Classification of diffeomorphism groups of 3-manifolds through Ricci flow

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(joint work with Bruce Kleiner, NYU)

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Structure of Talk

- Part 0: Diffeomorphism Groups
- Part I: Uniqueness of singular Ricci flows
- Part II: Applications of Ricci flow to diffeomorphism groups
- Part III: Further Questions

Part 0: Diffeomorphism Groups

Diffeomorphism groups

M mostly 3-dimensional compact manifold

Goal of this talk:

Understand $\mathrm{Diff}(M) = \{\phi : M \to M \text{ diffeomorphism}\}$ (with C^∞-topology).

Main theme:

Pick a "nice" Riemannian metric g on M (e.g. constant sectional curvature) and compare $\mathrm{Diff}(M)$ with $\mathrm{Isom}(M)$.

$$\mathrm{Isom}(M) \longrightarrow \mathrm{Diff}(M)$$

Smale 1958

$O(3) = \mathrm{Isom}(S^2) \longrightarrow \mathrm{Diff}(S^2)$ is a homotopy equivalence.

Diffeomorphism groups

Smale Conjecture

$O(4) = \mathrm{Isom}(S^3) \longrightarrow \mathrm{Diff}(S^3)$ is a homotopy equivalence.

Cerf 1964: Isomorphism on π_0
Hatcher 1983: Homotopy equivalence

Remark: Smale conjecture is equivalent to $\mathrm{Diff}(D^3 rel \partial D^3) \simeq *$

Generalized Smale Conjecture

$\mathrm{Isom}(S^3/\Gamma) \longrightarrow \mathrm{Diff}(S^3/\Gamma)$ is a homotopy equivalence.

Ivanov 1984, Hong, Kalliongis, McCullough, Rubinstein 2012:
Lens spaces (except $\mathbb{R}P^3$), prism and quaternionic case
remaining cases: $\mathbb{R}P^3$, tetrahedral, octahedral and icosahedral case

Diffeomorphism groups

Non-spherical cases

Gabai 2001

If M is closed hyperbolic, then $\mathrm{Isom}(M) \longrightarrow \mathrm{Diff}(M)$ is homotopy equivalence.

Assume that M is irreducible, geometric, non-spherical, $g =$ metric of maximal symmetry.

Generalized Smale Conjecture for geometric manifolds

$\mathrm{Isom}(M) \longrightarrow \mathrm{Diff}(M)$ is a homotopy equivalence.

Gabai, Ivanov, Hatcher, McCullough, Soma:
Verified for all cases except for M non-Haken infranil.

Main Results

Using Ricci flow

Theorem A (Ba., Kleiner 2017)

The Generalized Smale Conjecture holds for all spherical space forms $M = S^3/\Gamma$ except for (possibly) $M = \mathbb{R}P^3$ (and S^3):

$$\mathrm{Diff}(M) \simeq \mathrm{Isom}(M) \qquad (*)$$

Theorem B (Ba., Kleiner 2017)

$(*)$ also holds for all closed hyperbolic 3-manifold M. (Gabai's Theorem)

Remarks:

- Proof provides a uniform treatment of Thms A, B on fewer than 30 pages.
- Proof relies on Hatcher's Theorem for $M = S^3$.
- $M = \mathbb{R}P^3$ and $M = S^3$ (without Hatcher's Theorem) and other topologies still work in progress.

Part I: Uniqueness of singular Ricci flows

Basics of Ricci flow

Ricci flow: $(M^n, g(t)),\ t \in [0, T)$

$$\partial_t g(t) = -2\,\mathrm{Ric}_{g(t)}, \qquad g(0) = g_0 \qquad (*)$$

Theorem (Hamilton 1982)

- $(*)$ has a unique solution $(g(t))_{t\in[0,T)}$ for maximal $T > 0$ if M is compact.
- If $T < \infty$, then

