 4, 1945–1988.

[HrW14] ———, *Finiteness properties of cubulated groups*, Compos. Math. **150** (2012), no. 3, 453–506.

[HsW99] T. Hsu, D. Wise, *On linear and residual properties of graph products*, Michigan Math. J. **46** (1999), no. 2, 251–259.

[HsW12] ———, *Cubulating malnormal amalgams*, 20 pages, Preprint (2012), to appear in Inv. Math., `http://dx.doi.org/10.1007/s00222-014-0513-4`

[Hu13] Y. Hu, *The left-orderability and the cyclic branched coverings*, preprint, 2013.

[HuR01] G. Huck, S. Rosebrock, *Aspherical labelled oriented trees and knots*, Proc. Edinb. Math. Soc. (2) **44** (2001), no. 2, 285–294.

[IT14] K. Ichihara, Y. Temma, *Non left-orderable surgeries and generalized Baumslag-Solitar relators*, preprint, 2014.

[It13] T. Ito, *Non-left-orderable double branched coverings*, Alg. Geom. Top. **13** (2013), 1937–1965.

[Iva76] N. Ivanov, *Research in Topology,* II, Notes of LOMI scientific seminars **66** (1976), 172–176.

[Iva80] ———, *Spaces of surfaces in Waldhausen manifolds*, preprint LOMI P-5-80, 1980.

[Iva92] ———, *Subgroups of Teichmüller modular groups*, Translations of Mathematical Monographs, vol. 115, Amer. Math. Soc., Providence, RI, 1992.

[Ivb05] S. V. Ivanov, *On the asphericity of LOT-presentations of groups*, J. Group Theory **8** (2005), no. 1, 135–138.

[IK01] S. V. Ivanov, A. A. Klyachko, *The asphericity and Freiheitssatz for certain lot-presentations of groups*, Internat. J. Algebra Comput. **11** (2001), no. 3, 291–300.

[Iw43] K. Iwasawa, *Einige Sätze über freie Gruppen*, Proc. Imp. Acad. Tokyo **19** (1943), 272–274.

[Ja69] W. Jaco, *Heegaard splittings and splitting homomorphisms*, Trans. Amer. Math. Soc. **144** (1969), 365–379.

[Ja71] ———, *Finitely presented subgroups of three-manifold groups*, Invent. Math. **13** (1971), 335–346.

[Ja72] ———, *Geometric realizations for free quotients*, J. Austral. Math. Soc. **14** (1972), 411–418.

[Ja75] ———, *Roots, relations and centralizers in three-manifold groups*, in: *Geometric Topology,* pp. 283–309, Lecture Notes in Mathematics, vol. 438, Springer-Verlag, Berlin-New York, 1975.

[Ja80] ———, *Lectures on Three-Manifold Topology*, CBMS Regional Conference Series in Mathematics, vol. 43, American Mathematical Society, Providence, RI, 1980.

[JLR02] W. Jaco, D. Letscher, H. Rubinstein, *Algorithms for essential surfaces in* 3*-manifolds*, in: *Topology and Geometry: Commemorating SISTAG,* pp. 107–124, Contemporary Mathematics, vol. 314, American Mathematical Society, Providence, RI, 2002.

[JM79] W. Jaco, R. Myers, *An algebraic determination of closed orientable* 3*-manifolds*, Trans. Amer. Math. Soc. **253** (1979), 149–170.

[JO84] W. Jaco, U. Oertel, *An algorithm to decide if a* 3*-manifold is a Haken manifold*, Topology **23** (1984), no. 2, 195–209.

[JR89] W. Jaco, H. Rubinstein, *PL equivariant surgery and invariant decompositions of* 3*-manifolds*, Adv. in Math. **73** (1989), no. 2, 149–191.

[JR03] ———, 0*-efficient triangulations of* 3*-manifolds*, J. Differential Geom. **65** (2003), no. 1, 61–168.

[JS76] W. Jaco, P. Shalen, *Peripheral structure of* 3*-manifolds*, Invent. Math. **38** (1976), 55–87.

[JS78] ———, *A new decomposition theorem for irreducible sufficiently-large* 3*-manifolds*, in: *Algebraic and Geometric Topology,* part 2, pp. 71–84, Proceedings of Symposia in Pure Mathematics, vol. XXXII, American Mathematical Society, Providence, RI, 1978.

[JS79] ———, *Seifert fibered spaces in* 3*-manifolds*, Mem. Amer. Math. Soc. **21** (1979), no. 220.

[JT95] W. Jaco, J. L. Tollefson, *Algorithms for the complete decomposition of a closed* 3*-manifold*, Illinois J. Math. **39** (1995), no. 3, 358–406.

[JD83] M. Jenkins, W. D. Neumann, *Lectures on Seifert Manifolds*, Brandeis Lecture Notes, vol. 2, Brandeis University, Waltham, MA, 1983.

[Ji12] L. Ji, *Curve complexes versus Tits buildings: structures and applications*, in: *Buildings, Finite Geometries and Groups,* pp. 93–152, Springer Proceedings in Mathematics, vol. 10, Springer, New York, 2012.

[Jos35] I. Johansson, *Über singuläre Elementarflächen und das Dehnsche Lemma*, Math. Ann. **110** (1935), no. 1, 312–320.

[Jon75] K. Johannson, *Équivalences d'homotopie des variétés de dimension* 3, C. R. Acad. Sci. Paris Sér. A-B **281** (1975), no. 23, Ai, A1009–A1010.

[Jon79a] ———, *Homotopy Equivalences of 3-Manifolds with Boundaries*, Lecture Notes in Mathematics, vol. 761, Springer-Verlag, Berlin, 1979.

[Jon79b] ———, *On the mapping class group of simple* 3*-manifolds*, in: *Topology of Low-dimensional Manifolds,* pp. 48–66, Lecture Notes in Mathematics, vol. 722, Springer-Verlag, Berlin, 1979.

[Jon79c] ———, *On exotic homotopy equivalences of* 3*-manifolds*, in: *Geometric Topology,* pp. 101–111, Academic Press, New York-London, 1979.

[Jon94] ———, *On the loop- and sphere theorem*, in: *Low-dimensional Topology,* pp. 47–54, Conference Proceedings and Lecture Notes in Geometry and Topology, vol. III, International Press, Cambridge, MA, 1994.

[Joh80] D. Johnson, *Homomorphs of knot groups*, Proc. Amer. Math. Soc. **78** (1980), no. 1, 135–138.

[JKS95] D. L. Johnson, A. C. Kim, H. J. Song, *The growth of the trefoil group*, in: *Groups—Korea '94,* pp. 157-161, Walter de Gruyter & Co., Berlin, 1995.

[JL89] D. Johnson, C. Livingston, *Peripherally specified homomorphs of knot groups*, Trans. Amer. Math. Soc. **311** (1989), no. 1, 135–146.

[JW72] F. E. A. Johnson, C. T. C. Wall, *On groups satisfying Poincaré Duality*, Ann. of Math. (2) **96** (1972), no. 3, 592–598.

[Joy82] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra **23** (1982), no. 1, 37–65.

[Jun94] D. Jungreis, *Gaussian random polygons are globally knotted*, J. Knot Theory Ramifications **3** (1994), no. 4, 455–464.

[Juh08] A. Juhasz, *Knot Floer homology and Seifert surfaces*, Algebr. Geom. Topol. **8** (2008), no. 1, 603–608.

[KM12a] J. Kahn, V. Markovic, *Immersing almost geodesic surfaces in a closed hyperbolic three manifold*, Ann. of Math. (2) **175** (2012), 1127–1190.

[KM12b] ———, *Counting essential surfaces in a closed hyperbolic three-manifold*, Geom. Top. **16** (2012), 601–624.

[Kak91] O. Kakimizu, *Doubled knots with infinitely many incompressible spanning surfaces*, Bull. London Math. Soc. **23** (1991), 300–302.

[Kak92] ———, *Finding disjoint incompressible spanning surfaces for a link*, Hiroshima Math. J. **22** (1992), no. 2, 225–236.

[Kak05] ———, *Classification of the incompressible spanning surfaces for prime knots of 10 or less crossings*, Hiroshima Math. J. **35** (2005), no. 1, 47–92.

[Kay57] I. Kaplansky, *Problems in the theory of rings*, in: *Report of a Conference on Linear Algebras,* pp. 1-3, National Academy of Sciences-National Research Council, publ. 502, Washington, 1957.

[Kay70] ———, *"Problems in the theory of rings" revisited*, Amer. Math. Monthly **77** (1970), 445–454.

[KMW05] I. Kapovich, A. Miasnikov, R. Weidmann, *Foldings, graphs of groups and the membership problem*, Internat. J. Algebra Comput. **15** (2005), no. 1, 95–128.

[KR14] I. Kapovich, K. Rafi, *On hyperbolicity of free splitting and free factor complexes*, Groups Geom. Dyn. **8** (2014), no. 2, 391–414.

[Kap01] M. Kapovich, *Hyperbolic Manifolds and Discrete Groups*, Progress in Mathematics, vol. 183, Birkhäuser Boston, Inc., Boston, MA, 2001.

[KKl04] M. Kapovich, B. Kleiner, *Coarse fibrations and a generalization of the Seifert fibered space conjecture*, preprint, 2004. `http://www.math.ucdavis.edu/~kapovich/EPR/sei3.pdf`

[KaL97] M. Kapovich, B. Leeb, *Quasi-isometries preserve the geometric decomposition of Haken manifolds*, Invent. Math. **128** (1997), 393–416.

[KaL98] ———, 3*-manifold groups and nonpositive curvature*, Geom. Funct. Anal. **8** (1998), 841–852.

[KaS96] I. Kapovich, H. Short, *Greenberg's theorem for quasi-convex subgroups of word-hyperbolic groups*, Canad. J. Math. **48** (1996), no. 6, 1224–1244.

[KaN12] A. Kar, N. Nikolov, *Rank gradient for Artin groups and their relatives*, Groups Geom. Dyn., to appear.

[Kaw89a] A. Kawauchi, *Almost identical imitations of* $(3,1)$*-dimensional manifold pairs*, Osaka J. Math. **26** (1989), 743–758.

[Kaw89b] ______, *An imitation theory of manifolds*, Osaka J. Math. **26** (1989), 447-464.

[Kaw90] ______, *A Survey of Knot Theory*, Birkhäuser Verlag, Basel, 1996.

[Kaw93] ______, *Almost identical imitations of* $(3,1)$*-dimensional manifold pairs and the manifold mutation*, J. Aust. Math. Soc. Ser. A **55** (1993), 100–115.

[Kaw94] ______, *A survey of topological imitations of* $(3,1)$*-dimensional manifold pairs*, Proc. Applied Math. Workshop **4** (1994), 43–52.

[Kaw97] ______, *Topological imitations*, in: *Lectures at KNOTS '96,* pp. 19–37, Series on Knots and Everything, vol. 15, World Scientific Publishing Co., Inc., River Edge, NJ, 1997.

[KKo0] A. Kawauchi, S. Kojima, *Algebraic classification of linking pairings on* 3*-manifolds*, Math. Ann. **253** (1980), 29–42.

[Kaz75] D. A. Kazhdan, *On arithmetic varieties*, in: *Lie Groups and their Representations,* pp. 151–217, John Wiley & Sons, New York-Toronto, Ont., 1975.

[Ken04] R. Kent, *Bundles, handcuffs, and local freedom*, Geom. Dedicata **106** (2004), 145–159.

[Kea73] C. Kearton, *Classification of simple knots by Blanchfield duality*, Bull. Amer. Math. Soc. **79** (1973), 952–955.

[Ker60] M. Kervaire, *A manifold which does not admit any differentiable structure*, Comm. Math. Helv. **34** (1960), 257–270.

[Ker65] ______, *On higher dimensional knots*, in: *Differential and Combinatorial Topology,* pp. 105–119, Princeton University Press, Princeton, NJ, 1965.

[KeM63] M. Kervaire, J. Milnor, *Groups of homotopy spheres,* I, Ann. of Math. (2) **77** (1963), 504–537.

[KiK13] S. H. Kim, T. Koberda, *Anti-trees and right-angled Artin subgroups of braid groups*, preprint, 2013.

[KiS12] S. Kionke, J. Schwermer, *On the growth of the first Betti number of arithmetic hyperbolic* 3*-manifolds*, Groups Geom. Dyn., to appear.

[Kin05] S. King, *Skript zur Vorlesung Dreidimensionale Topologie*, Technische Universität Darmstadt, 2005.
`http://www.nuigalway.ie/maths/sk/pub/skript.pdf`

[Kir97] R. Kirby, *Problems in low-dimensional topology*, in: *Geometric Topology,* 2, pp. 35–473, AMS/IP Studies in Advanced Mathematics, vol. 2.2, American Mathematical Society, Providence, RI; International Press, Cambridge, MA, 1997.

[KyS77] R. C. Kirby, L. C. Siebenmann, *Foundational Essays on Topological Manifolds, Smoothings, and Triangulations*, Annals of Mathematics Studies, vol. 88, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1977.

[KMT03] T. Kitano, T. Morifuji, M. Takasawa, L^2*-torsion invariants and homology growth of a torus bundle over* S^1, Proc. Japan Acad. Ser. A Math. Sci. **79** (2003), no. 4, 76–79.

[KlL08] B. Kleiner, J. Lott, *Notes on Perelman's papers*, Geom. Topol. **12** (2008), no. 5, 2587–2855.

[Kn29] H. Kneser, *Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten*, Jber. Deutsch. Math.-Verein. **38** (1929), 248–260.

[Koi88] T. Kobayashi, *Casson–Gordon's rectangle condition of Heegaard diagrams and incompressible tori in* 3*-manifolds*, Osaka J. Math. **25** (1988), 553–573.

[Koi89] ______, *Uniqueness of minimal genus Seifert surfaces for links*, Topology Appl. **33** (1989), no. 3, 265–279.

[Kob12a] T. Koberda, *Mapping class groups, homology and finite covers of surfaces*, Ph.D. thesis, Harvard University, 2012.

[Kob12b] ______, *Right-angled Artin groups and a generalized isomorphism problem for finitely generated subgroups of mapping class groups*, Geom. Funct. Anal. **22** (2012), 1541–1590.

[Kob13] ______, *Residual properties of fibered and hyperbolic* 3*-manifolds*, Top. Appl. **160** (2013), 875–886.

[Kob14] ______, *Alexander varieties and largeness of finitely presented groups*, J. Homotopy Relat. Struct. **9** (2014), no. 2, 513–531.

[KZ07] D. Kochloukova, P. Zalesskii, *Tits alternative for 3-manifold groups*, Arch. Math. (Basel) **88** (2007), no. 4, 364–367.

[Koj84] S. Kojima, *Bounding finite groups acting on 3-manifolds*, Math. Proc. Cambridge Philos. Soc. **96** (1984), no. 2, 269–281.

