The Borsuk-Ulam Theorem

Mark Powell

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Abstract

I give a proof of the Borsuk-Ulam Theorem which I claim is a simplified version of the proof given in Bredon [1], using chain complexes explicitly rather than homology. Of course this is a matter of taste, and the mathematical content is identical, but in my opinion this proof highlights precisely where and how the contradiction arises.

1 Statements and Preliminaries

Theorem 1.1 (Borsuk-Ulam)**.** *Let $f\colon S^n \to \mathbb{R}^n$ be a continuous map for $n \in \mathbb{N}$. Then there exists a point $x \in S^n$ such that $f(x) = f(-x)$ (*cf.* Theorem 20.2 of Bredon [1]).*

Lemma 1.2. *Let $\phi\colon S^n \to S^m$ be a continuous map which is equivariant with respect to the antipodal maps on S^n and S^m. Then $n \leq m$. (*cf.* Theorems 20.1, 20.6 of Bredon [1].)*

Proof of Theorem 1.1 assuming Lemma 1.2. Suppose not. That is suppose that $f\colon S^n \to \mathbb{R}^n$ such that $f(x) \neq f(-x)$ for all $x \in S^n$. Then define

$$\phi\colon S^n \to S^{n-1} \text{ by:}$$

$$x \mapsto \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}.$$

This satisfies $\phi(-x) = -\phi(x)$ *i.e.* ϕ is continuous and equivariant with respect to the antipodal maps which contradicts Lemma 1.2 $\Rightarrow\Leftarrow$. $\square$

Prior to giving a proof of Lemma 1.2, we explain the main ingredient, namely a $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex of S^r, which arises by considering it as the universal covering space of $\mathbb{RP}^r$.

Proposition 1.3. *There exists a cell decomposition of S^r with two cells in each dimension $0, \dots, r$ with $S^{r-1} \subset S^r$ embedded as the equator. There is a $\mathbb{Z}_2 \cong \langle T \mid T^2 = 1 \rangle$ action corresponding to the antipodal map which transposes the two cells in each dimension. With respect to this decomposition the $\mathbb{Z}_2$-equivariant cellular chain complex*

$$W[0,r] := C_*(\widetilde{\mathbb{RP}^r}; \mathbb{Z}) = C_*(\mathbb{RP}^r; \mathbb{Z}[\mathbb{Z}_2])$$

of S^r can be written as:

$$\begin{aligned} 0 \to \mathbb{Z}[\mathbb{Z}_2] \cong W_r &\xrightarrow{1+(-1)^r T} \mathbb{Z}[\mathbb{Z}_2] \cong W_{r-1} \xrightarrow{1+(-1)^{r-1}T} \dots \\ \dots \to \mathbb{Z}[\mathbb{Z}_2] \cong W_i &\xrightarrow{1+(-1)^i T} \mathbb{Z}[\mathbb{Z}_2] \cong W_{i-1} \to \dots \\ \dots \to \mathbb{Z}[\mathbb{Z}_2] \cong W_2 &\xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \cong W_1 \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \cong W_0 \to 0. \end{aligned}$$

That is:

$$W_i \cong \begin{cases} \mathbb{Z}[\mathbb{Z}_2] & i = 0, 1, \dots, r \\ 0 & \textit{otherwise} \end{cases}$$

$$\partial_i \colon W_i \to W_{i-1}; \;\; \partial_i(x) = (1 + (-1)^i T)(x).$$

Proof. To construct such a cell decomposition, begin with two 0-cells, which gives us S^0. Attach two 1-cells, both with one end at each 0-cell, to give us S^1. Next, attaching two 2-cells with a map of degree 1 gives a 2-sphere S^2. The process iterates: each sphere S^{r-1} can be embedded as the equator in the sphere S^r. It then bounds the two hemispheres on either side, both of which are homeomorphic to disks D^r, and therefore to r-cells. As claimed, the antipodal map sends one hemisphere to the other, inducing a $\mathbb{Z}_2$-action on the set of cells, and a $\mathbb{Z}[\mathbb{Z}_2]$-action on the cellular chain complex. The cellular chain complex, if considered as $\mathbb{Z}$-modules, would have two summands in each dimension, one for each cell. However, in view of the $\mathbb{Z}_2$-action we can abbreviate this to a single $\mathbb{Z}[\mathbb{Z}_2]$ summand. Here is the cellular chain complex of S^2 as $\mathbb{Z}$-modules, derived by considering the attaching of each cell with an orientation:

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \cong W_2 \xrightarrow{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z} \cong W_1 \xrightarrow{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z} \cong W_0 \to 0.$$