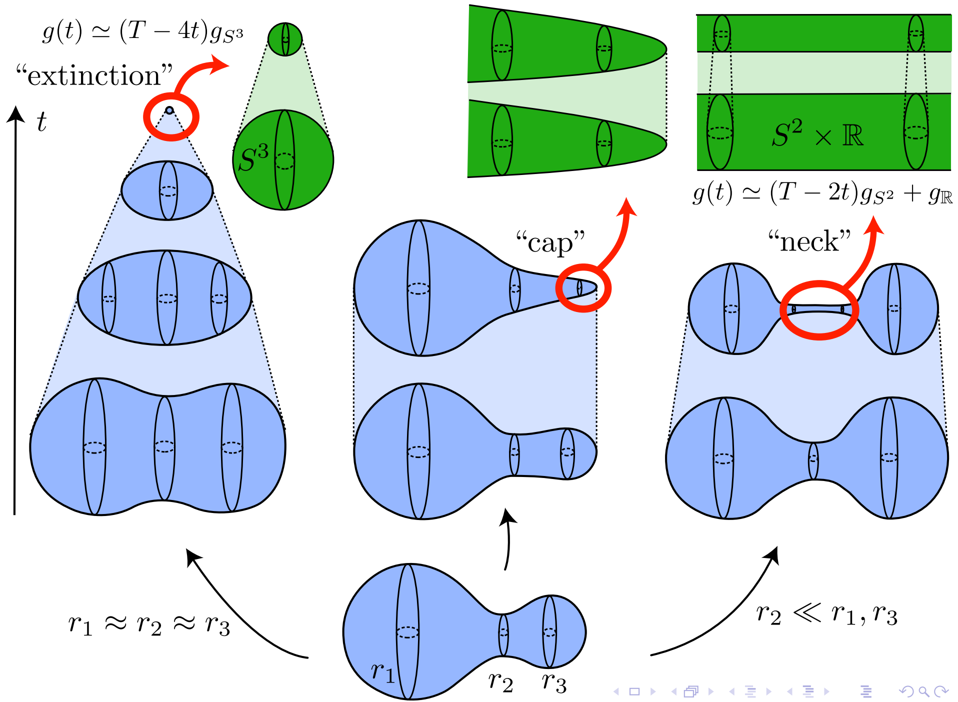
$$\lim_{t\to T} \max_M |\mathrm{Rm}_{g(t)}| = \infty$$

Speak: "$g(t)$ develops a singularity at time T".

Goal of Part I:

Theorem (Ba., Kleiner, 2016)

Any (compact) 3-dimensional (M^3, g_0) can be evolved into a **unique** (canonical), singular Ricci flow defined for all $t \geq 0$ that "flows through singularities".


$g(t) \simeq (T - 4t) g_{S^3}$
"extinction"
t
S^3
$S^2 \times \mathbb{R}$
$g(t) \simeq (T - 2t) g_{S^2} + g_{\mathbb{R}}$
"cap"
"neck"
$r_1 \approx r_2 \approx r_3$
$r_2 \ll r_1, r_3$
r_1
r_2
r_3

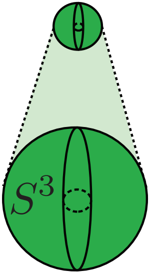
Singularities in 3d

Theorem (Perelman 2002)

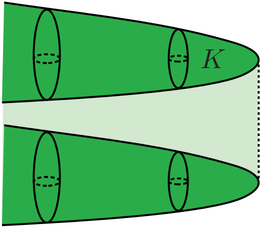
The singularity models in dimension 3 are κ-solutions.