[Koj87] ———, *Finite covers of 3-manifolds containing essential surfaces of Euler characteristic* = 0, Proc. Amer. Math. Soc. **101** (1987), no. 4, 743–747.

[Koj88] ———, *Isometry transformations of hyperbolic 3-manifolds*, Topology Appl. **29** (1988), no. 3, 297–307.

[KLg88] S. Kojima, D. Long, *Virtual Betti numbers of some hyperbolic 3-manifolds*, in: *A Fête of Topology,* pp. 417–437, Academic Press, Boston, MA, 1988.

[KLs14] P. Konstantis, F. Loose, *A classification of Thurston geometries without compact quotients*, preprint, 2014.

[KoM12] P. Korablev, S. Matveev, *Five lectures on 3-manifold topology*, in: *Strasbourg Master Class on Geometry,* pp. 255–284, IRMA Lectures in Mathematics and Theoretical Physics, vol. 18, European Mathematical Society, Zürich, 2012.

[Kos58] K. Koseki, *Poincarésche Vermutung in Topologie*, Math. J. Okayama Univ. **8** (1958), 1–106.

[Kot12] D. Kotschick, *Three-manifolds and Kähler groups*, Ann. Inst. Fourier (Grenoble) **62** (2012), no. 3, 1081–1090.

[Kot13] ———, *Kählerian three-manifold groups*, Math. Res. Lett. **20** (2013), no. 3, 521–525.

[KdlH14] D. Kotschick, P. de la Harpe, *Presentability by products for some classes of groups*, preprint, 2014.

[KoL09] D. Kotschick, C. Löh, *Fundamental classes not representable by products*, J. London Math. Soc. (2) **79** (2009), no. 3, 545–561.

[KoL13] ———, *Groups not presentable by products*, Groups Geom. Dyn. **7** (2013), no. 1, 181–204.

[KoN13] D. Kotschick, C. Neofytidis, *On three-manifolds dominated by circle bundles*, Math. Z. **274** (2013), 21–32.

[Kow08] E. Kowalski, *The large sieve and its applications*, Cambridge Tracts in Mathematics, vol. 175, Cambridge University Press, Cambridge, 2008.

[Kow] ———, *On the complexity of Dunfield-Thurston random 3-manifolds*, unpublished note. `http://www.math.ethz.ch/~kowalski/complexity-dunfield-thurston.pdf`

[KrL09] M. Kreck, W. Lück, *Topological rigidity for non-aspherical manifolds*, Pure Appl. Math. Q. **5** (2009), no. 3, 873–914.

[KrM04] P. B. Kronheimer, T. S. Mrowka, *Dehn surgery, the fundamental group and SU(2)*, Math. Res. Lett. **11** (2004), no. 5-6, 741–754.

[Kr90a] P. Kropholler, *A note on centrality in 3-manifold groups*, Math. Proc. Cambridge Philos. Soc. **107** (1990), no. 2, 261–266.

[Kr90b] ———, *An analogue of the torus decomposition theorem for certain Poincaré duality groups*, Proc. London Math. Soc. (3) **60** (1990), no. 3, 503–529.

[KLM88] P. H. Kropholler, P. A. Linnell, J. A. Moody, *Applications of a new K-theoretic theorem to soluble group rings*, Proc. Amer. Math. Soc. **104** (1988), no. 3, 675–684.

[KAG86] S. Krushkal, B. Apanasov, N. Gusevskij, *Kleinian Groups and Uniformization in Examples and Problems*, Translations of Mathematical Monographs, vol. 62, Amer. Math. Soc., Providence, RI, 1986.

[Kui79] N. H. Kuiper, *A short history of triangulation and related matters*, in: *Proceedings. Bicentennial Congress Wiskundig Genootschap,* I, pp. 61–79, Mathematical Centre Tracts, vol. 100, Mathematisch Centrum, Amsterdam, 1979.

[Kui99] ———, *A short history of triangulation and related matters*, in: *History of Topology,* pp. 491–502, North-Holland, Amsterdam, 1999.

[Kul05] O. V. Kulikova, *On the fundamental groups of the complements to Hurwitz curves,* Izv. Math. **69** (2005), no. 1, 123–130.

[Kup14] G. Kuperberg, *Knottedness is in NP, modulo GRH*, Adv. in Math. **256** (2014), 493–506.

[KwL88] S. Kwasik, K. B. Lee, *Locally linear actions on 3-manifolds*, Math. Proc. Cambridge Philos. Soc. **104** (1988), no. 2, 253–260.

[LaS86] J. Labesse, J. Schwermer, *On liftings and cusp cohomology of arithmetic groups*, Invent. Math. **83** (1986), 383–401.

[Lac00] M. Lackenby, *Word hyperbolic Dehn surgery*, Invent. Math. **140** (2000), 243–282.

[Lac04] ———, *The asymptotic behavior of Heegaard genus*, Math. Res. Lett. **11** (2004), no. 2–3, 139–149.

[Lac05] ———, *Expanders, rank and graphs of groups*, Israel J. Math. **146** (2005), 357–370.

[Lac06] ———, *Heegaard splittings, the virtually Haken conjecture and property* (τ), Invent. Math. **164** (2006), no. 2, 317–359.

[Lac07a] ———, *Some* 3*-manifolds and* 3*-orbifolds with large fundamental group*, Proc. Amer. Math. Soc. **135** (2007), no. 10, 3393–3402.

[Lac07b] ———, *Covering spaces of* 3*-orbifolds*, Duke Math. J. **136** (2007), no. 1, 181–203.

[Lac09] ———, *New lower bounds on subgroup growth and homology growth*, Proc. London Math. Soc. (3) **98** (2009), no. 2, 271–297.

[Lac10] ———, *Surface subgroups of Kleinian groups with torsion*, Invent. Math. **179** (2010), no. 1, 175–190.

[Lac11] ———, *Finite covering spaces of* 3*-manifolds*, in: *Proceedings of the International Congress of Mathematicians,* II, pp. 1042–1070, World Scientific Publishing Co. Pte. Ltd., Singapore, 2010.

[LaLR08a] M. Lackenby, D. Long, A. Reid, *Covering spaces of arithmetic* 3*-orbifolds*, Int. Math. Res. Not. 2008, no. 12.

[LaLR08b] ———, *LERF and the Lubotzky-Sarnak conjecture*, Geom. Topol. **12** (2008), no. 4, 2047–2056.

[LaM13] M. Lackenby, R. Meyerhoff, *The maximal number of exceptional Dehn surgeries*, Invent. Math. **191** (2013), no. 2, 341–382.

[Lan95] S. Lang, *Differential and Riemannian Manifolds*, 3rd ed., Graduate Texts in Mathematics, vol. 160, Springer-Verlag, New York, 1995.

[Lau74] F. Laudenbach, *Topologie de la dimension trois: homotopie et isotopie*, Astérisque, vol. 12, Société Mathématique de France, Paris, 1974.

[Lau85] ———, *Les* 2*-sphéres de* $\mathbb{R}^3$ *vues par A. Hatcher et la conjecture de Smale* $\mathrm{Diff}(S^3) \sim O(4)$, in: *Séminaire Bourbaki,* vol. 1983/84, exposés 615–632, pp. 279–293, Astérisque vol. 121-122, Société Mathématique de France, Paris, 1985.

[Le14a] T. Le, *Homology torsion growth and Mahler measure*, Comment. Math. Helv. **89** (2014), no. 3, 719–757.

[Le14b] ———, *Growth of homology torsion in finite coverings and hyperbolic volume*, preprint, 2014.

[LRa10] K. B. Lee, F. Raymond, *Seifert Fiberings*, Mathematical Surveys and Monographs, vol. 166, American Mathematical Society, Providence, RI, 2010.

[Lee73] R. Lee, *Semicharacteristic classes*, Topology **12** (1973), 183–199.

[Leb95] B. Leeb, 3*-manifolds with(out) metrics of nonpositive curvature*, Invent. Math. **122** (1995), 277–289.

[Ler02] C. Leininger, *Surgeries on one component of the Whitehead link are virtually fibered*, Topology **41** (2002), no. 2, 307–320.

[LRe02] C. Leininger, A. Reid, *The co-rank conjecture for* 3*-manifold groups*, Alg. Geom. Top. **2** (2002), 37–50.

[Les12] C. Lescop, *An introduction to finite type invariants of knots and* 3*-manifolds*, ICPAM-ICTP Research School, Symplectic Geometry and Geometric Topology, 2012.

[LeL12] A. Levine, S. Lewallen, *Strong L-spaces and left orderability*, Math. Res. Lett. **19** (2012), no. 6, 1237–1244.

[Lev69] J. Levine, *Knot cobordism groups in codimension two*, Comment. Math. Helv. **44** (1969), 229–244.

[Lev78] ———, *Some results on higher dimensional knot groups*, in: *Knot Theory,* pp. 243–273, Lecture Notes in Mathematics, vol. 685, Springer-Verlag, Berlin-New York, 1978.

[Lev85] ———, *Lectures on groups of homotopy spheres*, in: *Algebraic and Geometric Topology,* pp. 62–95, Lecture Notes in Mathematics, vol. 1126, Springer-Verlag, Berlin, 1985.

[LiM93] J. Li, J. Millson, *On the first Betti number of a hyperbolic manifold with an arithmetic fundamental group*, Duke Math. J. **71** (1993), no. 2, 365–401.

[Li02] T. Li, *Immersed essential surfaces in hyperbolic* 3*-manifolds*, Comm. Anal. Geom. **10** (2002), no. 2, 275–290.

[Lia13] ———, *Rank and genus of* 3*-manifolds*, J. Amer. Math. Soc. **26** (2013), 777–829.

[LiZ06] W. Li, W. Zhang, *An L^2-Alexander invariant for knots*, Commun. Contemp. Math. **8** (2006), 167–187.

[Lib09] Y. Li, 2-*string free tangles and incompressible surfaces*, J. Knot Theory Ramifications **18** (2009), no. 8, 1081–1087.

[LiW14] Y. Li, L. Watson, *Genus one open books with non-left-orderable fundamental group*, Proc. Am. Math. Soc. **142** (2014), no. 4, 1425–1435.

[Lic62] W. B. R. Lickorish, *A representation of orientable combinatorial* 3-*manifolds*, Ann. of Math. (2) **76** (1962), 531–540.

[Lic97] ———, *An Introduction to Knot Theory*, Graduate Texts in Mathematics, vol. 175, Springer-Verlag, New York, 1997.

[LiR91] M. Lien, J. Ratcliffe, *On the uniqueness of HNN decompositions of a group*, J. Pure Appl. Algebra **75** (1991), no. 1, 51–62.

[LiN08] X.-S. Lin, S. Nelson, *On generalized knot groups*, J. Knot Theory Ramifications **17** (2008), no. 3, 263–272.

[Lin01] X. S. Lin, *Representations of knot groups and twisted Alexander polynomials*, Acta Math. Sin. (Engl. Ser.) **17** (2001), no. 3, 361–380.

[Lil93] P. A. Linnell, *Division rings and group von Neumann algebras*, Forum Math. **5** (1993), vol. 6, 561–576.

[Lil98] ———, *Analytic versions of the zero divisor conjecture*, Kropholler, Peter H. (ed.) et al., Geometry and cohomology in group theory. Cambridge: Cambridge University Press. Lond. Math. Soc. Lect. Note Ser. 252, 209–248 (1998).

[Lil06] ———, *Noncommutative localization in group rings*, in: *Non-commutative Localization in Algebra and Topology,* pp. 40–59, London Mathematical Society Lecture Note Series, vol. 330, Cambridge University Press, Cambridge, 2006.

[LLS11] P. Linnell, W. Lück, R. Sauer, *The limit of $\mathbb{F}_p$-Betti numbers of a tower of finite covers with amenable fundamental groups*, Proc. Amer. Math. Soc. **139** (2011), 421–434.

[LiS07] P. Linnell and P. Schick, *Finite group extensions and the Atiyah conjecture*, J. Amer. Math. Soc. **20** (2007), no. 4, 1003–1051.

[Liu13] Y. Liu, *Virtual cubulation of nonpositively curved graph manifolds*, J. Topol. **6** (2013), no. 4, 793–822.

[Liu14] ———, *A characterization of virtually embedded subsurfaces in* 3-*manifolds*, preprint, 2014.

[LMa13] Y. Liu, V. Markovic, *Homology of curves and surfaces in closed hyperbolic* 3-*manifolds*, preprint, 2013.

[Lö31] F. Löbell, *Beispiele geschlossener dreidimensionaler Clifford-Kleinscher Räume negativer Krümmung*, Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl. **83** (1931), 167–174.

[Lo87] D. Long, *Immersions and embeddings of totally geodesic surfaces*, Bull. London Math. Soc. **19** (1987), no. 5, 481–484.

[LLuR08] D. Long, A. Lubotzky, A. Reid, *Heegaard genus and Property τ for hyperbolic* 3-*manifolds*, J. Topol. **1** (2008), 152–158.

[LoN91] D. Long, G. Niblo, *Subgroup separability and* 3-*manifold groups*, Math. Z. **207** (1991), no. 2, 209–215.

[LO97] D. Long, U. Oertel, *Hyperbolic surface bundles over the circle*, in: *Progress in Knot Theory and Related Topics,* pp. 121–142, Travaux en Cours, vol. 56, Hermann, Paris, 1997.

[LoR98] D. Long, A. Reid, *Simple quotients of hyperbolic* 3-*manifold groups*, Proc. Amer. Math. Soc. **126** (1998), no. 3, 877–880.

[LoR01] ———, *The fundamental group of the double of the figure-eight knot exterior is GFERF*. Bull. London Math. Soc. **33** (2001), no. 4, 391–396.

[LoR05] ———, *Surface subgroups and subgroup separability in* 3-*manifold topology*, Publicações Matemáticas do IMPA, Instituto Nacional de Matemática Pura e Aplicada, Rio de Janeiro, 2005.

[LoR08a] ———, *Subgroup separability and virtual retractions of groups*, Topology **47** (2008), no. 3, 137–159.

[LoR08b] ———, *Finding fibre faces in finite covers*, Math. Res. Lett. **15** (2008), no. 3, 521–524.

[LoR11] ———, *Grothendieck's problem for 3-manifold groups*, Groups Geom. Dyn. **5** (2011), no. 2, 479–499.

[Lop92] L. Lopez, *Alternating knots and non-Haken 3-manifolds*, Topology Appl. **48** (1992), no. 2, 117–146.

[Lop93] ———, *Small knots in Seifert fibered 3-manifolds*, Math. Z. **212** (1993), no. 1, 123–139.

[Lop94] ———, *Residual finiteness of surface groups via tessellations*, Discrete Comput. Geom. **11** (1994), no. 2, 201–211.

[Lor08] K. Lorensen, *Groups with the same cohomology as their profinite completions*, J. Algebra **320** (2008), no. 1, 1704–1722.

[Lot07] J. Lott, *The work of Grigory Perelman*, in: *International Congress of Mathematicians,* vol. I, pp. 66–76, European Mathematical Society, Zürich, 2007.