Translating this to $\mathbb{Z}[\mathbb{Z}_2]$-modules gives:

$$0 \to \mathbb{Z}[\mathbb{Z}_2] \cong W_2 \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \cong W_1 \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \cong W_0 \to 0,$$

which, when extended to the left as far as required gives the desired chain complex. As ever one should check that this really is a chain complex. $\mathbb{Z}[\mathbb{Z}_2]$ is commutative so there is only one check:

$$(1 - T)(1 + T) = 1 - T + T - T^2 = 1 - 1 = 0.$$

□

2 Proof of Lemma 1.2

We will now proceed to give the proof of the lemma, which we restate here:

Lemma 2.1. *Let $\phi\colon S^n \to S^m$ be a continuous map which is equivariant with respect to the antipodal maps on S^n and S^m. Then $n \leq m$.*

Proof. We suppose for a contradiction that $n > m$. By Cellular Approximation (see e.g. Theorem 4.8 of Hatcher [2]), we can assume that ϕ is cellular, which is to say that the k-skeleton of the domain is mapped into the k-skeleton of the codomain: $\phi((S^n)^{(k)}) \subset (S^m)^{(k)}$, for all k. This, and the equivariance of ϕ implies that ϕ induces a $\mathbb{Z}[\mathbb{Z}_2]$-module chain map $\phi_*\colon W[0,n] \to W[0,m]$.

To define the map $\phi_*\colon W[0,n]_i \to W[0,m]_i$ using degrees, take regular points y_1 and y_2, one in each i-cell of S^m, and count the number of points with sign of the inverse image sets $\phi^{-1}(y_1)$ and $\phi^{-1}(y_2)$ which live in *one* of the i-cells of S^n. As in the proof of Theorem 20.6 of Bredon [1] we can assume the map to be smooth in order to use transversality. Since ϕ is $\mathbb{Z}_2$-equivariant, it is sufficient to discover the induced map ϕ_* on a generator of $\mathbb{Z}[\mathbb{Z}_2]$ - that is on just one of the i-cells of S^n.

To see that this works consider the composition, for $i > 0$:

$$S^i \xrightarrow{(\mathrm{Id},0)} \frac{(S^n)^{(i)}}{(S^n)^{(i-1)}} = S^i \vee S^i \xrightarrow{\phi^{(i)}} \frac{(S^m)^{(i)}}{(S^m)^{(i-1)}} = S^i \vee S^i \xrightarrow{(c,0) \text{ or } (0,c)} S^i$$

where $(\mathrm{Id},0)$ is the inclusion into one of the wedge summands and c is the collapse map which collapses one of the wedge summands to a point. The choice of the final map gives two maps overall, from S^i to S^i. The degrees of these two maps, given by the orders with sign of the inverse image sets described above, yield the coefficients of 1 and T in ϕ_*.

The derivation above needs modifying for $i = 0$. $\phi_*\colon W[0,n]_0 \to W[0,m]_0$ as above. Here we have take one hemisphere of S^0, a single point D^0, and consider the map:

$$D^0 \xrightarrow{\phi|_{D^0}} S^0 \xrightarrow{(c,0) \text{ or } (0,c)} D^0.$$

The possible maps which arise as this pair of maps (given by the choice of two maps $S^0 \to D^0$) are ($\pm\,\mathrm{Id}$ and 0) or (0 and $\pm\,\mathrm{Id}$), which in the algebraic world are the $\mathbb{Z}[\mathbb{Z}_2]$ module homomorphisms ± 1 and $\pm T$. The sign choices correspond to orientations. This restriction of possibilities, which comes about from the fact that a geometric map must send a single point to a single point, and not to zero points and not to two points, will be used to derive our contradiction in due course.

Now since the argument from now on will only involve parity, to simplify the argument we only consider the coefficients modulo 2. Our ring is now $\mathbb{Z}_2[\mathbb{Z}_2] = \{0, 1, T, 1+T\}$ as a set. We therefore have the following chain map $\phi_*\colon W[0,n] \to W[0,m]$:

$$\begin{array}{ccccccccccc}
\mathbb{Z}_2[\mathbb{Z}_2] & \xrightarrow{(1+T)\cdot} & \cdots & \longrightarrow & \mathbb{Z}_2[\mathbb{Z}_2] & \xrightarrow{(1+T)\cdot} & \mathbb{Z}_2[\mathbb{Z}_2] & \xrightarrow{(1+T)\cdot} & \mathbb{Z}_2[\mathbb{Z}_2] & \xrightarrow{(1+T)\cdot} \cdots \xrightarrow{(1+T)\cdot} & \mathbb{Z}_2[\mathbb{Z}_2] \\
\downarrow{\scriptstyle \phi_*^n} & & & & \downarrow{\scriptstyle \phi_*^{m+1}} & & \downarrow{\scriptstyle \phi_*^{m}} & & \downarrow{\scriptstyle \phi_*^{m-1}} & & \downarrow{\scriptstyle \phi_*^{0}} \\
0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}_2[\mathbb{Z}_2] & \xrightarrow{(1+T)\cdot} & \mathbb{Z}_2[\mathbb{Z}_2] & \xrightarrow{(1+T)\cdot} \cdots \longrightarrow & \mathbb{Z}_2[\mathbb{Z}_2].
\end{array}$$

