

The mapping class group of connect sums of $S^2 \times S^1$

Tara Brendle Nathan Broaddus Andrew Putman*

Abstract

Let M_n be the connect sum of n copies of $S^2 \times S^1$. A classical theorem of Laudenbach says that the mapping class group $\text{Mod}(M_n)$ is an extension of $\text{Out}(F_n)$ by a group $(\mathbb{Z}/2)^n$ generated by sphere twists. We prove that this extension splits, so $\text{Mod}(M_n)$ is the semidirect product of $\text{Out}(F_n)$ by $(\mathbb{Z}/2)^n$, which $\text{Out}(F_n)$ acts on via the dual of the natural surjection $\text{Out}(F_n) \rightarrow \text{GL}_n(\mathbb{Z}/2)$. Our splitting takes $\text{Out}(F_n)$ to the subgroup of $\text{Mod}(M_n)$ consisting of mapping classes that fix the homotopy class of a trivialization of the tangent bundle of M_n . Our techniques also simplify various aspects of Laudenbach's original proof, including the identification of the twist subgroup with $(\mathbb{Z}/2)^n$.

1 Introduction

The *mapping class group* of a closed oriented 3-manifold M^3 , denoted $\text{Mod}(M^3)$, is the group of isotopy classes of orientation-preserving diffeomorphisms of M^3 . In this paper, we study the mapping class group of the connect sum M_n of n copies of $S^2 \times S^1$. The fundamental group $\pi_1(M_n)$ is the free group F_n on n letters. Since diffeomorphisms of M_n do not fix a basepoint, the action of $\text{Mod}(M_n)$ on $\pi_1(M_n)$ is only defined up to conjugation, so it gives a homomorphism

$$\rho: \text{Mod}(M_n) \longrightarrow \text{Out}(\pi_1(M)) \cong \text{Out}(F_n).$$

It follows from work of Whitehead [25, 26] that ρ is surjective. Its kernel was described by Laudenbach [20, 21].

Sphere twists. The kernel of ρ is the subgroup generated by sphere twists, which are defined as follows. Let M^3 be a closed oriented 3-manifold and let $S \subset M^3$ be a smoothly embedded 2-sphere. Fix a tubular neighborhood $U \cong S \times [0, 1]$ of S . Recall that $\pi_1(\text{SO}(3), \text{id}) \cong \mathbb{Z}/2$ is generated by a loop $\ell: [0, 1] \rightarrow \text{SO}(3)$ that rotates \mathbb{R}^3 about an axis by one full turn. Identifying S with $S^2 \subset \mathbb{R}^3$, the *sphere twist* about S , denoted T_S , is the isotopy class of the diffeomorphism $\tau: M^3 \rightarrow M^3$ that is the identity outside U and on $U \cong S \times [0, 1]$ takes the form $\tau(s, t) = (\ell(t) \cdot s, t)$. The isotopy class of T_S only depends on the isotopy class of S . In fact, Laudenbach [20, 21] proved that if S and S' are homotopic 2-spheres in M^3 that are non-nullhomotopic, then S and S' are isotopic, so T_S actually only depends on the homotopy class of S .

*Supported in part by NSF grant DMS-1811210

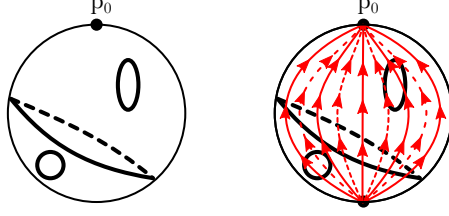


Figure 1: On the left is S along with p_0 and $Z \cap S$ (which here is 1-dimensional, so $\dim(Z) = 2$). On the right we show how to homotope Z such that $Z \cap S = \{p_0\}$ – choose a point $q \in S \setminus (\{p_0\} \cup Z)$, and homotope Z so as to push its intersection with S along paths from q to p_0 until it is entirely contained in p_0 .

Actions of sphere twists. We have $\mathrm{SO}(3) \cong \mathbb{RP}^3$, so $\pi_1(\mathrm{SO}(3), \mathrm{id}) \cong \mathbb{Z}/2$. It follows that the mapping class T_S has order at most 2. It follows from Laudenbach’s work that in the case $M^3 = M_n$, the sphere twist T_S is trivial if S separates M_n and has order 2 if S is nonseparating. Showing that T_S is ever nontrivial is quite subtle since T_S fixes the homotopy class of any loop or surface Z in M^3 , and thus cannot be detected by most basic algebro-topological invariants. To see this, let $U \cong S \times [0, 1]$ be the tubular neighborhood used to construct T_S and let $p_0 \in S$ be one of the two points of S lying on the axis of rotation used to construct T_S . Homotope Z to be transverse to S . The intersection $Z \cap S$ is then either a collection of circles (if $\dim(Z) = 2$) or points (if $\dim(Z) = 1$). As is shown in Figure 1, we can then homotope Z such that $Z \cap U \subset p_0 \times [0, 1]$, so T_S fixes Z .