[LoL95] J. Lott, W. Lück, *L^2-topological invariants of 3-manifolds*, Invent. Math. **120** (1995), no. 1, 15–60.

[Lou14] L. Louder, *Simple loop conjecture for limit groups*, Israel J. Math. **199** (2014), 527–545.

[Lub83] A. Lubotzky, *Group presentation, p-adic analytic groups and lattices in* $\mathrm{SL}_2(\mathbb{C})$, Ann. of Math. (2) **118** (1983), 115–130.

[Lu88] ———, *A group theoretic characterization of linear groups*, J. Algebra **113** (1988), no. 1, 207–214.

[Lub94] ———, *Discrete Groups, Expanding Graphs and Invariant Measures*, Progress in Mathematics, vol. 125, Birkhäuser Verlag, Basel, 1994.

[Lub95] ———, *Subgroup growth and congruence subgroups*, Invent. Math. **119** (1995), no. 2, 267–295.

[Lub96a] ———, *Eigenvalues of the Laplacian, the first Betti number and the congruence subgroup problem*, Ann. of Math. (2) **144** (1996), no. 2, 441–452.

[Lub96b] ———, *Free quotients and the first Betti number of some hyperbolic manifolds*, Transform. Groups **1** (1996), no. 1-2, 71–82.

[LMW14] A. Lubotzky, J. Maher, C. Wu, *Random methods in 3-manifold theory*, preprint, 2014.

[LMe11] A. Lubotzky, C. Meiri, *Sieve methods in group theory* II*: the mapping class group*, Geom. Dedicata **159** (2012), 327–336.

[LuSe03] A. Lubotzky, D. Segal, *Subgroup Growth*, Progress in Mathematics, vol. 212, Birkhäuser Verlag, Basel, 2003.

[LuSh04] A. Lubotzky, Y. Shalom, *Finite representations in the unitary dual and Ramanujan groups*, in: *Discrete Geometric Analysis,* pp. 173–189, Contemporary Mathematics, vol. 347, American Mathematical Society, Providence, RI, 2004.

[LuZ03] A. Lubotzky, A. Zuk, *On Property τ*, preliminary version of a book, 2003. `http://www.ma.huji.ac.il/~alexlub/BOOKS/Onproperty/Onproperty.pdf`

[Lü94] W. Lück, *Approximating L^2-invariants by their finite-dimensional analogues*, Geom. Funct. Anal. **4** (1994), no. 4, 455–481.

[Lü02] ———, *L^2-Invariants: Theory and Applications to Geometry and K-Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, vol. 44, Springer-Verlag, Berlin, 2002.

[Lü13] ———, *Approximating L^2-invariants and homology growth*, Geom. Funct. Anal. **23** (2013), no. 2, 622–663.

[Lü15] ———, *Survey on approximating L^2-invariants by their classical counterparts: Betti numbers, torsion invariants and homological growth*, preprint, 2015.

[LüS99] W. Lück, T. Schick, *L^2-torsion of hyperbolic manifolds of finite volume*, Geom. Funct. Anal. **9** (1999), no. 3, 518–567.

[Lue88] J. Luecke, *Finite covers of 3–manifolds containing essential tori*, Trans. Amer. Math. Soc. **310** (1988), 381–391.

[LuW93] J. Luecke, Y.-Q. Wu, *Relative Euler number and finite covers of graph manifolds*, in: *Geometric Topology,* pp. 80–103, AMS/IP Studies in Advanced Mathematics, vol. 2.1, Amer. Math. Soc., Providence, RI; International Press, Cambridge, MA, 1997.

[Luo12] F. Luo, *Solving Thurston equation in a commutative ring*, preprint, 2012.

[LMo99] M. Lustig, Y. Moriah, *Closed incompressible surfaces in complements of wide knots and links*, Topology Appl. **92** (1999), no. 1, 1–13.

[LyS77] R. C. Lyndon, P. E. Schupp, *Combinatorial Group Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 89, Springer-Verlag, Berlin, 1977.

[Ly71] H. C. Lyon, *Incompressible surfaces in knot spaces*, Trans. Amer. Math. Soc. **157** (1971), 53–62.

[Ly74a] ———, *Simple knots with unique spanning surfaces*, Topology **13** (1974), 275–279.

[Ly74b] ———, *Simple knots without unique minimal surfaces*, Proc. Amer. Math. Soc. **43** (1974), 449–454.

[Ma12] J. Ma, *Homology-genericity, horizontal Dehn surgeries and ubiquity of rational homology* 3*-spheres*, Proc. Amer. Math. Soc. **140** (2012), no. 11, 4027–4034.

[Ma14] ———, *The closure of a random braid is a hyperbolic link*, Proc. Am. Math. Soc. **142** (2014), no. 2, 695–701.

[MQ05] J. Ma, R. Qiu, 3*-manifold containing separating incompressible surfaces of all positive genera*, Acta Math. Sin. (Engl. Ser.) **21** (2005), no. 6, 1315–1318.

[MaM08] M. Macasieb, T. W. Mattman, *Commensurability classes of* $(2;3;n)$*-pretzel knot complements*, Algebr. Geom. Topol. **8** (2008), 1833–1853.

[MaR03] C. MacLachlan, A. Reid, *The Arithmetic of Hyperbolic* 3*-Manifolds*, Graduate Text in Mathematics, vol. 219, Springer-Verlag, Berlin, 2003.

[Mah05] J. Maher, *Heegaard gradient and virtual fibers*, Geom. Topol. **9** (2005), 2227–2259.

[Mah10a] ———, *Random Heegaard splittings*, J. Topol. **3** (2010), no. 4, 997–1025.

[Mah10b] ———, *Asymptotics for pseudo-Anosov elements in Teichmüller lattices*, Geom. Funct. Anal. **20** (2010), no. 2, 527–544.

[Mah11] ———, *Random walks on the mapping class group*, Duke Math. J. **156** (2011), no. 3, 429–468.

[Mah12] ———, *Exponential decay in the mapping class group*, J. London Math. Soc. **86** (2012), no. 2, 366–386.

[Mai01] S. Maillot, *Quasi-isometries of groups, graphs and surfaces*, Comment. Math. Helv. **76** (2001), no. 1, 29–60.

[Mai03] ———, *Open* 3*-manifolds whose fundamental groups have infinite center, and a torus theorem for* 3*-orbifolds*, Trans. Amer. Math. Soc. **355** (2003), no. 11, 4595–4638.

[Mal40] A. I. Mal′cev, *On faithful representations of infinite groups of matrices*, Mat. Sb. **8** (1940), 405–422.

[Mal65] ———, *On faithful representations of infinite groups of matrices*, Amer. Math. Soc. Transl. **45** (1965), no. 2, 1–18.

[MlS13] J. Malestein, J. Souto, *On genericity of pseudo-Anosovs in the Torelli group*, Int. Math. Res. Notices 2013, no. 6, 1434–1449.

[Mnna11] A. Mann, *The growth of free products*, J. Algebra **326** (2011), 208–217.

[Mnna12] ———, *How Groups Grow*, London Math. Soc. Lecture Note Series, vol. 395, Cambridge University Press, 2012.

[Mnnb14] K. Mann, *A counterexample to the simple loop conjecture for* $\mathrm{PSL}(2,\mathbb{R})$, Pacific J. Math. **269** (2014), no. 2, 425–432.

[Mng02] J. Manning, *Algorithmic detection and description of hyperbolic structures on closed* 3*-manifolds with solvable word problem*, Geom. Topol. **6** (2002), 1–25.

[MMP10] J. F. Manning, E. Martinez-Pedroza, *Separation of relatively quasi-convex subgroups*, Pacific J. Math. **244** (2010), no. 2, 309–334.

[Mau13] C. Manolescu, Pin(2)-equivariant Seiberg-Witten Floer homology and the Triangulation Conjecture, preprint, 2013.

[Man74] A. Marden, *The geometry of finitely generated Kleinian groups*, Ann. of Math. (2) **99** (1974), 383–462.

[Man07] ———, *Outer Circles*, Cambridge University Press, Cambridge, 2007.

[MMt79] A. Marden, B. Maskit, *On the isomorphism theorem for Kleinian groups*, Invent. Math. **51** (1979), 9–14.

[MrS81] G. Margulis, G. Soifer, *Maximal subgroups of infinite index in finitely generated linear groups*, J. Algebra **69** (1981), no. 1, 1–23.

[Mav58] A. A. Markov, *The insolubility of the problem of homeomorphy*, Dokl. Akad. Nauk SSSR **121** (1958), 218–220.

[Mav60] ———, *Insolubility of the problem of homeomorphy*, in: *Proceedings of the International Congress of Mathematicians 1958,* pp. 300–306, Cambridge University Press, New York, 1960.

[Mar14] M. Marengon, *On d-invariants and generalised Kanenobu knots*, preprint, 2014.

[Mac13] V. Markovic, *Criterion for Cannon's Conjecture*, Geom. Funct. Anal. **23** (2003), 1035–1061.

[MP09] E. Martinez-Pedroza, *Combination of quasiconvex subgroups of relatively hyperbolic groups*, Groups Geom. Dyn. **3** (2009), no. 2, 317–342.

[Mao07] A. Martino, *A proof that all Seifert* 3*-manifold groups and all virtual surface groups are conjugacy separable*, J. Algebra **313** (2007), no. 2, 773–781.

[MMn12] A. Martino, A. Minasyan, *Conjugacy in normal subgroups of hyperbolic groups*, Forum Math. **24** (2012), no. 5, 889–909.

[Mai14] H. Masai, *Fibered commensurability and arithmeticity of random mapping tori*, preprint, 2014.

[Msy81] W. Massey, *Algebraic Topology*, Graduate Texts in Mathematics, vol. 56, Springer-Verlag, New York-Heidelberg-Berlin, 1981.

[Mas00] J. Masters, *Virtual homology of surgered torus bundles*, Pacific J. Math. **195** (2000), no. 1, 205–223.

[Mas02] ———, *Virtual Betti numbers of genus* 2 *bundles*, Geom. Topol. **6** (2002), 541–562 (electronic).

[Mas06a] ———, *Heegaard splittings and* 1*-relator groups*, unpublished paper, 2006.

[Mas06b] ———, *Thick surfaces in hyperbolic* 3*-manifolds*, Geom. Dedicata **119** (2006), 17–33.

[Mas07] ———, *Virtually Haken surgeries on once-punctured torus bundles*, Comm. Anal. Geom. **15** (2007), no. 4, 733–756.

[MaZ08] J. Masters, X. Zhang, *Closed quasi-Fuchsian surfaces in hyperbolic knot complements*, Geom. Topol. **12** (2008), 2095–2171.

[MaZ09] ———, *Quasi-Fuchsian surfaces in hyperbolic link complements*, unpublished paper, `arXiv:0909.4501`, 2009.

[MMZ04] J. Masters, W. Menasco, X. Zhang, *Heegaard splittings and virtually Haken Dehn filling*, New York J. Math. **10** (2004), 133–150.

[MMZ09] ———, *Heegaard splittings and virtually Haken Dehn filling,* I, New York J. Math. **15** (2009), 1–17.

[MMy99] H. A. Masur, Y. N. Minsky, *Geometry of the complex of curves* I*: Hyperbolicity*, Invent. Math. **138** (1998), 103–139.

[MMy00] ———, *Geometry of the complex of curves* II*: Hierarchical structure*, Geom. Funct. Anal. **10** (2000), 902–974.

[Mad04] H. Matsuda, *Small knots in some closed Haken* 3*-manifolds*, Topology Appl. **135** (2004), no. 1-3, 149–183.

[Mat97a] S. Matsumoto, *A* 3*-manifold with a non-subgroup-separable fundamental group*, Bull. Austral. Math. Soc. **55** (1997), no. 2, 261–279.

[Mat97b] ———, *Non-separable surfaces in cubed manifolds*, Proc. Amer. Math. Soc. **125** (1997), no. 11, 3439–3446.

[MOP08] M. Matthey, H. Oyono-Oyono and W. Pitsch, *Homotopy invariance of higher signatures and* 3*-manifold groups*, Bull. Soc. Math. France **136** (2008), no. 1, 1–25.

[Mae82] S. V. Matveev, *Distributive groupoids in knot theory*, Mat. Sb. (N.S.) **119 (161)** (1982), no. 1, 78–88.

[Mae03] ———, *Algorithmic Topology and Classification of* 3*-Manifolds*, Algorithms and Computation in Mathematics, vol. 9, Springer-Verlag, Berlin, 2003.

[May72] E. Mayland, *On residually finite knot groups*, Trans. Amer. Math. Soc. **168** (1972), 221–232.

[May74] ———, *Two-bridge knots have residually finite groups*, in: *Proceedings of the Second International Conference on the Theory of Groups,* pp. 488–493, Lecture Notes in Mathematics, vol. 372, Springer-Verlag, Berlin-New York, 1974.

[May75a] ———, *The residual finiteness of the classical knot groups*, Canad. J. Math. **27** (1975), no. 5, 1092–1099.

[May75b] ———, *The residual finiteness of the groups of classical knots*, in: *Geometric Topology*, pp. 339–342, Lecture Notes in Mathematics, vol. 438, Springer-Verlag, Berlin-New York, 1975.

[MMi76] E. Mayland, K. Murasugi, *On a structural property of the groups of alternating links*, Canad. J. Math. **28** (1976), no. 3, 568–588.

[McC86] D. McCullough, *Mappings of reducible* 3*-manifolds*, in: *Geometric and Algebraic Topology,* pp. 61–76, Banach Center Publications, vol. 18, PWN—Polish Scientific Publishers, Warsaw, 1986.

[McC90] ______, *Topological and algebraic automorphisms of* 3*-manifolds*, in: *Groups of Self-equivalences and Related Topics*, pp. 102–113, Lecture Notes in Mathematics, vol. 1425, Springer-Verlag, Berlin, 1990.

[McC91] ______, *Virtually geometrically finite mapping class groups of* 3*-manifolds*, J. Differential Geom. **33** (1991), no. 1, 1–65.

[McC95] ______, 3*-Manifolds and their Mappings*, Lecture Notes Series, vol. 26, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1995.

[McM92] C. McMullen, *Riemann surfaces and the geometrization of* 3*-manifolds*, Bull. Amer. Math. Soc. (N.S.) **27** (1992), no. 2, 207–216.

[McM96] ______, *Renormalization and* 3*-Manifolds which fiber over the Circle*, Annals of Mathematics Studies, vol. 142, Princeton University Press, Princeton, NJ, 1996.

[McM02] ______, *The Alexander polynomial of a* 3*–manifold and the Thurston norm on cohomology*, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 2, 153–171.

[McM11] ______, *The evolution of geometric structures on* 3*-manifolds*, Bull. Amer. Math. Soc. (N.S.) **48** (2011), no. 2, 259–274.

