Whitney's First Embedding Theorem

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Abstract

This is an outline of the proof of Whitney's first embedding theorem which states that every n-dimensional differentiable manifold can be embedded in $\mathbb{R}^{2n+1}$ as a closed subset. Although the proof of this theorem has been considerably simplified since it was first published in 1936, this presentation follows closely the steps outlined in [1] which in turn is inspired from Whitney's original proof.

1 Introduction

We will start by listing some definitions and recalling few well known elementary facts from measure theory and from the theory of topology of smooth manifolds.

1.1 Definition

We define the n-cube $C^n(x,r)$ centred at x with edge length r as the cartesian product of n open intervals (a,b) of length r centred at x. One can think of $C^n(x,r)$ as being the generalization of the usual 3-dimensional cube. The *Lebesgue measure* of an n-dimensional cube is defined as its volume in the usual way and it is equal to r^n.

1.2 Definition

We say that a subset $A \in \mathbb{R}^n$ has *Lebesgue measure zero* if for any $\epsilon > 0$ there exists a countable collection of n-cubes which covers A

$$A \subset \bigcup_j C^n(x_j, r_j)$$

such that

$$\sum_j r_j^n < \epsilon$$

1.3 Lemma

If $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a smooth function and $A \subset U$ is an open subset with measure zero, then its image $f(A)$ has measure zero as well.
To see this, take an n-cube $C \subset \overline{C} \subset U$; then $A \cap C \subset C \Longrightarrow m(A \cap C) = 0$ from properties of Lebesgue measure. So this intersection can be covered by a countable family of n-cubes whose total volume is arbitrarily small

$$A \cap C \subset \bigcup_{i=1}^{\infty} C^n(x_i, r_i)\,; \quad \sum_j r_j^n < \epsilon$$

Since f is smooth, its first partial derivatives are bounded on the compact set $\overline{C}$. Let

$$b = \max_{x \in \overline{C},\ i,j} \left| \left(\frac{\partial f_i}{\partial x_j} \right)_x \right|$$

Then by the mean value theorem we get

$$\|f(x) - f(y)\| \leq \left(n^2 b^2\right)^{1/2} \|x - y\| = nb \|x - y\|$$

This means the image of of an n-cube of radius r is contained in an n-cube of radius nbr. Then for any $\epsilon > 0$, by covering $A \cap C$ with a countable number of n-cubes $C_k^n\left(x_k, \epsilon/2^k\right)$, the image $f(A \cap C)$ has measure zero. Repeat the same argument for the countable covering of A and the result follows.
The corollary below will use the fact that any k-subspace $\mathbb{R}^k \subset \mathbb{R}^n$ for $k < n$, has measure zero. This is due to the fact that for any $\epsilon > 0$ the subspace $\mathbb{R}^k$ can be covered with a countable number of n-cubes with all edges of length 1 except for one edge along a dimension outside of $\mathbb{R}^k$ which will have length equal to $\epsilon/2^m$. Then the volume of each n-cube is $\epsilon/2^m$ and their total volume is

$$\sum_{m=1}^{\infty} \epsilon/2^m = \epsilon$$

1.4 Corollary

Given $n < p$, let U be an open subset of $\mathbb{R}^n$, and $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^p$ be a smooth map. Then the image of U under f has measure zero.

Proof. Consider the map

$$g : U \times \mathbb{R}^{p-n} \xrightarrow{\pi} U \xrightarrow{f} \mathbb{R}^p$$

where π is the natural projection map. Then g is a smooth map since it is a composition of two smooth maps. Since $U \cong U \times \{0\} \subset U \times \mathbb{R}^{p-n}$ has measure zero then $f(U) = g(U \times \{0\})$ also has measure zero by the previous lemma. □

1.5 Corollary

If $f : N^n \longrightarrow M^m$ is a smooth math of manifolds of degree n and m respectively, and if $n < m$ the $f(N^n)$ has measure zero in M^m.

1.6 Lemma

Let $M(p, n)$ denote the set of all real $p \times n$ matrices with the product topology $\mathbb{R}^{pn}$, and $M_k(p, n) \subset M(p, n)$ denote the subset of matrices with rank k. Then $M_k(p, n)$ has dimension $k(p + n - k)$ as a submanifold.