Qualitative classification of κ-solutions

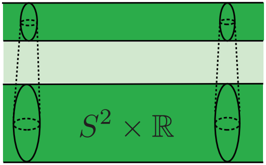
"extinction"



"cap"



"neck"



Given (M, g_0) construct
Ricci flow with surgery:

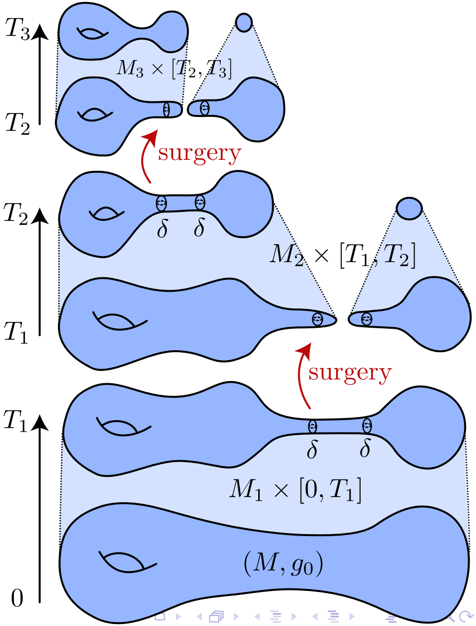
$$(M_1, g_1(t)), t \in [0, T_1],$$
$$(M_2, g_2(t)), t \in [T_1, T_2],$$
$$(M_3, g_3(t)), t \in [T_1, T_2], \quad \ldots$$

surgery scale $\approx \delta \ll 1$

Perelman 2003

- process can be continued indefinitely
- no accumulation of T_i.
- extinction if $\pi_1(M) < \infty$.

$M_k \approx$ connected sums components of M_{k+1} and copies of $S^2 \times S^1$.



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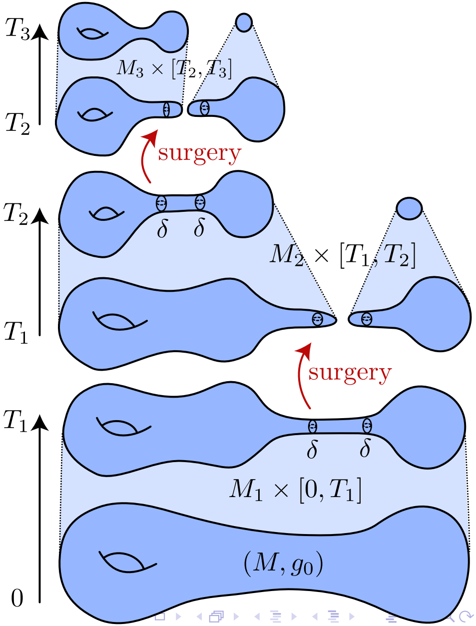
surgery scale $\approx \delta \ll 1$

high curvature regions are ε-close to κ-solutions:

- necks $\approx S^2 \times \mathbb{R}$
- spherical components
- caps

"ε-canonical neighborhood assumption"



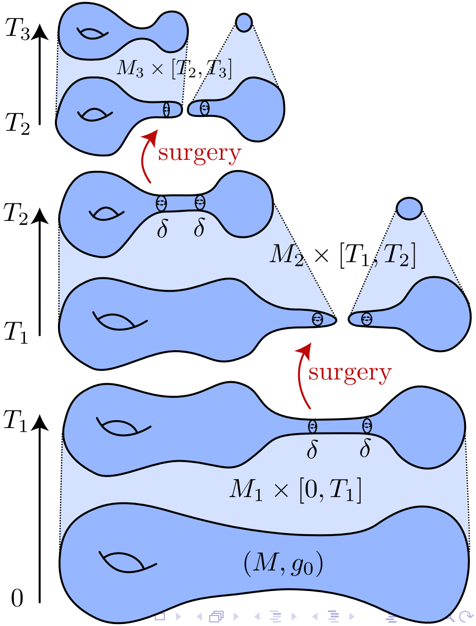
Ricci flow with surgery

Note:

surgery process is not canonical
(depends on surgery parameters)

Perelman:

- *It is likely that [. . .] one would get a canonically defined Ricci flow through singularities, but at the moment I don't have a proof of that.*
- *Our approach [. . .] is aimed at eventually constructing a canonical Ricci flow, [. . .] - a goal, that has not been achieved yet in the present work.*