[MeZ04] M. Mecchia, B. Zimmermann, *On finite groups acting on* $\mathbb{Z}_2$*-homology* 3*-spheres*, Math. Z. **248** (2004), no. 4, 675–693.

[MeZ06] ______, *On finite simple groups acting on integer and mod* 2 *homology* 3*-spheres*, J. Algebra **298** (2006), no. 2, 460–467.

[MeS86] W. H. Meeks, P. Scott, *Finite group actions on* 3*-manifolds*, Invent. Math. **86** (1986), 287–346.

[MSY82] W. H. Meeks, L. Simon, S. Yau, *Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature*, Ann. of Math. (2) **116** (1982), no. 3, 621–659.

[MFP14] P. Menal-Ferrer, J. Porti, *Higher dimensional Reidemeister torsion invariants for cusped hyperbolic* 3*-manifolds*, J. Topol. **7** (2014), no. 1, 69–119.

[Men84] W. Menasco, *Closed incompressible surfaces in alternating knot and link complements*, Topology **23** (1984), no. 1, 37–44.

[Mes88] G. Mess, *The Seifert conjecture and groups which are coarse quasiisometric to planes*, unpublished paper, 1988.
`https://lamington.files.wordpress.com/2014/08/mess_seifert_conjecture.pdf`

[Mes90] ______, *Finite covers of* 3*-manifolds and a theorem of Lubotzky*, I.H.E.S. preprint, 1990.

[Mila92] C. Miller, *Decision problems for groups: survey and reflections*, in: *Algorithms and Classification in Combinatorial Group Theory,* pp. 1–59, Mathematical Sciences Research Institute Publications, vol. 23, Springer-Verlag, New York, 1992.

[Milb82] R. Miller, *Geodesic laminations from Nielsen's viewpoint*, Adv. in Math. **45** (1982), no. 2, 189–212.

[Milb84] ______, *A new proof of the homotopy torus and annulus theorem*, in: *Four-Manifold Theory*, pp. 407–435, Contemporary Mathematics, vol. 35, American Mathematical Society, Providence, RI, 1984.

[Mie09] P. Milley, *Minimum volume hyperbolic* 3*-manifolds*, J. Topol. **2** (2009), no. 1, 181–192.

[Mis76] J. Millson, *On the first Betti number of a constant negatively curved manifold*, Ann. of Math. (2) **104** (1976), no. 2, 235–247.

[Mil56] J. Milnor, *On manifolds homeomorphic to the* 7*-sphere*, Ann. of Math. (2) **64** (1956), 399–405.

[Mil57] ______, *Groups which act on* S^n *without fixed points*, Amer. J. Math. **79** (1957), 623–630.

[Mil62] ______, *A unique factorization theorem for* 3*-manifolds*, Amer. J. Math. **84** (1962), 1–7.

[Mil66] ______, *Whitehead torsion*, Bull. Amer. Math. Soc. **72** (1966), 358–426.

[Mil68] ______, *A note on curvature and fundamental group*, J. Differential Geom. **2** (1968), 1–7.

[Mil82] ______, *Hyperbolic geometry: the first 150 years*, Bull. Amer. Math. Soc. (N.S.) **6** (1982), no. 1, 9–24.

[Mil03] ———, *Towards the Poincaré conjecture and the classification of* 3*-manifolds*, Notices Amer. Math. Soc. **50** (2003), 1226–1233.

[Mil04] ———, *The Poincaré conjecture one hundred years later*, in the Collected Papers of John Milnor. IV: Homotopy, homology and manifolds. Edited by John McCleary. Providence, RI: American Mathematical Society (AMS). (2009).

[Min12] A. Minasyan, *Hereditary conjugacy separability of right angled Artin groups and its applications*, Groups Geom. Dyn. **6** (2012), 335–388.

[MO13] A. Minasyan, D. Osin, *Acylindrical hyperbolicity of groups acting on trees*, Math. Ann., to appear.

[Miy94] Y. Minsky, *On Thurston's ending lamination conjecture*, in: *Low-Dimensional Topology*, pp. 109–122, Conference Proceedings and Lecture Notes in Geometry and Topology, vol. III, International Press, Cambridge, MA, 1994.

[Miy00] ———, *Short geodesics and end invariants*, in: *Fukuso rikigakkei to sho kanren bunya no sōgōteki kenkyū,* pp. 1–19, Kyoto University, Research Institute for Mathematical Sciences, Kyoto, 2000.

[Miy03] ———, *End invariants and the classification of hyperbolic* 3*-manifolds*, in: *Current Developments in Mathematics, 2002,* pp. 181–217, International Press, Somerville, MA, 2003.

[Miy06] ———, *Curve complexes, surfaces and* 3*-manifolds*, in: *International Congress of Mathematicians,* II, pp. 1001–1033, European Mathematical Society, Zürich, 2006.

[Miy10] ———, *The classification of Kleinian surface groups,* I*. Models and bounds*, Ann. of Math. (2) **171** (2010), no. 1, 1–107.

[Miz08] N. Mizuta, *A Bozejko-Picardello type inequality for finite-dimensional CAT(0) cube complexes*, J. Funct. Anal. **254** (2008), no. 3, 760–772.

[Moi52] E. Moise, *Affine structures in* 3*-manifolds,* V*. The triangulation theorem and Hauptvermutung*, Ann. of Math. (2) **56** (1952), 96–114.

[Moi77] ———, *Geometric Topology in Dimensions* 2 *and* 3, Graduate Texts in Mathematics, vol. 47, Springer-Verlag, New York-Heidelberg, 1977.

[Moi80] ———, *Statically tame periodic homeomorphisms of compact connected* 3*-manifolds,* II*. Statically tame implies tame*, Trans. Amer. Math. Soc. **259** (1980), 255–280.

[Mon74] J. Montesinos, *A representation of closed, orientable* 3*-manifolds as* 3*-fold branched coverings of* S^3, Bull. Amer. Math. Soc **80** (1974), 845–846.

[Mon83] ———, *Representing* 3*-manifolds by a universal branching set*, Math. Proc. Cambridge Philos. Soc. **94** (1983), no. 1, 109–123.

[Mon86] ———, *On* 3*-manifolds having surface-bundles as branched coverings*, in: *Contribuciones Matemáticas,* pp. 197–201, Editorial de la Universidad Complutense de Madrid, Facultad de Ciencias Matemáticas, Madrid, 1986.

[Mon87] ———, *On* 3*-manifolds having surface bundles as branched coverings*, Proc. Amer. Math. Soc. **101** (1987), no. 3, 555–558.

[Mon89] ———, *Discrepancy between the rank and the Heegaard genus of a* 3*-manifold*, in: *Giornate di Studio su Geometria Differenziale Topologia,* pp. 101–117, Note Mat. **9** (1989), suppl., Università degli Studi di Lecce, Lecce, 1989.

[Moo05] M. Moon, *A generalization of a theorem of Griffiths to* 3*-manifolds*, Topology Appl. **149** (2005), no. 1-3, 17–32.

[Mor84] J. Morgan, *Thurston's uniformization theorem for three dimensional manifolds*, in: *The Smith Conjecture,* pp. 37–125, Pure and Applied Mathematics, vol. 112, Academic Press, Inc., Orlando, FL, 1984.

[Mor05] ———, *Recent progress on the Poincaré conjecture and the classification of* 3*-manifolds*, Bull. Amer. Math. Soc. (N.S.) **42** (2005), no. 1, 57–78.

[MTi07] J. Morgan, G. Tian, *Ricci Flow and the Poincaré Conjecture*, Clay Mathematics Monographs, vol. 3, Amer. Math. Soc., Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2007.

[MTi14] ———, *The Geometrization Conjecture*, Clay Mathematics Monographs, vol. 5, Amer. Math. Soc., Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2014.

[Moa86] S. Morita, *Finite coverings of punctured torus bundles and the first Betti number*, Sci. Papers College Arts Sci. Univ. Tokyo **35** (1986), no. 2, 109–121.

[Mos68] G. D. Mostow, *Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms,* Inst. Hautes Études Sci. Publ. Math. **34** (1968), 53–104.

[MTe13] K. Motegi, M. Teragaito, *Left-orderable, non-L-space surgeries on knots*, preprint, 2013.

[Muk80] H. J. Munkholm, *Simplices of maximal volume in hyperbolic space, Gromov's norm, and Gromov's proof of Mostow's rigidity theorem* (*following Thurston*), in: *Topology Symposium, Siegen 1979,* pp. 109–124, Lecture Notes in Mathematics, vol. 788, Springer, Berlin, 1980.

[Mun59] J. Munkres, *Obstructions to the smoothing of piecewise-differentiable homeomorphisms*, Bull. Amer. Math. Soc. **65** (1959), 332–334.

[Mun60] ———, *Obstructions to the smoothing of piecewise-differentiable homeomorphisms*, Ann. of Math. (2) **72** (1960), 521–554.

[Mur65] K. Murasugi, *On the center of the group of a link*, Proc Amer. Math. Soc. **16** (1965), 1052–1057.

[MyR96] A. G. Myasnikov, V. N. Remeslennikov, *Exponential groups,* II. *Extensions of centralizers and tensor completion of CSA-groups,* Internat. J. Algebra Comput. **6** (1996), no. 6, 687–711.

[Mye82] R. Myers, *Simple knots in compact, orientable* 3-*manifolds*, Trans. Amer. Math. Soc. **273** (1982), no. 1, 75–91.

[Mye00] ———, *Splitting homomorphisms and the geometrization conjecture*, Math. Proc. Cambridge Philos. Soc. **129** (2000), no. 2, 291–300.

[Myr41] P. J. Myrberg, *Die Kapazität der singulären Menge der linearen Gruppen*, Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. (1941), no. 10.

[Nak13] Y. Nakae, *A good presentation of* $(-2, 3, 2s+1)$*-type pretzel knot group and* $\mathbb{R}$*-covered foliation*, J. Knot Theory Ramifications **22** (2013), no. 1, 1250143.

[Nag14] M. Nagel, *Minimal genus in circle bundles over* 3-*manifolds*, preprint, 2014.

[NTY14] Y. Nakagawa, M. Tamura, Y. Yamashita, *The growth of torus link groups*, preprint, 2014.

[NaS09] H. Namazi, J. Souto, *Heegaard splittings and pseudo-Anosov maps*, Geom. Funct. Anal. **19** (2009), 1195–1228.

[NaR14] G. Naylor, D. Rolfsen, *Generalized torsion in knot groups*, preprint, 2014.

[NN08] S. Nelson, W. Neumann, *The* 2-*generalized knot group determines the knot*, Commun. Contemp. Math. **10** (2008), suppl. 1, 843–847.

[Nema49] B. H. Neumann, *On ordered groups*, Amer. J. Math. **71** (1949), 1–18.

[Nemb76] D. A. Neumann, 3-*manifolds fibering over* S^1, Proc. Amer. Math. Soc. **58** (1976), 353–356.

[Nemc73] P. Neumann, *The SQ-universality of some finitely presented groups*, J. Austral. Math. Soc. **16** (1973), 1–6.

[Nemd79] W. Neumann, *Normal subgroups with infinite cyclic quotient*, Math. Sci. **4** (1979), no. 2, 143–148.

[Nemd96] ———, *Commensurability and virtual fibration for graph manifolds*, Topology **39** (1996), 355–378.

[Nemd99] ———, *Notes on geometry and* 3-*manifolds*, in: *Low Dimensional Topology,* pp. 191–267, Bolyai Society Mathematical Studies, vol. 8, János Bolyai Mathematical Society, Budapest, 1999.

[NeR92] W. Neumann, A. W. Reid, *Arithmetic of hyperbolic manifolds*, in: *Topology '90,* pp. 273–310, Ohio State University Mathematical Research Institute Publications, vol. 1, Walter de Gruyter & Co., Berlin, 1992.

[Neh60] L. Neuwirth, *The algebraic determination of the genus of knots*, Amer. J. Math. **82** (1960), 791–798.

[Neh61a] ———, *The algebraic determination of the topological type of the complement of a knot*, Proc. Amer. Math. Soc. **12** (1961), 904–906.

[Neh61b] ———, *An alternative proof of a theorem of Iwasawa on free groups*, Proc. Cambridge Philos. Soc. **57** (1961), 895–896.

[Neh63a] ———, *A remark on knot groups with a center*, Proc. Amer. Math. Soc. **14** (1963), 378–379

[Neh63b] ———, *On Stallings fibrations*, Proc. Amer. Math. Soc. **14** (1963), 380–381.

[Neh65] ———, *Knot Groups*, Annals of Mathematics Studies, vol. 56, Princeton University Press, Princeton, NJ, 1965.

[Neh68] ———, *An algorithm for the construction of* 3*-manifolds from* 2*-complexes*, Proc. Cambridge Philos. Soc. **64** (1968), 603–613.

[Neh70] ———, *Some algebra for* 3*-manifolds*, in: *Topology of Manifolds,* pp. 179–184, Markham, Chicago, IL, 1970.

[Neh74] ———, *The status of some problems related to knot groups*, in: *Topology Conference*, pp. 209–230, Lecture Notes in Mathematics, vol. 375, Springer-Verlag, Berlin-New York, 1974.

[New85] M. Newman, *A note on Fuchsian groups,* Illinois J. Math. **29** (1985), no. 4, 682–686.

[Nib90] G. Niblo, *H.N.N. extensions of a free group by* $\mathbb{Z}$ *which are subgroup separable*, Proc. London Math. Soc. (3) **61** (1990), no. 1, 18–32.

[Nib92] G. Niblo, *Separability properties of free groups and surface groups*, J. Pure Appl. Algebra **78** (1992), no. 1, 77–84.

[NW98] G. A. Niblo, D. T. Wise, *The engulfing property for* 3*-manifolds*, in: *The Epstein Birthday Schrift,* pp. 413-418, Geometry & Topology Monographs, vol. 1, Geometry & Topology Publications, Coventry, 1998.

[NW01] ———, *Subgroup separability, knot groups and graph manifolds*, Proc. Amer. Math. Soc. **129** (2001), 685–693.

[Nic13] B. Nica, *Linear groups—Malcev's theorem and Selberg's lemma*, preprint, 2013.

[Nie44] J. Nielsen, *Surface transformation classes of algebraically finite type*, Danske Vid. Selsk. Math.-Phys. Medd. **21** (1944), no. 2.

[NiS03] N. Nikolov, D. Segal, *Finite index subgroups in profinite groups*, C. R. Math. Acad. Sci. Paris **337** (2003), no. 5, 303–308.

[NiS07] ———, *On finitely generated profinite groups,* I. *Strong completeness and uniform bounds*, Ann. of Math. (2) **165** (2007), no. 1, 171–238.

[Nis40] V. L. Nisnevîc, Über Gruppen, die durch Matrizen über einem kommutativen Feld isomorph darstellbar sind, Mat. Sbornik (N.S.) **8 (50)** (1940), 395–403.

[No67] D. Noga, *Über den Aussenraum von Produktknoten und die Bedeutung der Fixgruppen*, Math. Z. **101** (1967), 131–141.