Twist subgroup. The *twist subgroup* of $\mathrm{Mod}(M^3)$, denoted $\mathrm{Twist}(M^3)$, is the subgroup generated by all sphere twists. For $f \in \mathrm{Mod}(M^3)$ and a sphere twist $T_S \in \mathrm{Twist}(M^3)$, we have

$$fT_Sf^{-1} = T_{f(S)}.$$

This implies that $\mathrm{Twist}(M^3)$ is a normal subgroup of $\mathrm{Mod}(M^3)$. Also, if T_S and $T_{S'}$ are sphere twists, we saw above that $T_{S'}(S)$ is homotopic to S . Since a sphere twist only depends on the homotopy class of the sphere along which we are twisting, setting $f = T_{S'}$ in the above relation we get

$$T_{S'}T_S T_{S'}^{-1} = T_S.$$

In other words, $\mathrm{Twist}(M^3)$ is abelian.

Laudenbach sequence. Laudenbach’s theorem can thus be summarized as saying that there is a short exact sequence

$$1 \longrightarrow \mathrm{Twist}(M_n) \longrightarrow \mathrm{Mod}(M_n) \xrightarrow{\rho} \mathrm{Out}(F_n) \longrightarrow 1. \quad (1.1)$$

He also proved that $\mathrm{Twist}(M_n) \cong (\mathbb{Z}/2)^n$ and is generated by the sphere twists about the core spheres $S^2 \times *$ of the n different $S^2 \times S^1$ summands of M_n . This theorem raises two natural questions:

1. Does the extension (1.1) split?
2. The conjugation action of $\text{Mod}(M_n)$ on its normal abelian group $\text{Twist}(M_n) \cong (\mathbb{Z}/2)^n$ induces an action of $\text{Out}(F_n)$ on $\text{Twist}(M_n) \cong (\mathbb{Z}/2)^n$. What action is this?

Main theorem. Our main theorem answers both of these questions. It says that the extension (1.1) does split, and in fact the image of the splitting $\text{Out}(F_n) \rightarrow \text{Mod}(M_n)$ has a simple geometric description: it is the stabilizer of the homotopy class of a trivialization of the tangent bundle of M_n . A precise statement is as follows.

Theorem A. *Let $[\sigma_0]$ be the homotopy class of a trivialization σ_0 of the tangent bundle of M_n and let $(\text{Mod}(M_n))_{[\sigma_0]}$ be the $\text{Mod}(M_n)$ -stabilizer of $[\sigma_0]$. The following then hold:*

- $\text{Mod}(M_n) = \text{Twist}(M_n) \rtimes (\text{Mod}(M_n))_{[\sigma_0]}$.
- $\text{Twist}(M_n) \cong H^1(M_n; \mathbb{Z}/2)$ as a $\text{Mod}(M_n)$ -module.
- $(\text{Mod}(M_n))_{[\sigma_0]} \cong \text{Out}(F_n)$.

Remark 1.1. Before Laudenbach's work, Gluck [6] proved that $\text{Mod}(M_1) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. The first factor was a sphere twist, and the second factor was $\text{Out}(F_1) = \mathbb{Z}/2$. This is of course a special case of Theorem A.

Sphere complex and $\text{Out}(F_n)$. Laudenbach's exact sequence (1.1) plays an important role in the study of $\text{Out}(F_n)$. In his seminal paper [10], Hatcher defined the *sphere complex* \mathbb{S}_n to be the simplicial complex whose k -simplices are sets $\{S_0, \dots, S_k\}$ of isotopy classes of non-nullhomotopic smoothly embedded 2-spheres in M_n that can be realized disjointly. One of his main theorems says that \mathbb{S}_n is contractible. The group $\text{Mod}(M_n)$ acts on \mathbb{S}_n , and since sphere twists fix the isotopy class of any smoothly embedded 2-sphere the twist subgroup $\text{Twist}(M_n)$ acts trivially. By (1.1), we thus get an action of $\text{Out}(F_n)$ on \mathbb{S}_n .