Since this is a chain map, the diagram commutes. From the square:

$$\begin{array}{ccc} \mathbb{Z}_2[\mathbb{Z}_2] & \xrightarrow{(1+T)\cdot} & \mathbb{Z}_2[\mathbb{Z}_2] \\ {\scriptstyle \phi_*^{m+1}}\downarrow & & \downarrow{\scriptstyle \phi_*^{m}} \\ 0 & \longrightarrow & \mathbb{Z}_2[\mathbb{Z}_2] \end{array}$$

we have that

$$0 = \phi_*^m \circ (1+T)\cdot .$$

As ϕ is $\mathbb{Z}_2$-equivariant recall that ϕ_* is a $\mathbb{Z}_2[\mathbb{Z}_2]$-module homomorphism, so it suffices to obtain $\phi_*^m(1)$. Suppose $\phi_*^m(1) = a + bT$. Then

$$0 = 0 + 0T = (a+bT)(1+T) = (a+b) + (a+b)T \in \mathbb{Z}_2[\mathbb{Z}_2],$$

so that $a = b \mod 2$. Therefore

$$\phi_*^m = 0\cdot \text{ or } (1+T)\cdot .$$

Next, consider the next square to the right:

$$\begin{array}{ccc} \mathbb{Z}_2[\mathbb{Z}_2] & \xrightarrow{(1+T)\cdot} & \mathbb{Z}_2[\mathbb{Z}_2] \\ {\scriptstyle \phi_*^{m}}\downarrow & & \downarrow{\scriptstyle \phi_*^{m-1}} \\ \mathbb{Z}_2[\mathbb{Z}_2] & \xrightarrow{(1+T)\cdot} & \mathbb{Z}_2[\mathbb{Z}_2] \end{array}$$

Since

$$(1+T)(1+T) = 2 + 2T = 0 \in \mathbb{Z}_2[\mathbb{Z}_2],$$

in either instance of the map ϕ_*^m, we have that

$$0 = \phi_*^{m-1} \circ (1+T)\cdot .$$

The same argument applies once more to show that

$$\phi_*^{m-1} = 0\cdot \text{ or } (1+T)\cdot .$$

The argument then iterates thus, so that by induction we deduce that:

$$\phi_*^0 = 0\cdot \text{ or } (1+T)\cdot$$

which produces our desired contradiction; recall from above that the only possible homomorphisms which can arise as induced maps on the zero chains, from a *geometric* map, are $1\cdot$ and $T\cdot$.

$$\Rightarrow\Leftarrow$$

□

Remark 2.2. A similarly phrased proof of the Brouwer fixed point theorem involves a much simpler algebraic argument: the work is in showing, in the absence of a fixed point, the existence of a *retract* $r\colon D^n \to S^{n-1}$, that is a continuous map which is identity on S^{n-1}. This would induce a chain map $r_*\colon D^n \to S^{n-1}$ as follows:

$$\begin{array}{ccccccccccc}
\mathbb{Z} & \xrightarrow{1} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} \\
\downarrow & & \downarrow{\scriptstyle r_*=1} & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle r_*=1} \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z},
\end{array}$$

for which the failure of the left hand square to commute yields the contradiction.

Remark 2.3. In order to better parallel the proof in the previous remark, and in Bredon [1], perhaps the induction could run from right to left instead of from left to right: starting with the fact that the rightmost map ϕ^0_* is 1 or T, use commutativity to show that ϕ^m_* is 1 or T so that the failure of the square

$$\begin{array}{ccc}
\mathbb{Z}_2[\mathbb{Z}_2] & \xrightarrow{(1+T)\cdot} & \mathbb{Z}_2[\mathbb{Z}_2] \\
{\scriptstyle \phi^{m+1}_*}\downarrow & & \downarrow{\scriptstyle \phi^m_*=1 \text{ or } T} \\
0 & \longrightarrow & \mathbb{Z}_2[\mathbb{Z}_2]
\end{array}$$

to commute yields the desired contradiction. I conjecture that it would be desirable to associate an integral to each passage around this square, perhaps with one of them requiring m applications of Stoke's theorem to get to the far right. ??

Remark 2.4. The chain complex W (of S^∞, the direct limit of the $W[0,r]$) is of great importance in topology; in the first instance in the construction of the Steenrod Squares, and thence for the definition of the Stiefel-Whitney classes, through to its significance in the symmetric and quadratic constructions in the Algebraic Theory of Surgery [3].

References

[1] Glen Bredon. *Topology and Geometry.* Springer-Verlag, 1993.

[2] Allen Hatcher. *Algebraic Topology.* CUP, 2001.

[3] A.A. Ranicki. The algebraic theory of surgery I and II. *Proc. London Math. Soc.*, (3) 40:87–192, 1980.