[NuR11] V. Nuñez, J. Remigio, *A note on Heegaard genera of covering spaces of* 3*-manifolds*, preprint, 2011.
`http://www.cimat.mx/~victor/noteheecov2.pdf`

[Oe84] U. Oertel, *Closed incompressible surfaces in complements of star links*, Pacific J. Math. **111** (1984), no. 1, 209–230.

[Oe86] ———, *Homology branched surfaces: Thurston's norm on* $H_2(M)$, in: *Low-dimensional Topology and Kleinian Groups,* pp. 253–272, London Mathematical Society Lecture Note Series, vol. 112, Cambridge University Press, Cambridge, 1986.

[Oh02] K. Ohshika, *Discrete Groups*, Translations of Mathematical Monographs, vol. 207, Iwanami Series in Modern Mathematics, Amer. Math. Soc., Providence, RI, 2002.

[Or72] P. Orlik, *Seifert Manifolds*, Lecture Notes in Mathematics, vol. 291, Springer-Verlag, Berlin-New York, 1972.

[OVZ67] P. Orlik, E. Vogt, H. Zieschang, *Zur Topologie gefaserter dreidimensionaler Mannigfaltigkeiten*, Topology **6** (1967), 49–64.

[Osb78] R. Osborne, *Simplifying spines of* 3*-manifolds*, Pacific J. Math. **74** (1978), no. 2, 473–480.

[OsS74] R. Osborne, R. Stevens, *Group presentations corresponding to spines of* 3*-manifolds,* I, Amer. J. Math. **96** (1974), 454–471.

[OsS77a] ———, *Group presentations corresponding to spines of* 3*-manifolds*, III, Trans. Amer. Math. Soc. **234** (1977), no. 1, 245–251.

[OsS77b] ———, *Group presentations corresponding to spines of* 3*-manifolds*, II, Trans. Amer. Math. Soc. **234** (1977), no. 1, 213–243.

[Osi06] D. Osin, *Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems*, Mem. Amer. Math. Soc. **179** (2006), no. 843.

[Osi07] ———, *Peripheral fillings of relatively hyperbolic groups*, Invent. Math. **167** (2007), no. 2, 295–326.

[Ot96] J. Otal, *Le théorème d'hyperbolisation pour les variétés fibrées de dimension* 3, Astérisque **235** (1996).

[Ot98] ______, *Thurston's hyperbolization of Haken manifolds*, in: *Surveys in Differential Geometry,* vol. III, pp. 77–194, International Press, Boston, MA, 1998.

[Ot01] ______, *The Hyperbolization Theorem for Fibered* 3*-Manifolds*, SMF/AMS Texts and Monographs, vol. 7, American Mathematical Society, Providence, RI, Société Mathématique de France, Paris, 2001.

[Ot14] ______, *William P. Thurston: "Three-dimensional manifolds, Kleinian groups and hyperbolic geometry"*, Jahresber. Dtsch. Math.-Ver. **116** (2014), 3–20.

[Oza08] N. Ozawa, *Weak amenability of hyperbolic groups*, Groups Geom. Dyn. **2** (2008), no. 2, 271–280.

[Ozb08] M. Ozawa, *Morse position of knots and closed incompressible surfaces*, J. Knot Theory Ramifications **17** (2008), no. 4, 377–397.

[Ozb09] ______, *Closed incompressible surfaces of genus two in* 3*-bridge knot complements*, Topology Appl. **156** (2009), no. 6, 1130–1139.

[Ozb10] ______, *Rational structure on algebraic tangles and closed incompressible surfaces in the complements of algebraically alternating knots and links*, Topology Appl. **157** (2010), no. 12, 1937–1948.

[OzS04a] P. Ozsváth, Z. Szabó, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. of Math. (2) **159** (2004), 1159–1245.

[OzS04b] ______, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2) **159** (2004), 1027–1158.

[OzS04c] ______, *Holomorphic disks and genus bounds*, Geom. Topol. **8** (2004), 311–334.

[OzS05] ______, *On knot Floer homology and lens space surgeries*, Topology **44** (2005), no. 6, 1281–1300.

[PP12] D. Panov and A. Petrunin, *Telescopic actions*, Geom. Funct. Anal. **22** (2012), 1814–1831.

[Pao13] L. Paoluzzi, *The notion of commensurability in group theory and geometry*, RIMS Kôkyûroku, 2013.

[Pap57a] C. Papakyriakopoulos, *On Dehn's lemma and the asphericity of knots*, Ann. of Math. (2) **66** (1957), 1–26.

[Pap57b] ______, *On solid tori*, Proc. London Math. Soc. (3) **7** (1957), 281–299.

[Pas13] N. Pappas, *Rank gradient and p-gradient of amalgamated free products and HNN extensions*, preprint, 2013.

[Par92] W. Parry, *A sharper Tits alternative for* 3*-manifold groups*, Israel J. Math. **77** (1992), no. 3, 265–271.

[Par07] ______, *Examples of growth series of torus bundle groups*, J. Group Theory **10** (2007), no. 2, 245–266.

[Pas77] D. S. Passman, *The Algebraic Structure of Group Rings*, Pure and Applied Mathematics, Wiley-Interscience, New York-London-Sydney, 1977.

[Pat14] P. Patel, *On a theorem of Peter Scott*, Proc. Am. Math. Soc. **142** (2014), no. 8, 2891–2906.

[Per02] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, `arXiv:math/0211159`, 2002.

[Per03a] ______, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, `arXiv:math/0307245`, 2003.

[Per03b] ______, *Ricci flow with surgery on three-manifolds*, `arXiv:math/0303109`, 2003.

[PR03] B. Perron, D. Rolfsen, *On orderability of fibered knot groups*, Math. Proc. Cambridge Philos. Soc. **135** (2003), no. 1, 147–153.

[PR06] ______, *Invariant ordering of surface groups and* 3*-manifolds which fiber over* S^1, Math. Proc. Cambridge Philos. Soc. **141** (2006), no. 2, 273–280.

[Pes93] L. Person, *A piecewise linear proof that the singular norm is the Thurston norm*, Topology Appl. **51** (1993), no. 3, 269–289.

[Pet09] T. Peters, *On L-spaces and non left-orderable* 3*-manifold groups*, preprint, 2009.

[Pf13] J. Pfaff, *Exponential growth of homological torsion for towers of congruence subgroups of Bianchi groups*, preprint, 2013.

[Pin03] M. Picantin, *Automatic structures for torus link groups*, J. Knot Theory Ramifications **12** (2003), no. 6, 833–866.

[Pil74] P. Pickel, *Metabelian groups with the same finite quotients*, Bull. Austral. Math. Soc. **11** (1974), 115–120.

[Pie84] J.-P. Pier, *Amenable Locally Compact Groups*, Pure and Applied Mathematics, John Wiley & Sons, Inc., New York, 1984.

[Pla66] V. Platonov, *The Frattini subgroup of linear groups and finite approximability*, Dokl. Akad. Nauk SSSR **171** (1966), 798–801.

[Pla68] ———, *A certain problem for finitely generated groups*, Dokl. Akad. Nauk BSSR **12** (1968), 492–494.

[PT86] V. Platonov, O. Tavgen′, *On the Grothendieck problem of profinite completions of groups*, Dokl. Akad. Nauk SSSR **288** (1986), no. 5, 1054–1058.

[Plo80] S. Plotnick, *Vanishing of Whitehead groups for Seifert manifolds with infinite fundamental group*, Comment. Math. Helv. **55** (1980), no. 4, 654–667.

[Poi04] H. Poincaré, *Cinquiéme complément à l'analysis situs*, Rend. Circ. Mat. Palermo **18** (1904), 45–110.

[Poi96] ———, *Œuvres,* VI, Les Grands Classiques Gauthier-Villars, Éditions Jacques Gabay, Sceaux, 1996.

[Poi10] ———, *Papers on Topology,* History of Mathematics, vol. 37, American Mathematical Society, Providence, RI; London Mathematical Society, London, 2010.

[Pos48] M. M. Postnikov, *The structure of intersection rings of* 3*-manifolds*, Doklady AS USSR **61** (1948), 795–797.

[PV00] L. Potyagailo, S. Van, *On the co-Hopficity of* 3*-manifold groups*, St. Petersburg Math. J. **11** (2000), no. 5, 861–881.

[Pow75] R. T. Powers, *Simplicity of the* C^**-algebra associated with the free group on two generators*, Duke Math. J. **42** (1975), 151–156.

[Pra73] G. Prasad, *Strong rigidity of* $\mathbb{Q}$*-rank* 1 *lattices*, Invent. Math. **21** (1973), 255–286.

[Pre05] J.-P. Préaux, *Conjugacy problem in groups of non-oriented geometrizable* 3*-manifolds*, unpublished paper, 2005.
http://www.cmi.univ-mrs.fr/~preaux/PDF/cpno3m.pdf

[Pre06] ———, *Conjugacy problem in groups of oriented geometrizable* 3*-manifolds*, Topology **45** (2006), no. 1, 171–208.

[Pre14] ———, *A survey on Seifert fibre space conjecture*, ISRN Geom. 2014, Article ID 694106.

[Prz79] J. Przytycki, *A unique decomposition theorem for* 3*-manifolds with boundary*, Bull. Acad. Polon. Sci. Ser. Sci. Math. **27** (1979), no. 2, 209–215.

[PY03] J. Przytycki, A. Yasuhara, *Symmetry of links and classification of lens spaces*, Geom. Dedicata **98** (2003), 57–61.

[PW12] P. Przytycki, D. Wise, *Mixed* 3*-manifolds are virtually special*, preprint, 2012.

[PW14a] ———, *Graph manifolds with boundary are virtually special*, J. Topology **7** (2014), 419–435.

[PW14b] ———, *Separability of embedded surfaces in* 3*-manifolds*, Compos. Math. **150** (2014), no. 9, 1623–1630.

[Pur98] G. Putinar, *Nilpotent quotients of fundamental groups of special* 3*-manifolds with boundary*, Bull. Austral. Math. Soc. **58** (1998), no. 2, 233–237.

[Pun06] A. Putman, *The rationality of Sol-manifolds*, J. Algebra **304** (2006), no. 1, 190–215.

[QW04] R. Qiu, S. Wang, *Simple, small knots in handlebodies*, Topology Appl. **144** (2004), no. 1-3, 211–227.

[Rab58] M. Rabin, *Recursive unsolvability of group theoretic problems*, Ann. of Math. (2) **67** (1958), 172–194.

[Rad25] T. Radó, *Über den Begriff der Riemannschen Fläche*, Acta Sci. Math. (Szeged) **2** (1925), 101–121.

[Rai12a] J. Raimbault, *Exponential growth of torsion in abelian coverings*, Algebr. Geom. Topol. **12** (2012), no. 3, 1331–1372.

[Rai12b] J. Raimbault, *Asymptotics of analytic torsion for hyperbolic three–manifolds*, preprint, 2012.

[Rai13] ———, *Analytic, Reidemeister and homological torsion for congruence three–manifolds*, preprint, 2013.

[Raj04] C. S. Rajan, *On the non-vanishing of the first Betti number of hyperbolic three manifolds*, Math. Ann. **330** (2004), no. 2, 323–329.

[Rak81] J. Rakovec, *A theorem about almost sufficiently large* 3*-manifolds*, Glas. Mat. Ser. III **16(36)** (1981), no. 1, 151–156.

[Rat87] J. Ratcliffe, *Euler characteristics of* 3*-manifold groups and discrete subgroups of* $SL(2, \mathbb{C})$, J. Pure Appl. Algebra **44** (1987), no. 1-3, 303–314.

[Rat90] ———, *On the uniqueness of amalgamated product decompositions of a group*, in: *Combinatorial Group Theory,* pp. 139–146, Contemporary Mathematics, vol. 109, American Mathematical Society, Providence, RI, 1990.

[Rat06] ———, *Foundations of Hyperbolic Manifolds*, 2nd ed., Graduate Texts in Mathematics, vol. 149, Springer-Verlag, New York, 2006.

[Ray80] F. Raymond, *The Nielsen theorem for Seifert fibered spaces over locally symmetric spaces*, J. Korean Math. Soc. **16** (1979/80), no. 1, 87–93.

[RaS77] F. Raymond, L. Scott, *Failure of Nielsen's theorem in higher dimensions*, Arch. Math. (Basel) **29** (1977), no. 6, 643–654.

[Ree04] M. Rees, *The Ending Laminations Theorem direct from Teichmüller geodesics*, unpublished paper, `math/0404007`, 2004.

[Red91] A. W. Reid, *Arithmeticity of knot complements*, J. London Math. Soc. **43** (1991), 171–184.

[Red92] ———, *Some remarks on* 2*-generator hyperbolic* 3*-manifolds*, in: *Discrete Groups and Geometry,* pp. 209–219, London Mathematical Society Lecture Note Series, vol. 173, Cambridge University Press, Cambridge, 1992.

[Red95] ———, *A non-Haken hyperbolic* 3*-manifold covered by a surface bundle*, Pacific J. Math. **167** (1995), 163–182.

[Red07] ———, *The geometry and topology of arithmetic hyperbolic* 3*-manifolds*, in: *Proceedings of the Symposium on Topology, Complex Analysis and Arithmetic of Hyperbolic Spaces,* pp. 31–58, RIMS Kôkyûroku Series **1571** (2007).

[Red13] ———, *Profinite properties of discrete groups*, lecture notes at Groups St Andrews 2013.
`http://www.ma.utexas.edu/users/areid/StAndrews3.pdf`

[ReW08] A. Reid, G. Walsh, *Commensurability classes of* 2*-bridge knot complements*, Algebr. Geom. Topol. **8** (2008), 1031–1057.

[Rer35] K. Reidemeister, *Homotopieringe und Linsenräume*, Abh. Math. Sem. Univ. Hamburg **11** (1935), no. 1, 102–109.

[Rer36] ———, *Kommutative Fundamentalgruppen*, Monatsh. Math. Phys. **43** (1936), no. 1, 20–28.

[Rev69] V. N. Remeslennikov, *Conjugacy in polycyclic groups*, Algebra i Logika **8** (1969), 712–725.

[Ren10] C. Renard, *Gradients de Heegaard sous-logarithmiques d'une variété hyperbolique de dimension* 3 *et fibres virtuelles*, Actes du Séminaire Théorie Spectrale et Géométrie de Grenoble **29** (2010-2011), 97–131.

[Ren14a] ———, *Circular characteristics and fibrations of hyperbolic closed* 3*-manifolds*, Proc. Amer. Math. Soc. **142** (2014), no. 10, 3649–3664.

[Ren14b] ———, *Detecting surface bundles in finite covers of hyperbolic closed* 3*-manifolds*, Trans. Amer. Math. Soc. **366** (2014), no. 2, 979–1027.

[Reni01] M. Reni, *On finite groups acting on homology* 3*-spheres*, J. London Math. Soc. (2) **63** (2001), no. 1, 226–246.