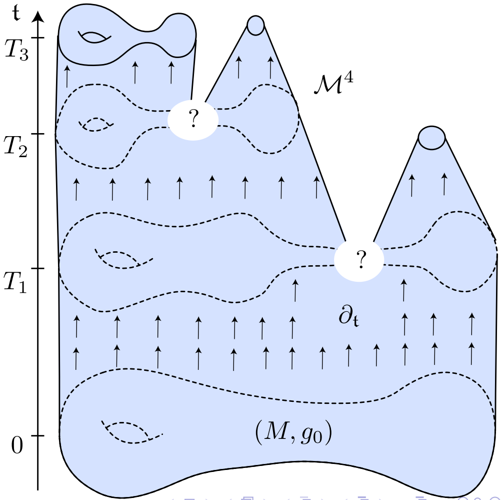


Space-time picture

- Space-time 4-manifold:

  $\mathcal{M}^4 = \big(M_1 \times [0, T_1] \;\cup\; M_2 \times [T_1, T_2] \;\cup\; M_3 \times [T_2, T_3] \;\cup\; \ldots\big) - \text{surgery points}$

- Time function: $\mathfrak{t} : \mathcal{M} \to [0, \infty)$.
- Time-slice: $\mathcal{M}_t = \mathfrak{t}^{-1}(t)$
- Time vector field:
  $\partial_{\mathfrak{t}}$ on $\mathcal{M}$ (with $\partial_{\mathfrak{t}} \cdot \mathfrak{t} = 1$).
- Metric g: on the distribution
  $\{d\mathfrak{t} = 0\} \subset T\mathcal{M}$
- Ricci flow equation:
  $\mathcal{L}_{\partial_{\mathfrak{t}}} g = -2\,\mathrm{Ric}_g$



$(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ is called a
Ricci flow space-time.

Note: there are "holes" at scale $\approx \delta$
space-time is δ-complete

Theorem (Kleiner, Lott 2014)

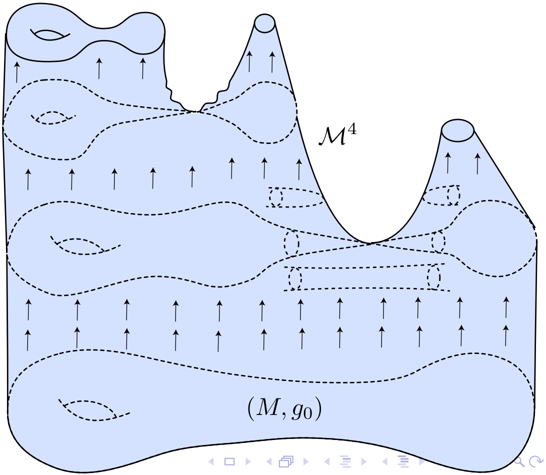
Given a compact (M^3, g_0), there is a Ricci flow space-time $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ s.t.:

- initial time-slice: $(\mathcal{M}_0, g) = (M, g_0)$.
- $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ is 0-complete (i.e. "singularity scale $\delta = 0$")
- $\mathcal{M}$ satisfies the ε-canonical nbhd assumption at small scales for all $\varepsilon > 0$.

$(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ flows "through singularities at infinitesimal scale"

Remarks:

- $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ is smooth everywhere and not defined at singularities
- singular times may accumulate
- $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ arises as limit for $\delta_i \to 0$.



Uniqueness

Theorem (Ba., Kleiner, 2016)

There is a constant $\varepsilon_{\text{can}} > 0$ such that:
Every Ricci flow space-time $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ is uniquely determined by its initial time-slice $(\mathcal{M}_0, g_0)$, provided that it

- is 0-complete and
- satisfies the ε_{can}-canonical neighborhood assumption below some positive scale.