2 Whitney's First Embedding Theorem

Every n-dimensional differentiable manifold can be embedded in $\mathbb{R}^{2n+1}$ as a closed subset.

To prove this theorem, we will first establish some sort of a hierarchy of maps. We will show that under certain conditions any smooth function between two manifolds can be arbitrarily approximated by an immersion, which in turn can be arbitrarily approximated by an injective immersion. The latter, then can be approximated by embedding. We begin by the following theorem,

2.1 Theorem

Let $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^p$ be a smooth map with $2n \leq p$. Then for any $\epsilon > 0$ there exists a $p \times n$ matrix $A = (a_{ij})$ such that

1. $|a_{ij}| < \epsilon$
2. The map $g : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^p$ defined by $g(x) = f(x) + Ax$ is an immersion.

Proof. We need to show that there exists a matrix A such that $|a_{ij}| < \epsilon$ and

$$rank\,(Dg\,(x)) = rank\,(Df\,(x) + A) = n$$

Given any $\epsilon > 0$, let $\Lambda = \{A_{p\times n} : \ |a_{ij}| < \epsilon\}$. Then the measure $m\,(\Lambda) > 0$ since $m\,((-\epsilon, \epsilon)) = 2\epsilon > 0$ and thus $m\,(\Lambda) = (2\epsilon)^{p\times n} > 0$.
$g\,(x) = f\,(x) + Ax \Longrightarrow Dg\,(x) = Df\,(x) + A$. Consider the subset $\Omega \subset \Lambda$ such that

$$\Omega = \{A \in \Lambda : \ A = B - Df\,(x)\,; \ rank\,(B) < n\}$$

We will show that Ω has measure zero which will then imply the existence of the desired matrix A. To do so, we consider the

$$F_k : \ M_k\,(p, n) \times U \longrightarrow M\,(p, n)\,; \ F_k\,(B, x) = B - Df\,(x)$$

where $rank\,(B) < n$. Notice that $dim\,(M_k\,(p, n) \times U) = k\,(p + n - k) + n$. If we let $\phi\,(k) = k\,(p + n - k) + n$, then $\phi'\,(k) = -2k + p + n > 0$ since $2k < 2n \leq p$. So ϕ is increasing. Then

$$\begin{aligned} k \leq n - 1 \Longrightarrow k\,(p + n - k) &\leq (n - 1)\,(p + n - (n - 1)) \\ &= (2n - p) + pn - 1 \\ &< pn \\ &= dim\,(M\,(p, n)) \end{aligned}$$

By corollary 1.5, $m\,(F_k\,(M_k\,(p, n) \times U)) = 0$. The existence of the matrix A follows"' □

We can restate the preceding theorem using the notion of δ-approximations which we define as follows:

2.2 Definition

Let X be a topological space and (Y, d) be a metric space with metric d. We say that a function $g : \ X \longrightarrow (Y, d)$ is a δ-approximation of a function $f : \ X \longrightarrow (Y, d)$ if for any $\epsilon > 0$, there exists a function $\delta : X \longrightarrow (0, \epsilon)$ such that

$$d\,(f\,(x)\,, g\,(s)) < \delta\,(x) \ \ \forall x \in X$$

Then the previous theorem states that for any n-dimensional manifold M and a smooth function $f : M \longrightarrow \mathbb{R}^p$ where $2n \leq p$, we can find an immersion $g : M \longrightarrow \mathbb{R}^p$ which is a δ-approximation to f. In fact, more can be said. If $rank\,(f) = n$ on some closed subset $N \subset M$, then g can be chosen to be homotopic to f relative to N.