[ReZ02] M. Reni, B. Zimmermann, *Finite simple groups acting on* 3*-manifolds and homology spheres*, Rend. Istit. Mat. Univ. Trieste **32** (2001), suppl. 1, 305–315 (2002).

[Rez97] A. Reznikov, *Three-manifolds class field theory* (*homology of coverings for a nonvirtually* b_1*-positive manifold*), Selecta Math. (N.S.) **3** (1997), no. 3, 361–399.

[Rh73] A. H. Rhemtulla, *Residually* F_p*-groups, for many primes p, are orderable,* Proc. Amer. Math. Soc. **41** (1973), 31–33.

[RiZ93] L. Ribes, P. Zalesskii, *On the profinite topology on a free group*, Bull. London Math. Soc. **25** (1993), no. 1, 37–43.

[RiZ10] ———, *Profinite Groups*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, vol. 40, Springer-Verlag, Berlin, 2010.

[Ril75a] R. Riley, *Discrete parabolic representations of link groups*, Mathematika **22** (1975), no. 2, 141–150.

[Ril75b] ———, *A quadratic parabolic group*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 281–288.

[Ril82] ———, *Seven excellent knots*, in: *Low-dimensional Topology,* pp. 81–151, London Mathematical Society Lecture Note Series, vol. 48, Cambridge University Press, Cambridge-New York, 1982.

[Ril90] ———, *Growth of order of homology of cyclic branched covers of knots*, Bull. London Math. Soc. **22** (1990), no. 3, 287–297.

[Ril13] ———, *A personal account of the discovery of hyperbolic structures on some knot complements*, Expo. Math. **31** (2013), no. 2, 104–115.

[Riv08] I. Rivin, *Walks on groups, counting reducible matrices, polynomials, and surface and free group automorphisms*, Duke Math. J. **142** (2008), 353–379.

[Riv09] ———, *Walks on graphs and lattices–effective bounds and applications*, Forum Math. **21** (2009), 673–685.

[Riv10] ———, *Zariski density and genericity*, Int. Math. Res. Not. **19** (2010), 3649–3657.

[Riv12] ———, *Generic phenomena in groups—some answers and many questions*, in: *Thin Groups and Superstrong Approximation,* to appear.

[Riv14] ———, *Statistics of random* 3*-manifolds occasionally fibering over the circle*, preprint, 2014.

[Rob13] L. Roberts, *On non-isotopic spanning surfaces for a class of arborescent knots*, preprint, 2013.

[RoS10] R. Roberts and J. Shareshian, *Non-right-orderable* 3*-manifold groups*, Canad. Math. Bull. **53** (2010), no. 4, 706–718.

[RSS03] R. Roberts, J. Shareshian, M. Stein, *Infinitely many hyperbolic* 3*-manifolds which contain no Reebless foliation*, J. Amer. Math. Soc. **16** (2003), no. 3, 639–679.

[Rol90] D. Rolfsen, *Knots and Links*, Mathematics Lecture Series, vol. 7, Publish or Perish, Inc., Houston, TX, 1990.

[Rol14a] ———, *Low dimensional topology and ordering groups*, Math. Slovaca **64** (2014), no. 3, 579–600.

[Rol14b] ———, *A topological view of ordered groups*, preprint, 2014.

[RoZ98] D. Rolfsen, J. Zhu, *Braids, orderings and zero divisors*, J. Knot Theory Ramifications **7** (1998), no. 6, 837–841.

[Rom69] N. S. Romanovskii, *On the residual finiteness of free products with respect to subgroups*, Izv. Akad. Nauk SSSR Ser. Mat. **33** (1969), 1324–1329.

[Ros94] S. Rosebrock, *On the realization of Wirtinger presentations as knot groups*, J. Knot Theory Ramifications **3** (1994), no. 2, 211–222.

[Ros07] S. Rosebrock, *The Whitehead Conjecture—an overview*, Sib. Èlektron. Mat. Izv. **4** (2007), 440–449.

[RsZ87] M. Rost, H. Zieschang, *Meridional generators and plat presentations of torus links*, J. London Math. Soc. (2) **35** (1987), 551–562.

[Rou04] S. K. Roushon, *Topology of* 3*-manifolds and a class of groups,* II, Bol. Soc. Mat. Mexicana (3) **10** (2004), special issue, 467–485.

[Rou08a] ———, *The Farrell-Jones isomorphism conjecture for* 3*-manifold groups*, J. K-Theory **1** (2008), no. 1, 49–82.

[Rou08b] ———, *The isomorphism conjecture for* 3*-manifold groups and K-theory of virtually poly-surface groups*, J. K-Theory, **1** (2008), no. 1, 83–93.

[Rou11] ———, *Vanishing structure set of* 3*-manifolds*, Topology Appl. **158** (2011), 810–812.

[Rov07] C. Röver, *On subgroups of the pentagon group*, Math. Proc. R. Ir. Acad. **107** (2007), no. 1, 11–13 (electronic).

[Row72] W. Row, *Irreducible* 3*-manifolds whose orientable covers are not prime*, Proc. Amer. Math. Soc. **34** (1972), 541–545.

[Row79] ———, *An algebraic characterization of connected sum factors of closed* 3*-manifolds*, Trans. Amer. Math. Soc. **250** (1979), 347–356.

[Rub01] D. Ruberman, *Isospectrality and* 3*-manifold groups*, Proc. Amer. Math. Soc. **129** (2001), no. 8, 2467–2471.

[RS90] J. Rubinstein, G. Swarup, *On Scott's core theorem*, Bull. London Math. Soc. **22** (1990), no. 5, 495–498.

[RuW98] J. H. Rubinstein, S. Wang, *On π_1-injective surfaces in graph manifolds*, Comment. Math. Helv. **73** (1998), 499–515.

[Rus10] B. Rushton, *Constructing subdivision rules from alternating links*, Conform. Geom. Dyn. **14** (2010), 1–13.

[Sag95] M. Sageev, *Ends of group pairs and non-positively curved cube complexes*, Proc. London Math. Soc. **71** (1995), no. 3, 585–617.

[Sag97] ———, *Codimension-*1 *subgroups and splittings of groups*, J. Algebra **189** (1997), no. 2, 377–389.

[Sag12] ———, *CAT(0) Cube Complexes and Groups*, lecture notes from Park City Mathematics Institute summer session on Geometric Group Theory, 2012. `http://www.math.utah.edu/pcmi12/lecture_notes/sageev.pdf`

[SaW12] M. Sageev, D. Wise, *Cubing cores for quasiconvex actions*, preprint, 2012.

[Sak81] M. Sakuma, *Surface bundles over S^1 which are* 2*-fold branched cyclic coverings of S^3*, Math. Sem. Notes Kobe Univ. **9** (1981), no. 1, 159–180.

[Sak94] ———, *Minimal genus Seifert surfaces for special arborescent links*, Osaka J. Math. **31** (1994), 861–905.

[Sal87] M. Salvetti, *Topology of the complement of real hyperplanes in $\mathbb{C}^N$*, Invent. Math. **88** (1987), no. 3, 603–618.

[Sar12] P. Sarnak, *Notes on Thin Matrix Groups*, lectures at the MSRI hot topics workshop on superstrong approximation, 2012. `http://web.math.princeton.edu/sarnak/NotesOnThinGroups.pdf`

[Sav12] N. Saveliev, *Lectures on the Topology of* 3*-Manifolds,* 2nd rev. ed., de Gruyter Textbook, Walter de Gruyter & Co., Berlin, 2012.

[SZ01] S. Schanuel, X. Zhang, *Detection of essential surfaces in* 3*-manifolds with SL_2-trees*, Math. Ann. **320** (2001), no. 1, 149–165.

[STh88] M. Scharlemann, A. Thompson, *Finding disjoint Seifert surfaces*, Bull. London Math. Soc. **20** (1988), no. 1, 61–64.

[Scf67] C. B. Schaufele, *The commutator group of a doubled knot*, Duke Math. J. **34** (1967), 677–681.

[Scv14] K. Schreve, *The Strong Atiyah conjecture for virtually compact special groups*, Math. Ann. **359** (2014), no. 3–4, 629–636.

[Sct49] H. Schubert, *Die eindeutige Zerlegbarkeit eines Knotens in Primknoten*, S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. **1949** (1949), no. 3, 57–104.

[Sct53] ———, *Knoten und Vollringe*, Acta Math. **90** (1953), 131–286.

[Sct54] ———, *Über eine numerische Knoteninvariante*, Math. Z. **61** (1954), 245–288.

[Scs14] J. Schultens, *Introduction to* 3*-Manifolds*, Graduate Studies in Mathematics, vol. 151, American Mathematical Society, Providence, RI, 2014.

[ScW07] J. Schultens, R. Weidmann, *On the geometric and the algebraic rank of graph manifolds*, Pacific J. Math. **231** (2007), 481–510.

[Scr04] J. Schwermer, *Special cycles and automorphic forms on arithmetically defined hyperbolic* 3*-manifolds*, Asian J. Math. **8** (2004), no. 4, 837–859.

[Scr10] ———, *Geometric cycles, arithmetic groups and their cohomology*, Bull. Amer. Math. Soc. (N.S.) **47** (2010), no. 2, 187–279.

[Sco72] P. Scott, *On sufficiently large* 3*-manifolds*, Quart. J. Math. Oxford Ser. (2) **23** (1972), 159–172.

[Sco73a] ———, *Finitely generated* 3*-manifold groups are finitely presented*, J. London Math. Soc. (2) **6** (1973), 437–440.

[Sco73b] ———, *Compact submanifolds of* 3*-manifolds*, J. London Math. Soc. (2) **7** (1973), 246–250.

[Sco74] ———, *An introduction to* 3*-manifolds*, Department of Mathematics, University of Maryland, Lecture Note, no. 11, Department of Mathematics, University of Maryland, College Park, MD, 1974.

[Sco78] ———, *Subgroups of surface groups are almost geometric*, J. London Math. Soc. **17** (1978), 555–565.

[Sco80] ———, *A new proof of the Annulus and Torus Theorems*, Amer. J. Math. **102** (1980), no. 2, 241–277.

[Sco83a] ———, *The geometries of* 3*-manifolds*, Bull. London Math. Soc. **15** (1983), no. 5, 401–487.

[Sco83b] ———, *There are no fake Seifert fiber spaces with infinite* π_1, Ann. of Math. (2) **117** (1983), 35–70.

[Sco84] ———, *Strong annulus and torus theorems and the enclosing property of characteristic submanifolds of* 3*-manifolds*, Quart. J. Math. Oxford Ser. (2) **35** (1984), no. 140, 485–506.

[Sco85a] ———, *Correction to: 'Subgroups of surface groups are almost geometric'*, J. London Math. Soc. (2) **32** (1985), no. 2, 217–220.

[Sco85b] ———, *Homotopy implies isotopy for some Seifert fibre spaces*, Topology **24** (1985), 341–351.

[SSh14] P. Scott, H. Short, *The homeomorphism problem for closed* 3*-manifolds*, Alg. Geom. Topology **14** (2014), 2431–2444.

[SSw01] P. Scott, G. A. Swarup, *Canonical splittings of groups and* 3*-manifolds*, Trans. Amer. Math. Soc. **353** (2001), no. 12, 4973–5001.

[SSw03] ———, *Regular neighbourhoods and canonical decompositions for groups*, Astérisque **289** (2003).

[SSw07] ———, *Annulus-Torus decompositions for Poincaré duality pairs*, preprint, `math/0703890`, 2007.

[STu89] P. Scott, T. Tucker, *Some examples of exotic noncompact* 3*-manifolds*, Quart. J. Math. Oxford Ser. (2) **40** (1989), no. 160, 481–499.

[Scri10] T. Scrimshaw, *Embeddings of right-angled Artin groups,* preprint, 2010.

[Sei33a] H. Seifert, *Topologie dreidimensionaler gefaserter Räume*, Acta Math. **60** (1933), 147–238.

[Sei33b] ———, *Verschlingungsinvarianten*, Sitzungsber. Preuss. Akad. Wiss., Phys.-Math. Kl. **1933** (1933), no. 26-2, 811–828.

[SeT30] H. Seifert, W. Threlfall, *Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes*, Math. Ann. **104** (1930), 1–70.

[SeT33] ———, *Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes II*, Math. Ann. 107 (1933), 543–586.

[SeT34] ———, *Lehrbuch der Topologie*, B. G. Teubner, Leipzig, 1934.

[SeT80] ———, *A Textbook of Topology*, Pure and Applied Mathematics, vol. 89, Academic Press, Inc., New York-London, 1980.

[SeW33] H. Seifert, C. Weber, *Die beiden Dodekaederräume*, Math. Z. **37** (1933), 237–253.

[Sel93] Z. Sela, *The conjugacy problem for knot groups*, Topology **32** (1993), no. 2, 363–369.

[Sel95] ———, *The isomorphism problem for hyperbolic groups,* I, Ann. of Math. (2) **141** (1995), no. 2, 217–283.

[Sen11] M. H. Şengün, *On the integral cohomology of Bianchi groups*, Exp. Math. **20** (2011), 487–505.

[Sen12] ———, *On the torsion homology of non-arithmetic hyperbolic tetrahedral groups*, Int. J. Number Theory **8** (2012), 311–320.

[Ser77] J.-P. Serre, *Arbres, amalgames,* SL_2, Astérisque, no. 46, Société Mathématique de France, Paris, 1977.

[Ser80] ———, *Trees*, Springer-Verlag, Berlin-New York, 1980.

[Ser97] ———, *Galois Cohomology*, Springer-Verlag, Berlin, 1997.

[Shn75] P. Shalen, *Infinitely divisible elements in* 3*-manifold groups*, in: *Knots, Groups, and* 3*-Manifolds,* pp. 293–335, Annals of Mathematics Studies, vol. 84, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1975.

[Shn79] ———, *Linear representations of certain amalgamated products*, J. Pure. Appl. Algebra **15** (1979), 187–197.

[Shn84] ———, *A "piecewise-linear" method for triangulating* 3*-manifolds*, Adv. in Math. **52** (1984), no. 1, 34–80.

[Shn01] ———, *Three-manifolds and Baumslag-Solitar groups*, Topology Appl. **110** (2001), no. 1, 113–118.

[Shn02] ———, *Representations of 3-manifold groups*, in: *Handbook of Geometric Topology,* pp. 955–1044, North-Holland, Amsterdam, 2002.

[Shn07] ———, *Hyperbolic volume, Heegaard genus and ranks of groups*, in: *Workshop on Heegaard Splittings,* pp. 335–349, Geometry & Topology Monographs, vol. 12, Geometry & Topology Publications, Coventry, 2007.

[Shn12] ———, *Orders of elements in finite quotients of Kleinian groups*, Pacific J. Math. **256** (2012), no. 1, 211–234.