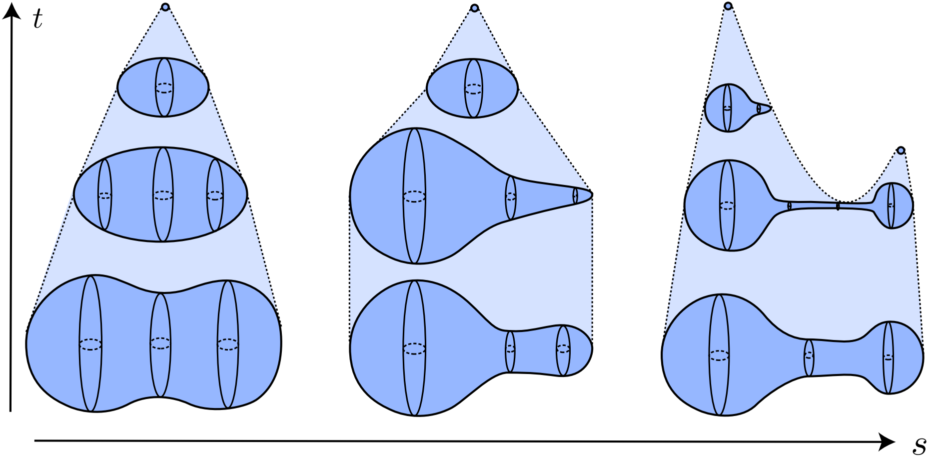
Corollary

For every compact (M^3, g_0) there is a unique, canonical singular Ricci flow space-time $\mathcal{M}$ with $\mathcal{M}_0 = (M^3, g_0)$.

Uniqueness $\longrightarrow$ Continuity

continuous family of metrics $(g^{(s)})_{s\in[0,1]}$ on M

$\rightsquigarrow \quad \{\mathcal{M}^{(s)}\}_{s\in[0,1]}$ singular RFs



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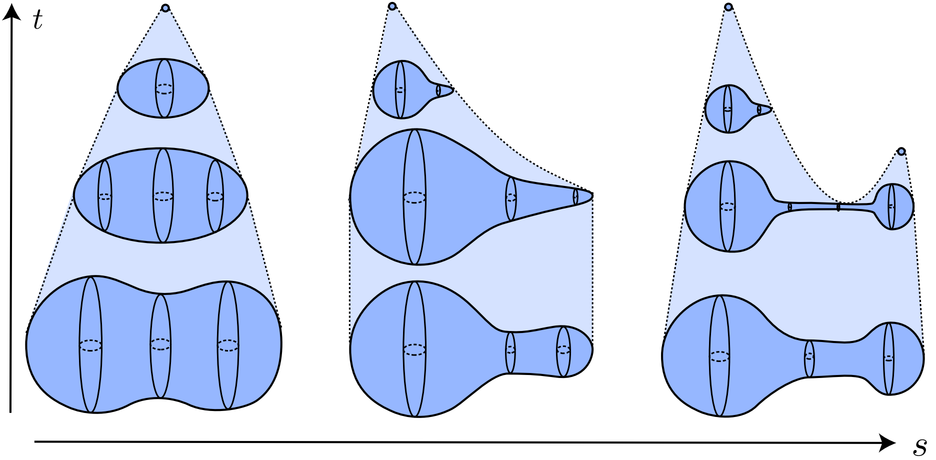


Continuity of singular RFs

Corollary

The singular Ricci flow space-time $\mathcal{M}$ depends continuously on its initial data $(\mathcal{M}_0, g_0)$ (in a certain sense).

Corollary

Every continuous/smooth family $(g^{(s)})_{s\in\Omega}$ of Riemannian metrics on a compact manifold M^3 can be evolved to a "continuous/smooth family of singular Ricci flows" $(\mathcal{M}^{(s)})_{s\in\Omega}$.