The proof of this last statement is rather tedious and is therefore omited here. The full proof is given in[1] for the interested reader.
We have thus far encountered the first bound on the dimension of the Euclidean space in which a manifold can be immersed. As we have seen above, this limitation is due to considerations related to measure theory. The next limitation will appear when we seek to make the above immersion injective. The proof relies on the fact that immersions are local embeddings, that is for each $x \in M$ there exists a neighborhood U_x on which the immersion is an embedding. By covering the manifold with such neighborhoods and taking a locally finite countable refinement, one can use induction to construct the desired function from cut-off fuctions. Similarly, one can choose such a function to be regularly homotopic to the immersion relative to some closed set. We will formally state this result in the lemma below and once again refer the reader to [1] for the proof.

2.3 Lemma

Let $f : M \longrightarrow \mathbb{R}^p$ be an immersion of an n-dimensional manifold M, with $2n < p$. Then there exists an injective immersion g which is a δ-approximation of f. Moreover, if f is injective on some open subset of a closed set $N \in M$, g can be chosen to be regularly homotopic to f relative to N.

2.4 Definition

Let $f : M \longrightarrow \mathbb{R}^p$ be a continuous function. We define the *limit set* of f, $L(f)$ by

$$L(f) = \left\{ y \in \mathbb{R}^p : \ y = \lim_{n\to\infty} f(x_n) \text{ for some sequence with no limit points in M} \right\}$$

2.5 Lemma

Let M be an n-dimensional manifold and $f : M \longrightarrow \mathbb{R}^n$ be an injective immersion. Then

1. The range $f(M)$ is closed in $\mathbb{R}^n$ if and only if $L(f) \subset f(M)$, and
2. f is an embedding if and only if $L(f) \cap f(M) = \emptyset$.

Proof. 1. ($\Longrightarrow$) Suppose $f(M)$ is closed. Let $y = \lim_{n\to\infty} f(x_n)$ with $\{x_n\} \in M$ being a sequence in M with no limit points. Then $f(M)$ closed implies $\lim_{n\to\infty} f(x_n) \in f(M)$. So $L(f) \subset f(M)$.

2. ($\Longleftarrow$) Let $y \in \overline{f(M)}$. Then y must be a limit to some sequence $y = \lim_{n\to\infty} f(x_n)$.
If $\{x_n\}$ has a converging subsequence $x_{n_k} \longrightarrow c$, then $f(c) = y$ and so $y \in f(M)$ and hence $f(M)$ is closed. Otherwise (ie: if $\{x_n\}$ has no converging subsequences), then $y \in L(f)$ which is in $f(M)$ by assumption. Hence $f(M)$ is closed.

3. ($\Longrightarrow$) Let f be an embedding and assume that $y \in L(f) \cap f(M)$. Then $y = \lim_{n\to\infty} f(x_n)$ for some sequence $\{x_n\} \in M$. But f is an embedding $\Longrightarrow x_n \longrightarrow f^{-1}(y) = c$ which is a contradiction since $y \in L(y)$ and thhus $\{x_n\}$ cannot converge. So no such y exists: $L(f) \cap f(M) = \emptyset$.
($\Longleftarrow$) Let $f(x_n) \longrightarrow y$ be a sequence in M converging to y. Then $x_n \longrightarrow c$ for some $c \in M$ because if $\{x_n\}$ has no limit point then y would necessarily lie in $L(f)$ and thus in $L(f) \cap f(M)$ which by assumption is empty. Also, note that the possiblity of $\{x_n\}$ having two distinct converging subsequences is excluded by virtue of injectivity of f. This completes the proof.

□

2.6 Remark

Note that in order for an injective immersion $f : M \longrightarrow \mathbb{R}^n$ to be an embedding with a closed image, both $L(f) \subset f(M)$ and $L(f) \cap f(M) = \emptyset$ must be satisfied, in which case $L(f)$ is necessarily empty.

We now state one more known fact about smooth manifolds. Detailed proof can be found in [2].

2.7 Lemma

Let M be an n-dimensional manifold. Then any open covering $\bigcup_\alpha \{(U_\alpha, \psi_\alpha)\} \supset M$ admits a countable, locally finite refinement $\{V_j, h_j\}$ such that

1. $h_j(V_j) = C^n(3)$
2. $M = \bigcup_j W_j$ where $W_j = h_j^{-1}(C^n(1))$ and has compact closure.