The space \mathbb{S}_n is also sometimes called the *free splitting complex* and has played an important role in a huge amount of subsequent work (see, e.g., [1, 3, 7, 8, 11, 14, 15, 16]). It is unsatisfying that the original construction of the action of $\text{Out}(F_n)$ on \mathbb{S}_n was so indirect: you first construct an action of $\text{Mod}(M_n)$, then notice that $\text{Twist}(M_n)$ acts trivially, and only then get an induced action of the quotient group $\text{Out}(F_n) = \text{Mod}(M_n)/\text{Twist}(M_n)$. It follows from Theorem A that $\text{Out}(F_n)$ can be embedded as a subgroup of $\text{Mod}(M_n)$, so there is no longer a need to perform this indirect construction.

Nontriviality of sphere twists. Our proof also gives a new and easier argument for seeing that the sphere twists T_S about nonseparating spheres in M_n (and other

3-manifolds) are nontrivial (cf. Corollary 5.2). This is not as easy as one might expect. The usual way that one studies a group like $\text{Mod}(M_n)$ is via its action on homotopy classes of submanifolds of M_n . However, the sphere twists T_S fix the homotopy classes of all loops and surfaces in M_n , so a new idea is needed. Laudendach used an argument involving framed cobordism and the Pontryagin–Thom construction, while as we describe below we study the action of $\text{Mod}(M_n)$ on trivializations of the tangent bundle.

Remark 1.2. The idea of using the Pontryagin–Thom construction to study sphere-twists in 3-manifolds goes back to early work of Pontryagin; see the example at the end of [24, §4].

Derivative crossed homomorphism. The heart of our proof of Theorem A is what we call the derivative crossed homomorphism. This is a cocycle on $\text{Mod}(M_n)$ constructed using the action of $\text{Mod}(M_n)$ on the set of trivializations of the tangent bundle of M_n . A similar idea (using stable trivializations rather than trivializations) has been used to study mapping class groups of high-dimensional manifolds, where it is used to understand various short exact sequences coming from surgery theory. The earliest appearance of this idea seems to be work of Krylov [18], and it was further developed by Crowley [4] and Krannich [17].

Automorphisms vs outer automorphisms. Let $M_{n,1}$ be M_n equipped with a basepoint $* \in M_n$, and define $\text{Mod}(M_{n,1})$ to be the group of isotopy classes of orientation-preserving diffeomorphisms of $M_{n,1}$ that fix $*$. The group $\text{Mod}(M_{n,1})$ then acts on $\pi_1(M_{n,1}, *) = F_n$, so we get a homomorphism $\text{Mod}(M_{n,1}) \rightarrow \text{Aut}(F_n)$. It follows from Laudendach’s work that we also have a short exact sequence

$$1 \longrightarrow \text{Twist}(M_{n,1}) \longrightarrow \text{Mod}(M_{n,1}) \longrightarrow \text{Aut}(F_n) \longrightarrow 1$$

and that $\text{Twist}(M_{n,1}) = (\mathbb{Z}/2)^n$. Our work shows that this sequence also splits, and a result identical to Theorem A holds. This can be proved either by adapting our proof (which needs almost no changes), or by using the exact sequence

$$1 \longrightarrow F_n \longrightarrow \text{Mod}(M_{n,1}) \longrightarrow \text{Mod}(M_n) \longrightarrow 1 \tag{1.2}$$

arising from the long exact sequence in homotopy groups of the fiber bundle

$$\text{Diff}^+(M_n, *) \longrightarrow \text{Diff}^+(M_n) \longrightarrow M_n.$$

The F_n in (1.2) is the image of $\pi_1(M_n)$ in $\text{Mod}(M_{n,1}) = \pi_0(\text{Diff}^+(M_n, *))$ and maps to the inner automorphisms in $\text{Aut}(F_n)$. We leave the details to the interested reader.

Other 3-manifolds. Let M^3 be an arbitrary closed orientable 3-manifold and let $\pi = \pi_1(M^3)$. The twist subgroup $\text{Twist}(M^3)$ is still an abelian normal subgroup of the

mapping class group $\text{Mod}(M^3)$, and it turns out that there is still an exact sequence

$$1 \longrightarrow \text{Twist}(M^3) \longrightarrow \text{Mod}(M^3) \longrightarrow G \longrightarrow 1, \quad (1.3)$$

where $G < \text{Out}(\pi)$ is the image of $\text{Mod}(M^3)$ in $\text{Out}(\pi)$. See [12, Proposition 2.1] for how to extract this from the literature; in fact, this reference also gives an appropriate but more complicated statement for 3-manifolds with boundary. It is often the case that $G = \text{Out}(\pi)$; see [12, Proposition 2.2] for some conditions that ensure this.