Part II: Applications of Ricci flow to diffeomorphism groups

$\text{Diff}(M) \longleftrightarrow \text{Met}(M)$

$\text{Met}(M) = \{g \text{ metric on } M\}$

$\text{Met}_{K \equiv k}(M) = \{g \in \text{Met}(M) \mid K_g \equiv k\}$

Lemma

For any $g_0 \in \text{Met}_{K\equiv k}(M)$:

$$\text{Diff}(M) \simeq \text{Isom}(M, g_0) \qquad \Longleftrightarrow \qquad \text{Met}_{K\equiv k}(M) \simeq *$$

Proof

$$\text{Isom}(M, g_0) \longrightarrow \text{Diff}(M) \longrightarrow \text{Met}_{K\equiv k}(M)$$

$$\phi \longmapsto \phi^* g_0$$

is a fibration

Main results, reworded

Theorems A + B (Ba., Kleiner 2017)

If $M \not\approx \mathbb{R}P^3, S^3$, then

$$\mathrm{Met}_{K \equiv \pm 1}(M) \simeq *$$

2d case

Smale 1958

$O(3) = \text{Isom}(S^2) \longrightarrow \text{Diff}(S^2)$ is a homotopy equivalence.

Proof (different from Smale's proof)

$$* \simeq \text{Met}(S^2) \longrightarrow \text{Met}_{K\equiv 1}(S^2)$$

$$g \longmapsto \text{limit of RF } (g_t)_{t\in[0,T)} \text{ (modulo rescaling) with initial condition } g_0 = g$$

is a deformation retraction

$$\Longrightarrow \quad \text{Met}_{K\equiv 1}(S^2) \simeq *$$

3d case

Assume $M = S^3/\Gamma$, $\Gamma \neq 1, \mathbb{Z}_2$ (hyperbolic case is similar)

Goal:

Theorem

$$\mathrm{Met}_{K\equiv 1}(M) \simeq *$$

Strategy:

- **Hope:** Construct retraction $\mathrm{Met}(M) \longrightarrow \mathrm{Met}_{K\equiv 1}(M)$.
- For any $g \in \mathrm{Met}(M)$ consider the (unique) $\mathcal{M}$ with $(\mathcal{M}_0, g_0) = (M, g)$.
- $\mathcal{M}$ goes extinct in finite time
- Analyze asymptotic behavior of $\mathcal{M}$ and extract limiting data, which "depends continuously on g".

Theorem

Given $\mathcal{M}$ with $(\mathcal{M}, g_0) = (M, g)$, there are $T_g^1 < T_g^2$ such that:

- for every $t \in [T_g^1, T_g^2)$ there is a unique component $\mathcal{C}_t \subset \mathcal{M}_t$ with $\mathcal{C}_t \approx M$.
- $(\mathcal{C}_t, g_t)$ converges to a round metric as $t \nearrow T_g^2$ (modulo rescaling).

Def: x survives until time t_0 if the ∂_t-trajectory through x intersects $\mathcal{M}_{t_0}$ in $x(t_0)$.

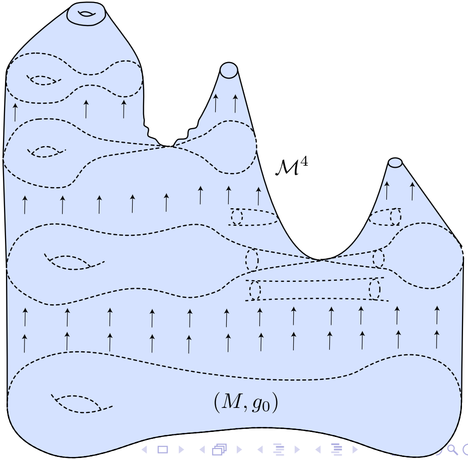
Lemma: All but finitely many bad points of $\mathcal{C}_{T_g^1}$ survive until time 0.

$W := \{x(0) \mid x \in \mathcal{C}_{T_g^1}\} \subset M$

$\overline{g}_t :=$ pushforward of g_t onto W by flow of $-\partial_t$

$\overline{g}_t \xrightarrow[t \nearrow T_g^2]{} \overline{g}$ modulo rescaling

$(W, \overline{g}) \cong (S^3/\Gamma - \{p_1, \ldots, p_N\}, g_{K\equiv 1})$



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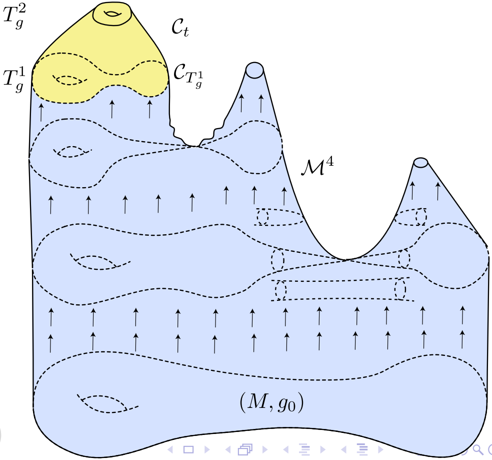