[ShW92] P. Shalen, P. Wagreich, *Growth rates, $\mathbb{Z}_p$-homology, and volumes of hyperbolic 3-manifolds*, Trans. Amer. Math. Soc. **331** (1992), no. 2, 895–917.

[SpW58] A. Shapiro, J. H. C. Whitehead, *A proof and extension of Dehn's lemma*, Bull. Amer. Math. Soc. **64** (1958), 174–178.

[Sho92] M. Shapiro, *Automatic structure and graphs of groups*, in: *Topology '90,* pp. 355–380, Ohio State University Mathematical Research Institute Publications, vol. 1, Walter de Gruyter & Co., Berlin, 1992.

[Sho94] ———, *Growth of a $PSL_2\mathbb{R}$ manifold group*, Math. Nach. **167** (1994), 279–312.

[Sht85] H. Short, *Some closed incompressible surfaces in knot complements which survive surgery*, in: *Low-dimensional Topology*, pp. 179–194, London Mathematical Society Lecture Note Series, vol. 95, Cambridge University Press, Cambridge, 1985.

[Sht04] M. Shtan′ko, *A theorem of A. A. Markov and algorithmically unrecognizable combinatorial manifolds*, Izv. Math. **68** (2004), no. 1, 205–221.

[Sik05] A. Sikora, *Cut numbers of 3-manifolds*, Trans. Amer. Math. Soc. **357** (2005), no. 5, 2007–2020.

[Siv87] J.–C. Sikorav, *Homologie de Novikov associée à une classe de cohomologie réelle de degré un*, thèse, Orsay, 1987.

[Sil96] D. Silver, *Nontorus knot groups are hyper-Hopfian*, Bull. London Math. Soc. **28** (1996), no. 1, 4–6.

[SWW10] D. Silver, W. Whitten, S. Williams, *Knot groups with many killers*, Bull. Austr. Math. Soc. **81** (2010), 507–513.

[SiW02a] D. Silver, S. Williams, *Mahler measure, links and homology growth*, Topology **41** (2002), no. 5, 979–991.

[SiW02b] ———, *Torsion numbers of augmented groups with applications to knots and links*, Enseign. Math. (2) **48** (2002), no. 3-4, 317–343.

[SiW09a] ———, *Nonfibered knots and representation shifts*, in: *Algebraic Topology—Old and New,* pp. 101–107, Banach Center Publications, vol. 85, Polish Academy of Sciences, Institute of Mathematics, Warsaw, 2009.

[SiW09b] ———, *Twisted Alexander Polynomials and Representation Shifts*, Bull. London Math. Soc. **41** (2009), 535–540.

[Sim76a] J. Simon, *Roots and centralizers of peripheral elements in knot groups*, Math. Ann. **222** (1976), no. 3, 205–209.

[Sim76b] ———, *On the problems of determining knots by their complements and knot complements by their groups*, Proc. Amer. Math. Soc. **57** (1976), no. 1, 140–142.

[Sim80] ———, *How many knots have the same group?*, Proc. Amer. Math. Soc. **80** (1980), no. 1, 162–166.

[Sis11a] A. Sisto, *3-manifold groups have unique asymptotic cones*, preprint, 2011.

[Sis11b] ———, *Contracting elements and random walks*, preprint, `arXiv:1112.2666`, 2011.

[Som91] T. Soma, *Virtual fibre groups in 3-manifold groups*, J. London Math. Soc. (2) **43** (1991), no. 2, 337–354.

[Som92] ———, *3-manifold groups with the finitely generated intersection property*, Trans. Amer. Math. Soc. **331** (1992), no. 2, 761–769.

[Som06a] ———, *Existence of ruled wrappings in hyperbolic 3-manifolds*, Geom. Top. **10** (2006), 1173–1184.

[Som06b] ———, *Scott's rigidity theorem for Seifert fibered spaces; revisited*, Trans. Amer. Math. Soc. **358** (2006), no. 9, 4057–4070.

[Som10] ———, *Geometric approach to Ending Lamination Conjecture*, preprint, 2010.

[Sou08] J. Souto, *The rank of the fundamental group of certain hyperbolic 3-manifolds fibering over the circle*, in: *The Zieschang Gedenkschrift,* pp. 505–518, Geometry & Topology Monographs, vol. 14, Geometry & Topology Publications, Coventry, 2008.

[Sp49] E. Specker, *Die erste Cohomologiegruppe von Überlagerungen und Homotopie-Eigenschaften dreidimensionaler Mannigfaltigkeiten*, Comment. Math. Helv. **23** (1949), 303–333.

[Sta59a] J. Stallings, *J. Grushko's theorem II, Kneser's conjecture*, Notices Amer. Math. Soc. **6** (1959), no. 559-165, 531–532.

[Sta59b] ———, *Some topological proofs and extensions of Grus̆ko's theorem*, Ph.D. Thesis, Princeton University, 1959.

[Sta60] ———, *On the loop theorem*, Ann. of Math. (2) **72** (1960), 12–19.

[Sta62] ———, *On fibering certain* 3*-manifolds*, in: *Topology of* 3*-Manifolds and Related Topics,* pp. 95–100, Prentice-Hall, Englewood Cliffs, NJ, 1962.

[Sta65] ———, *Homology and central series of groups*, J. Algebra **2** (1965), 170–181.

[Sta66] ———, *How not to prove the Poincaré conjecture*, in: *Topology Seminar, Wisconsin, 1965,* pp. 83–88, Annals of Mathematics Studies, vol. 60, Princeton University Press, Princeton, NJ, 1966.

[Sta68a] ———, *Groups of dimension* 1 *are locally free*, Bull. Amer. Math. Soc. **74** (1968), 361–364.

[Sta68b] ———, *On torsion-free groups with infinitely many ends*, Ann. of Math. (2) **88** (1968), 312–334.

[Sta71] ———, *Group Theory and Three-dimensional Manifolds*, Yale Mathematical Monographs, vol. 4, Yale University Press, New Haven, Conn.-London, 1971.

[Sta77] ———, *Coherence of* 3*-manifold fundamental groups*, in: *Séminaire Bourbaki,* vol. 1975/76, pp. 167–173, Lecture Notes in Mathematics, vol. 567, Springer-Verlag, Berlin (1977).

[Sta82] ———, *Topologically unrealizable automorphisms of free groups*, Proc. Amer. Math. Soc. **84** (1982), no. 1, 21–24.

[Sta85] ———, *Topology of finite graphs*, Invent. Math. **71** (1983), no. 3, 551–565.

[Ste68] P. Stebe, *Residual finiteness of a class of knot groups*, Comm. Pure Appl. Math. **21** (1968), 563–583.

[Ste72] ———, *Conjugacy separability of groups of integer matrices*, Proc. Amer. Math. Soc. **32** (1972), 1–7.

[Sts75] R. Stevens, *Classification of* 3*-manifolds with certain spines*, Trans. Amer. Math. Soc. **205** (1975), 151–166.

[Sti93] J. Stillwell, *Classical Topology and Combinatorial Group Theory*, 2nd ed., Graduate Texts in Mathematics, vol. 72, Springer-Verlag, New York, 1993.

[Sti12] ———, *Poincaré and the early history of* 3*-manifolds*, Bull. Amer. Math. Soc. **49** (2012), 555–576.

[Str74] R. Strebel, *Homological methods applied to the derived series of groups*, Comment. Math. Helv. **49** (1974), 302–332.

[Sula75] D. Sullivan, *On the intersection ring of compact three manifolds*, Topology **14** (1975), 275–277.

[Sulb00] M. Sullivan, *Knot factoring*, Amer. Math. Monthly **107** (2000), no. 4, 297–315.

[Sul81] ———, *Travaux de Thurston sur les groupes quasi-fuchsiens et les variétés hyperboliques de dimension* 3 *fibrées sur* S^1, in: *Séminaire Bourbaki,* vol. 1979/80, pp. 196–214, Lecture Notes in Mathematics, vol. 842, Springer-Verlag, Berlin-New York, 1981.

[Sun13] H. Sun, *Virtual homological torsion of closed hyperbolic* 3*-manifolds*, J. Differential Geom., to appear.

[Sun14] ———, *Virtual domination of* 3*-manifolds*, preprint, 2014.

[Sus13] T. Susse, *Stable commutator length in amalgamated free products*, preprint, 2013.

[Suz13] M. Suzuki, *On pseudo-meridians of the trefoil knot group*, RIMS Kôkyûroku, 2013.

[Sv04] P. Svetlov, *Graph manifolds of nonpositive curvature are vitually fibered over the circle*, J. Math. Sci. (N. Y.) **119** (2004), no. 2, 278–280.

[Swn67] R. Swan, *Representation of polycyclic groups*, Proc. Amer. Math. Soc. **18** (1967) no. 3, 573–574.

[Swp70] G. A. Swarup, *Some properties of* 3*-manifolds with boundary*, Quart. J. Math. Oxford Ser. (2) **21** (1970), 1–23.

[Swp73] ———, *Finding incompressible surfaces in* 3*-manifolds*, J. London Math. Soc. (2) **6** (1973), 441–452.

[Swp74] ———, *On incompressible surfaces in the complements of knots*, J. Indian Math. Soc. (N.S.) **37** (1973), 9–24 (1974).

[Swp75] ———, *Addendum to: 'On incompressible surfaces in the complements of knots'*, J. Indian Math. Soc. (N.S.) **38** (1974), no. 1–4, 411–413 (1975).

[Swp78] ———, *On a theorem of Johannson*, J. London Math. Soc. (2) **18** (1978), no. 3, 560–562.

[Swp80a] ———, *Two finiteness properties in* 3*-manifolds*, Bull. London Math. Soc. **12** (1980), no. 4, 296–302.

[Swp80b] ———, *Cable knots in homotopy* 3*-spheres*, Quart. J. Math. Oxford Ser. (2) **31** (1980), no. 121, 97–104.

[Swp86] ———, *A remark on cable knots*, Bull. London Math. Soc. **18** (1986), no. 4, 401–402.

[Swp93] ———, *Geometric finiteness and rationality*, J. Pure Appl. Algebra **86** (1993), no. 3, 327–333.

[Sz08] G. Szpiro, *Poincaré's Prize: The Hundred-Year Quest to Solve One of Math's Greatest Puzzles*, Plume (2008).

[TY99] Y. Takeuchi, M. Yokoyama, *The geometric realizations of the decompositions of* 3*-orbifold fundamental groups*, Topology Appl. **95** (1999), no. 2, 129–153.

[Ta57] D. Tamari, *A refined classification of semi-groups leading to generalized polynomial rings with a generalized degree concept*, in: *Proceedings of the International Congress of Mathematicians, Amsterdam, 1954,* vol. 3, pp. 439–440, Erven P. Noordhoff N. V., Groningen; North-Holland Publishing Co., Amsterdam, 1957.

[Tau92] C. H. Taubes, *The existence of anti-self-dual conformal structures*, J. Differential Geom. **36** (1992), 163–253.

[Tay13] S. Taylor, *Exceptional surgeries on knots with exceptional classes*, preprint, 2013.

[Tei97] P. Teichner, *Maximal nilpotent quotients of* 3*-manifold groups*, Math. Res. Lett. **4** (1997), no. 2–3, 283–293.

[Ter06] M. Teragaito, *Toroidal Dehn fillings on large hyperbolic* 3*-manifolds*, Comm. Anal. Geom. **14** (2006), no. 3, 565–601.

[Ter13] ———, *Left-orderability and exceptional Dehn surgery on twist knots,* Canad. Math. Bull. **56** (2013), no. 4, 850–859.

[Tho68] C. B. Thomas, *Nilpotent groups and compact* 3*-manifolds*, Proc. Cambridge Philos. Soc. **64** (1968), 303–306.

[Tho78] ———, *Free actions by finite groups on* S^3, Algebr. geom. Topol., Stanford/Calif. 1976, Proc. Symp. Pure Math., Vol. 32, Part 1, 125-130 (1978).

[Tho79] ———, *On* 3*-manifolds with finite solvable fundamental group*, Invent. Math. **52** (1979), no. 2, 187–197.

[Tho84] ———, *Splitting theorems for certain* PD^3*-groups*, Math. Z. **186** (1984), 201–209.

[Tho86] ———, *Elliptic structures on* 3*-manifolds*, London Mathematical Society Lecture Note Series, vol. 104, Cambridge University Press, Cambridge, 1986.

[Tho95] ———, 3*-manifolds and* PD(3)*-groups*, in: *Novikov Conjectures, Index Theorems and Rigidity,* vol. 2, pp. 301–308, London Mathematical Society Lecture Note Series, vol. 227, Cambridge University Press, Cambridge, 1995.

[Thp97] A. Thompson, *Thin position and bridge number for knots in the* 3*-sphere*, Topology **36** (1997), no. 2, 505–507.

[Thu79] W. P. Thurston, *The geometry and topology of* 3*-manifolds*, Princeton Lecture Notes (1979), available at `http://www.msri.org/publications/books/gt3m/`

[Thu82a] ———, *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. **6** (1982), 357–379.

[Thu82b] ———, *Hyperbolic geometry and* 3*-manifolds*, in: *Low-dimensional Topology,* pp. 9–25, London Mathematical Society Lecture Note Series, vol. 48, Cambridge University Press, Cambridge-New York, 1982.

[Thu86a] ———, *A norm for the homology of* 3*-manifolds*, Mem. Amer. Math. Soc. **59** (1986), no. 339, 99–130.

[Thu86b] ———, *Hyperbolic structures on* 3*-manifolds.* I. *Deformation of acylindrical manifolds*, Ann. of Math. (2) **124** (1986), no. 2, 203–246.

[Thu86c] ———, *Hyperbolic Structures on* 3*-manifolds,* II. *Surface groups and* 3*-manifolds which fiber over the circle*, unpublished preprint, `math/9801045`, 1986.

[Thu86d] ———, *Hyperbolic Structures on 3-manifolds,* III. *Deformations of 3-manifolds with incompressible boundary*, unpublished preprint, `math/9801058`, 1986.

[Thu88] ———, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. (N.S.) **19** (1988), no. 2, 417–431.

[Thu97] ———, *Three-Dimensional Geometry and Topology,* vol. 1., ed. by Silvio Levy, Princeton Mathematical Series, vol. 35, Princeton University Press, Princeton, NJ, 1997.

[Tie08] H. Tietze, *Über die topologischen Invarianten mehrdimensionaler Mannigfaltigkeiten*, Monatsh. Math. Phys. **19** (1908), 1–118.

[Tit72] J. Tits, *Free subgroups in linear groups*, J. Algebra **20** (1972), 250–270.

[Tis70] D. Tischler, *On fibering certain foliated manifolds over* S^1, Topology **9** (1970), 153–154.

[To69] J. Tollefson, *3-manifolds fibering over* S^1 *with nonunique connected fiber*, Proc. Amer. Math. Soc. **21** (1969), 79–80.

[Tra13a] A. Tran, *On left-orderable fundamental groups and Dehn surgeries on knots*, J. Math. Soc. Japan, to appear.