In light of Theorem A, it is natural to wonder whether (1.3) splits. Let $\text{Twist}_{\text{ns}}(M^3)$ be the subgroup of $\text{Twist}(M^3)$ generated by sphere twists about nonseparating spheres. Our proof of Theorem A can be generalized to show that we have a splitting

$$\text{Mod}(M^3) = \text{Twist}_{\text{ns}}(M^3) \rtimes \Gamma,$$

where Γ is a subgroup of $\text{Mod}(M^3)$. However, we cannot show that Γ can be taken to be the stabilizer of the homotopy class of a trivialization of TM^3 (c.f. the proof of Theorem 5.3 below).

Unfortunately, we do not know how to deal with separating sphere twists. For $M^3 = M_n$, separating sphere twists are always trivial, so $\text{Twist}_{\text{ns}}(M^3) = \text{Twist}(M^3)$ and separating twists can be ignored when studying (1.3). However, for general 3-manifolds the situation is very complicated. Let $S \subset M^3$ be a 2-sphere that separates M^3 .

- In [13], Hendriks gives a remarkable characterization of when T_S is homotopic to the identity. Namely, T_S is homotopic to the identity if and only if for one of the two components N of cutting M^3 open along S the following strange condition holds. Let P be a prime summand of the result of gluing a closed 3-ball to $\partial N = S^2$. Then either $P = S^2 \times S^1$, or P has a finite fundamental group whose Sylow 2-subgroup is cyclic.
- However, this is not the whole story. In [5], Friedman–Witt show that in some cases the separating sphere twists that Hendriks showed were homotopic to the identity are not *isotopic* to the identity, and thus still define nontrivial elements of $\text{Mod}(M^3)$.

What happens in the general case is unclear. See [12, Remark 2.4] for some further discussion of it.

Outline. The outline of our paper is as follows. We start in §2 by constructing the exact sequence (1.1). To make our paper self-contained, we give a mostly complete proof of this, simplifying some details from Laudenbach’s original paper. We then have a preliminary algebraic section §3 on crossed homomorphisms and their relationship to splitting exact sequences. We then construct the crossed homomorphisms we need in §4 and §5 before closing with §6, which takes care of a few final details.

Notational conventions. It will be important for us to distinguish between a diffeomorphism and its isotopy class. For $f \in \text{Diff}^+(M_n)$ we will write $[f] \in \text{Mod}(M_n)$ for the isotopy class of f . More generally, we will use square brackets frequently to indicate that something is being taken up to homotopy/isotopy, though we will try to be explicit about this whenever it might be confusing to the reader.

Acknowledgments. We would like to thank Jake Landgraf for helpful conversations and Allen Hatcher for some useful references, historical remarks, and corrections. We would also like to thank Oscar Randal-Williams for pointing out the work of Crowley [4], Krannich [17], and Krylov [18]. We additionally thank Diarmuid Crowley, Dan Margalit, and Richard Wade for comments and corrections.

2 Constructing the extension

This preliminary section discusses some aspects of Laudenbach’s work we will need for our proof.

What is needed. Recall from the introduction that Laudenbach [20] proved that $\text{Mod}(M_n)$ is an extension of $\text{Out}(F_n)$ by $\text{Twist}(M_n)$ and that $\text{Twist}(M_n) \cong (\mathbb{Z}/2)^n$. Theorem A strengthens this and its proof will give as a byproduct that $\text{Twist}(M_n) \cong (\mathbb{Z}/2)^n$, but it will depend on the following piece of Laudenbach’s work.

Theorem 2.1 (Laudenbach, [20, 21]). *The map*

$$\rho: \text{Mod}(M_n) \rightarrow \text{Out}(\pi_1(M_n)) = \text{Out}(F_n)$$

*is surjective with kernel $\ker(\rho) = \text{Twist}(M_n)$. Also, $\text{Twist}(M_n)$ is generated by the sphere twists about the core spheres $S^2 \times *$ of the n summands of $S^2 \times S^1$ in M_n .*

Remark 2.2. Theorem 2.1 does *not* assert that the indicated sphere twists are nontrivial. As we discussed in the introduction, Laudenbach proved that they are and that $\text{Twist}(M_n) \cong (\mathbb{Z}/2)^n$, but we will only establish this (in a stronger form) later; see Corollary 5.2.