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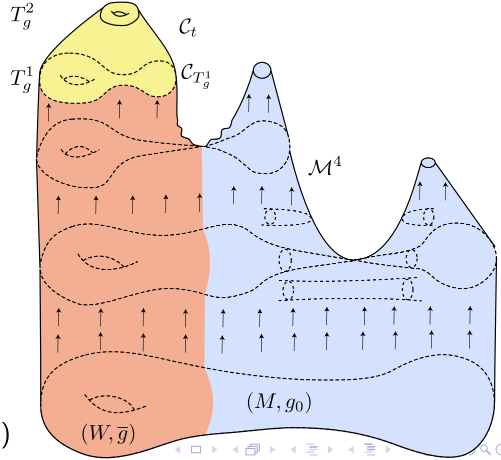
$$\overline{g}_t \xrightarrow[t \nearrow T_g^2]{} \overline{g} \text{ modulo rescaling}$$

$$(W, \overline{g}) \cong (S^3/\Gamma - \{p_1, \ldots, p_N\}, g_{K\equiv 1})$$



Conclusion

This process describes a continuous canonical map

$$\mathrm{Met}(M) \longrightarrow \mathrm{PartMet}_{K\equiv 1}(M)$$
$$g \longmapsto (W, \overline{g})$$

where $\mathrm{PartMet}_{K\equiv 1}(M)$ consists of pairs $(W, \overline{g})$ such that:

- $W \subset M$ open
- $\overline{g}$ is a metric on W
- $(W, \overline{g})$ is isometric to the round punctured S^3/Γ
- $M \setminus W$ can be covered finitely many pairwise disjoint disks
- If $K_g \equiv 1$, then $(W, \overline{g}) = (M, g)$.

Topology on $\mathrm{PartMet}_{K\equiv 1}(M)$: C^∞-convergence on compact subsets of W (not Hausdorff)

Proof of Main Theorem

Goal: Show $\mathrm{Met}_{K\equiv 1}(M) \simeq 1$, i.e. construct nullhomotopy for any

$$g : S^k = \partial D^{k+1} \longrightarrow \mathrm{Met}_{K\equiv 1}(M).$$

Solution:

1. extend g to continuous family

$$g' : D^{k+1} \longrightarrow \mathrm{Met}(M), \qquad g'|_{\partial D^{k+1}} = g$$

2. previous slide $\rightsquigarrow$ continuous family

$$(W(p), \widehat{g}(p)) \in \mathrm{PartMet}_{K\equiv 1}(M), \qquad p \in D^{k+1}$$

   such that $W(p) = M$ and $K_{\widehat{g}(p)} \equiv 1$ for $p \in \partial D^{k+1}$.

3. **Remaining:** "extend" $(W(p), \widehat{g}(p))$ to $\overline{g}(p) \in \mathrm{Met}_{K\equiv 1}(M)$ "up to contractible ambiguity".

Main ingredient:

Lemma

Let $A = A(1-\varepsilon, 1) \subset D(1) \subset \mathbb{R}^3$ and

$$h : D^k \longrightarrow \mathrm{Met}_{K\equiv 1}(A),$$

$$h_0 : \partial D^k \longrightarrow \mathrm{Met}_{K\equiv 1}(D(1))$$

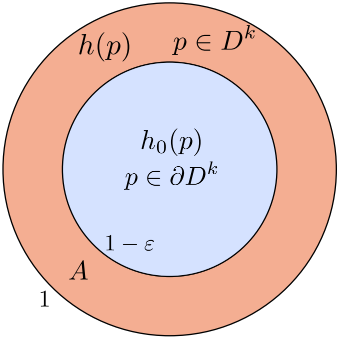
be continuous such that:

- $h_0(p)|_A = h(p)$ for all $p \in \partial D^k$.
- $(A, h(p))$ embeds into the round sphere for all $p \in D^k$.