[Tra13b] ———, *On left-orderability and cyclic branched coverings*, preprint, 2013.

[Tre90] M. Tretkoff, *Covering spaces, subgroup separability, and the generalized M. Hall property*, in: *Combinatorial Group Theory,* pp. 179–191, Contemporary Mathematics, vol. 109, American Mathematical Society, Providence, RI, 1990.

[Tri69] A. G. Tristram, *Some cobordism invariants for links*, Proc. Cambridge Philos. Soc. **66** (1969), 251–264.

[Ts85] C. M. Tsau, *Nonalgebraic killers of knots groups*, Proc. Amer. Math. Soc. **95** (1985), 139–146.

[Tuf09] C. Tuffley, *Generalized knot groups distinguish the square and granny knots*, J. Knot Theory Ramifications **18** (2009), no. 8, 1129–1157.

[Tuk88a] P. Tukia, *Homeomorphic conjugates of Fuchsian groups*, J. Reine Angew. Math. **391** (1988), 35–70.

[Tuk88b] ———, *Homeomorphic conjugates of Fuchsian groups: an outline*, in: *Complex Analysis, Joensuu 1987,* pp. 344–353, Lecture Notes in Mathematics, vol. 1351, Springer-Verlag, Berlin, 1988.

[Tur76] V. G. Turaev, *Reidemeister torsion and the Alexander polynomial*, Mat. Sb. (N.S.) **18(66)** (1976), no. 2, 252–270.

[Tur82] ———, *Nilpotent homotopy types of closed 3-manifolds*, in: *Topology,* pp. 355–366, Lecture Notes in Mathematics, vol. 1060, Springer-Verlag, Berlin, 1984.

[Tur84] ———, *Cohomology rings, linking forms and invariants of spin structures of three-dimensional manifolds*, Math. USSR, Sb. 48 (1984), 65–79.

[Tur83] ———, *Cohomology rings, linking coefficient forms and invariants of spin structures in three-dimensional manifolds*, Mat. Sb. (N.S.) **120(162)** (1983), no. 1, 68–83, 143.

[Tur88] ———, *Homeomorphisms of geometric three-dimensional manifolds*, Math. Notes **43** (1988), no. 3–4, 307–312.

[Tur90] ———, *Three-dimensional Poincaré complexes: homotopy classification and splitting*, Math. USSR-Sb. **67** (1990), no. 1, 261–282.

[Tur02] ———, *Introduction to Combinatorial Torsions*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2001.

[Us93] K. Usadi, *A counterexample to the equivariant simple loop conjecture*, Proc. Amer. Math. Soc. **118** (1993), no. 1, 321–329.

[Ven08] T. Venkataramana, *Virtual Betti numbers of compact locally symmetric spaces*, Israel J. Math. **166** (2008), 235–238.

[Ves87] A. Y. Vesnin, *Three-dimensional hyperbolic manifolds of Löbell type*, Sibirsk. Mat. Zh. **28** (1987), no. 5, 50–53.

[Vi49] A. A. Vinogradov, *On the free product of ordered groups*, Mat. Sbornik N.S. **25(67)** (1949), 163–168.

[Vo96] K. Volkert, *The early history of Poincaré's conjecture*, in: *Henri Poincaré: Science et Philosophie,* pp. 241–250, Publikationen des Henri-Poincaré-Archivs, Akademie Verlag, Berlin; Albert Blanchard, Paris, 1996.

[Vo02] ———, *Das Homöomorphieproblem insbesondere der 3-Mannigfaltigkeiten in der Topologie 1892–1935*, Philosophia Scientiae, Editions Kimé, Paris, 2002.

[Vo13a] ———, *Lens spaces in dimension 3: a history*, Bulletin of the manifold atlas (2013) http://www.boma.mpim-bonn.mpg.de/data/38print.pdf

[Vo13b] ———, *Poincaré's homology sphere*, Bulletin of the manifold atlas (2013) http://www.boma.mpim-bonn.mpg.de/data/40print.pdf

[Vo13c] ———, *Die Beiträge von Seifert und Threlfall zur dreidimensionalen Topologie* http://www2.math.uni-wuppertal.de/~volkert/SeifertThrelfall.pdf

[Vo14] ———, *Poincaré's Conjecture*, Bulletin of the manifold atlas (2014) http://www.boma.mpim-bonn.mpg.de/data/56screen.pdf

[Wan67a] F. Waldhausen, *Gruppen mit Zentrum und 3-dimensionale Mannigfaltigkeiten*, Topology **6** (1967), 505–517.

[Wan67b] ———, *Eine Verallgemeinerung des Schleifensatzes*, Topology **6** (1967), 501–504.

[Wan67c] ———, *Eine Klasse von 3-dimensionalen Mannigfaltigkeiten,* I, II, Invent. Math. **3** (1967), 308–333; ibid. **4** (1967), 87–117.

[Wan68a] ———, *On irreducible 3-manifolds which are sufficiently large*, Ann. of Math. (2) **87** (1968), 56–88.

[Wan68b] ———, *The word problem in fundamental groups of sufficiently large irreducible 3-manifolds*, Ann. of Math. (2) **88** (1968), 272–280.

[Wan69] ———, *On the determination of some bounded 3-manifolds by their fundamental groups alone*, in: *Proceedings of the International Symposium on Topology and its Applications,* pp. 331–332.

[Wan78a] ———, *Algebraic K-theory of generalized free products*, II, Ann. of Math. **108** (1978), 205–256.

[Wan78b] ———, *Some problems on 3-manifolds*, in: *Algebraic and Geometric Topology,* part 2, pp. 313–322, Proceedings of Symposia in Pure Mathematics, vol. XXXII, American Mathematical Society, Providence, RI, 1978.

[Wala83] C. T. C. Wall, *On the work of W. Thurston*, in: *Proceedings of the International Congress of Mathematicians,* vol. 1, pp. 11–14, PWN—Polish Scientific Publishers, Warsaw; North-Holland Publishing Co., Amsterdam, 1984.

[Wala04] ———, *Poincaré duality in dimension 3*, in: *Proceedings of the Casson Fest,* pp. 1–26, Geometry & Topology Monographs, vol. 7, Geometry & Topology Publications, Coventry, 2004.

[Walb09] L. Wall, *Homology growth of congruence subgroups*, PhD Thesis, Oxford University, 2009.

[Wac60] A. H. Wallace, *Modifications and cobounding manifolds*, Canad. J. Math. **12** (1960), 503-528.

[Wah05] G. Walsh, *Great circle links and virtually fibered knots*, Topology **44** (2005), no. 5, 947–958.

[Wah11] ———, *Orbifolds and commensurability*, in: *Interactions between Hyperbolic Geometry, Quantum Topology and Number Theory,* pp. 221–231, Contemporary Mathematics, vol. 541, American Mathematical Society, Providence, RI, 2011.

[Wag90] S. Wang, *The virtual $\mathbb{Z}$-representability of 3-manifolds which admit orientation reversing involutions*, Proc. Amer. Math. Soc. **110** (1990), no. 2, 499–503.

[Wag93] ———, *3-manifolds which admit finite group actions*, Trans. Amer. Math. Soc. **339** (1993), no. 1, 191–203.

[WW94] S. Wang, Y. Wu, *Covering invariants and co-Hopficity of 3-manifold groups*, Proc. London Math. Soc. (3) **68** (1994), no. 1, 203–224.

[WY94] S. Wang, F. Yu, *Covering invariants and cohopficity of 3-manifold groups*, Proc. London Math. Soc. (3) **68** (1994), 203–224.

[WY97] ———, *Graph manifolds with non-empty boundary are covered by surface bundles*, Math. Proc. Cambridge Philos. Soc. **122** (1997), no. 3, 447–455.

[WY99] ———, *Covering degrees are determined by graph manifolds involved*, Comment. Math. Helv. **74** (1999), 238–247.

[Weh73] B. A. F. Wehrfritz, *Infinite Linear Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 76, Springer-Verlag, New York-Heidelberg, 1973.

[Web71] C. Weinbaum, *The word and conjugacy problems for the knot group of any tame, prime, alternating knot*, Proc. Amer. Math. Soc. **30** (1971), 22–26.

[Wei02] R. Weidmann, *The Nielsen method for groups acting on trees*, Proc. London Math. Soc. (3) **85** (2002), no. 1, 93–118.

[Wei03] ______, *Some* 3*-manifolds with* 2*-generated fundamental group*, Arch. Math. (Basel) **81** (2003), no. 5, 589–595.

[Whd34] J. H. C. Whitehead, *Certain theorems about three-dimensional manifolds*, Q. J. Math. Oxford **5** (1934), 308–320.

[Whd35a] ______, *Three-dimensional manifolds (Corrigendum)*, Q. J. Math. Oxford **6**, (1935), 80.

[Whd35b] ______, *A certain open manifold whose group is unity*, Quart. J. Math. Oxford **6** (1935), 268–279.

[Whd41a] ______, *On incidence matrices, nuclei and homotopy types*, Ann. of Math. (2) **42** (1941), 1197–1239.

[Whd41b] ______, *On adding relations to homotopy groups*, Ann. of Math. (2) **42** (1941), 409–428.

[Whd58a] ______, *On* 2*-spheres in* 3*-manifolds*, Bull. Amer. Math. Soc. **64** (1958), 161–166.

[Whd58b] ______, *On finite cocycles and the sphere theorem*, Colloq. Math. **6** (1958) 271–281.

[Whd61] ______, *Manifolds with transverse fields in euclidean space*, Ann. of Math. (2) **73** (1961), 154–212.

[Whn73] W. Whitten, *Isotopy types of knot spanning surfaces*, Topology **12** (1973), 373–380.

[Whn86] ______, *Rigidity among prime-knot complements*, Bull. Amer. Math. Soc. (N.S.) **14** (1986), no. 2, 299–300.

[Whn87] ______, *Knot complements and groups*, Topology **26** (1987), no. 1, 41–44.

[Whn92] ______, *Recognizing nonorientable Seifert bundles*, J. Knot Theory Ramifications **1** (1992), no. 4, 471–475.

[Win98] J. Wilson, *Profinite Groups*, London Mathematical Society Monographs, New Series, vol. 19, The Clarendon Press, Oxford University Press, New York, 1998.

[Wil07] H. Wilton, *Elementarily free groups are subgroup separable*, Proc. London Math. Soc. (3) **95** (2007), no. 2, 473–496.

[Wil08] ______, *Residually free* 3*-manifolds*, Algebr. Geom. Topol. **8** (2008), no. 4, 2031–2047.

[WZ10] H. Wilton, P. Zalesskii, *Profinite properties of graph manifolds*, Geom. Dedicata **147** (2010), 29–45.

[WZ14] ______, *Distinguishing geometries using finite quotients*, preprint, 2014.

[Win94] B. Winters, *Infinite cyclic normal subgroups of fundamental groups of noncompact* 3*-manifolds*, Proc. Amer. Math. Soc. **120** (1994), no. 3, 959–963.

[Wim95] A. Wiman, *Ueber die hyperelliptischen Curven und diejenigen vom Geschlechte* $p = 3$*, welche eindeutige Transformationen in sich zulassen*, Stockh. Akad. Bihang XXI 1 . No. 1. **23** S (1895).

[Wis00] D. Wise, *Subgroup separability of graphs of free groups with cyclic edge groups*, Q. J. Math. **51** (2000), no. 1, 107–129.

[Wis06] ______, *Subgroup separability of the figure* 8 *knot group*, Topology **45** (2006), no. 3, 421–463.

[Wis09] ______, *The structure of groups with a quasi-convex hierarchy*, Electronic Res. Ann. Math. Sci **16** (2009), 44–55.

[Wis12a] ______, *The structure of groups with a quasi-convex hierarchy*, 189 pp., preprint, 2012, downloaded on Oct. 29, 2012 from `http://www.math.mcgill.ca/wise/papers.html`

[Wis12b] ______, *From Riches to RAAGS:* 3*-Manifolds, Right-angled Artin Groups, and Cubical Geometry,* CBMS Regional Conference Series in Mathematics, vol. 117, American Mathematical Society, Providence, RI, 2012.

[Wod15] D. Woodhouse, *Classifying Finite Dimensional Cubulations of Tubular Groups*, preprint, 2015.

[Won11] H. Wong, *Quantum invariants can provide sharp Heegaard genus bounds*, Osaka J. Math. **48** (2011), no. 3, 709–717.

[Wu92] Y.-Q. Wu, *Incompressibility of surfaces in surgered* 3*-manifolds*, Topology **31** (1992), 271–279.

[Wu04] ______, *Immersed essential surfaces and Dehn surgery*, Topology **43** (2004), no. 2, 319–342.

[Xu92] X. Xue, *On the Betti numbers of a hyperbolic manifold*, Geom. Funct. Anal. **2** (1992), no. 1, 126–136.

[Zha05] X. Zhang, *Trace fields of representations and virtually Haken* 3*-manifolds*, Q. J. Math. **56** (2005), no. 3, 431–442.

[Zhb12] Y. Zhang, *Lifted Heegaard surfaces and virtually Haken manifolds*, J. Knot Theory Ramifications **21** (2012), no. 8, 1250073, 26 pp.

[Zhu08] D. Zhuang, *Irrational stable commutator length in finitely presented groups*, J. Mod. Dyn. **2** (2008), no. 3, 499–507.

[Zie88] H. Zieschang, *On Heegaard diagrams of* 3*-manifolds*, in: *On the Geometry of Differentiable Manifolds,* pp. 247280, 283, Astérisque **163-164** (1988), Société Mathématique de France, Paris, 1989.

[ZZ82] H. Zieschang, B. Zimmermann, *Über Erweiterungen von* $\mathbb{Z}$ *und* $\mathbb{Z}_2 * \mathbb{Z}_2$ *durch nichteuklidische kristallographische Gruppen*, Math. Ann. **259** (1982), no. 1, 29–51.

[Zim79] B. Zimmermann, *Periodische Homöomorphismen Seifertscher Faserräume*, Math. Z. **166** (1979), 289–297.

[Zim82] ———, *Das Nielsensche Realisierungsproblem für hinreichend große* 3*-Mannigfaltigkeiten*, Math. Z. **180** (1982), 349–359.

[Zim91] ———, *On groups associated to a knot*, Math. Proc. Cambridge Philos. Soc. **109** (1991), no. 1, 79–82.

[Zim02a] ———, *Finite groups acting on homology* 3*-spheres: on a result of M. Reni*, Monatsh. Math. **135** (2002), no. 3, 253–258.

[Zim02b] ———, *On finite simple groups acting on homology* 3*-spheres*, Topology Appl. **125** (2002), no. 2, 199–202.

[Zim04] ———, *On the classification of finite groups acting on homology* 3*-spheres*, Pacific J. Math. **217** (2004), no. 2, 387–395.

[Zu97] L. Zulli, *Semibundle decompositions of* 3*-manifolds and the twisted cofundamental group*, Topology Appl. **79** (1997), 159–172.

Index