Moreover, for each W_j there exists a smooth function $\phi_j : M \longrightarrow \mathbb{R}^n$ which satifies the following

1. $\phi_j(x) = 1$ for all $x \in \overline{W_j}$,
2. $0 < \phi_j(x) < 1$ for all $x \in h_j^{-1}\left(C^n(2) - \overline{C^n(1)}\right)$, and
3. $\phi_j(x) = 0$ for all $x \in \mathbb{R}^n - C^n(2)$.

Such a function is called a *cut-off function.*

2.8 Lemma

Let M be an n-maifold. Then for any integer $p > 0$ there exists a smooth map $f : M \longrightarrow \mathbb{R}^p$ with $L(f) = \emptyset$.

Proof. Take a collection of cut-off functions $\{\phi_j\}$ associated with the covering $\{V_j, h_j\}$ as in lemma 2.7. Define the function $f : M \longrightarrow \mathbb{R}^n$ by

$$f(x) = \sum_j j\phi_j(x)$$

This function is well-defined because the covering is locally finite, so there are only finitely many cut-off functions which don't vanish at x. Since ϕ_j are smooth for all j, then f is smooth. To show that $L(f) = \emptyset$, we take a sequence $\{x_n\} \in M$ with no limit points. Then for any $k > 0$ there exists a positive integer $n > 0$ such that

$$x_n \notin \overline{W_1} \cup \overline{W_2} \cdots \cup \overline{W_k}$$

For if not, then there exists a k such that

$$x_n \in \bigcup_{j=1}^{k} \overline{W_j} \ \forall n \in \mathbb{N}$$

But this would lead to a contradiction since $\bigcup_{j=1}^{k} \overline{W_j}$ is compact and so the sequence $\{x_n\}$ must admit a converging subsequence which cannot happen by assumption. Therefore $x_n \in \overline{W_i}$ for some $i > k$. But the this means

$$f(x_n) = \sum_j \phi_j(x_n) \geq i\phi_i(x_n) = i > k$$

This argument shows that if a sequence $\{x_n\}$ does not have a limit point, then the corresponding sequence $f(x_n)$ diverges to infinity. Hence $L(f) = \emptyset$. $\square$

2.9 Remark

If $g : M \longrightarrow \mathbb{R}^{2n+1}$ is a smooth function with $L(g) = \emptyset$ and if $f : M \longrightarrow \mathbb{R}^{2n+1}$ is an injective immersion which is a δ-approximation of g, then $L(f) = \emptyset$ To see this, take a sequence $\{x_n\} \in M$ which has no limit points. Assume $y = \lim_{n\to\infty} f(x_n)$ exists. Since $L(g) = \emptyset$, there exists $\epsilon > 0$ and $N > 0$ such that $\|g(x_n) - y\| \geq \epsilon$ for all $n > N$. But then if we choose $\delta : M \longrightarrow \mathbb{R}_+$ such that $\delta(\epsilon/2)$, we get a contradiction since if $\lim_{n\to\infty} f(x_n) = y$ then

$$\|f(x_n) - g(x_n)\| < \epsilon \implies \epsilon > \lim_{n\to\infty} \|f(x_n) - g(x_n)\| = \|y - g(x_n)\| > \epsilon$$

We now have acquired the necessary tools to prove the main theorem of this presentation

Proof. There exists a smooth function $f :\longrightarrow \mathbb{R}^{2n+1}$ with $L(f) = \emptyset$. For any $\epsilon > 0$ let $\delta : M \longrightarrow \mathbb{R}_+$ such that $\delta(x) = \epsilon$ (constant). Then there exists an immersion $g : M \longrightarrow \mathbb{R}^{2n+1}$ which is a δ-approximation of f. Then there exists an injective immersion $h : M \longrightarrow \mathbb{R}^{2n+1}$ which is a δ-approximation of g with $L(h) = \emptyset$. Hence h is an embedding and $h(M)$ is closed.

□

3 References

[1] Milton Persson. The Whitney Embedding Theorem. Umea UniversityV̇T 2014
[2] William M. Boothby. An Introduction to Differentiable Manifolds and Riemannian Geometry. Elsevier.2008.