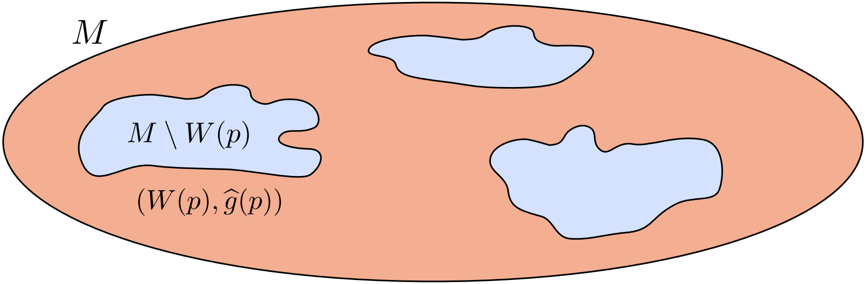
Then, after shrinking ε, there is a continuous map

$$\overline{h} : D^k \longrightarrow \mathrm{Met}_{K\equiv 1}(D(1))$$

with $\overline{h}(p)|_A = h(p)$ for all $p \in D^k$ and $\overline{h}(p) = h_0(p)$ for all $p \in \partial D^k$.

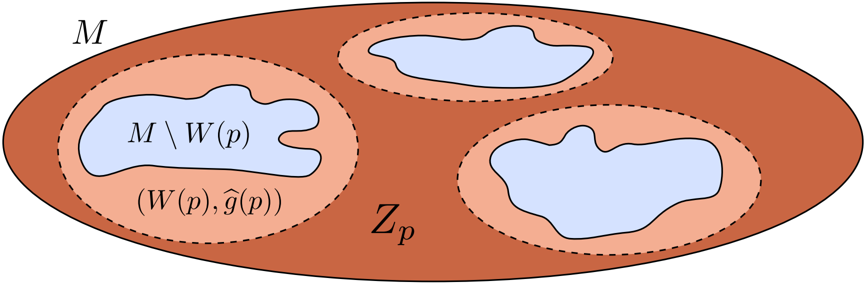


Proof: Hatcher's Theorem $\Longrightarrow \mathrm{Diff}(D^3 \, rel \, \partial D^3) \simeq 1$



Extending $(W(p), \widehat{g}(p))$ to $\overline{g}(p)$ on M:

- $p \in D^{k+1}$
- Choose compact domain $Z_p \subset W(p)$ such that $M \setminus \operatorname{Int} Z_p$ consists of finitely many disks.
- $Z_p \subset W(p')$ for p' close to p.
- Extend $\widehat{g}(p')|_{Z_p}$ to $\overline{g}(p) \in \mathrm{Met}_{K\equiv 1}(M)$, for p' close to p.
- $\overline{g}(p')$ is "unique up to contractible ambiguity" by previous Lemma
- Construct $\overline{g}(p')$ for all $p' \in D^{K+1}$ by induction over skeleta of a fine enough simplicial decomposition of D^{k+1}
- q.e.d.



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Part III: Further Questions

Further Questions

- $\mathbb{R}P^3$ case
- Reprove Hatcher's Theorem (S^3 case)
- (Re)prove Generalized Smale Conjecture for other geometric manifolds.

PSC Conjecture

$\mathrm{Met}_{R>0}(S^3) = \{g \in \mathrm{Met}(S^3) \mid R_g > 0\}$ is contractible.

Marques 2012: $\pi_0(\mathcal{R}^+(S^3)) = 0$.

Necessary Tools:

- Better understanding of continuous families of singular Ricci flows
- Asymptotic characterization of the flow.
  Does the flow always converge towards its geometric model?