To make this paper more self-contained, this section contains a mostly complete sketch of a proof of Theorem 2.1. We will follow the outline of Laudenbach’s original proof, but we will simplify one key step (see Theorem 2.4 below).

Homotopy vs isotopy for spheres. Our proof of Theorem 2.1 will depend on three preliminary results. The first is the following:

Theorem 2.3 (Laudenbach, [20, 21]). *Let M^3 be a closed oriented 3-manifold and let $\iota, \iota': \sqcup_{i=1}^k S^2 \rightarrow M^3$ be homotopic embeddings of disjoint smoothly embedded spheres. Assume that none of the components of the images of ι or ι' are nullhomotopic. Then ι and ι' are ambient isotopic.*

We omit the proof of Theorem 2.3 since it is lengthy and its details do not shed much light on our work.

Action on second homotopy group. The second preliminary result is the following theorem of Laudenbach. Our proof is much shorter than his proof. We remark that Hatcher-Wahl have given a different (but somewhat longer) simplified proof in [12, Appendix].

Theorem 2.4 (Laudenbach). *Let M^3 be a closed oriented 3-manifold equipped with a basepoint x_0 and let $f: (M^3, x_0) \rightarrow (M^3, x_0)$ be a basepoint-preserving diffeomorphism such that $f_*: \pi_1(M^3, x_0) \rightarrow \pi_1(M^3, x_0)$ is the identity. Then $f_*: \pi_2(M^3, x_0) \rightarrow \pi_2(M^3, x_0)$ is the identity.*

Proof. Let $(\widetilde{M}^3, \widetilde{x}_0) \rightarrow (M^3, x_0)$ be the universal cover of (M^3, x_0) . Let $\widetilde{f}: (\widetilde{M}^3, \widetilde{x}_0) \rightarrow (\widetilde{M}^3, \widetilde{x}_0)$ be the lift of f . To prove that f acts trivially on $\pi_2(M^3, x_0)$, it is enough to prove that \widetilde{f} acts trivially on $\pi_2(\widetilde{M}^3, \widetilde{x}_0) = H_2(\widetilde{M}^3)$. By Poincaré duality, we have

$$H_2(\widetilde{M}^3) \cong H_c^1(\widetilde{M}^3) = \varinjlim_K H^1(\widetilde{M}^3, \widetilde{M}^3 \setminus K) = \varinjlim_K \widetilde{H}^0(\widetilde{M}^3 \setminus K). \quad (2.1)$$

Here the limit is over compact subspaces K of \widetilde{M}^3 and the final equality comes from the long exact sequence of the pair $(\widetilde{M}^3, \widetilde{M}^3 \setminus K)$ and the fact that \widetilde{M}^3 is 1-connected. Elements of $\widetilde{H}^0(\widetilde{M}^3 \setminus K)$ can be interpreted as locally constant functions $\kappa: \widetilde{M}^3 \setminus K \rightarrow \mathbb{Z}$ modulo the globally constant functions. Fix such a $\kappa: \widetilde{M}^3 \setminus K \rightarrow \mathbb{Z}$, and let K' be a compact subspace of \widetilde{M}^3 containing $K \cup \widetilde{f}(K)$ such that no components of $\widetilde{M}^3 \setminus K'$ are bounded (i.e., have compact closure). The image under the homeomorphism \widetilde{f} of the element of $H_2(\widetilde{M}^3)$ represented by κ under (2.1) is represented by the function

$$\kappa \circ \widetilde{f}^{-1}: \widetilde{M}^3 \setminus K' \rightarrow \mathbb{Z}.$$

We must prove that $\kappa = \kappa \circ \widetilde{f}^{-1}$ on $\widetilde{M}^3 \setminus K'$. The key observation is that since f acts trivially on $\pi_1(M^3, x_0)$, the lift \widetilde{f} fixes each point in the $\pi_1(M^3, x_0)$ -orbit of the basepoint \widetilde{x}_0 . This orbit will contain points in each component of $\widetilde{M}^3 \setminus K'$, so κ and $\kappa \circ \widetilde{f}^{-1}$ agree on at least one point in each component of $\widetilde{M}^3 \setminus K'$. Since they are locally constant, we conclude that they are equal everywhere on $\widetilde{M}^3 \setminus K'$, as desired. \square

Mapping class groups of punctured spheres. The third and final preliminary result we need is as follows. For a 3-manifold M^3 with boundary, we define $\text{Mod}(M^3)$ to be $\pi_0(\text{Diff}^+(M^3, \partial M^3))$, i.e., the group of isotopy classes of orientation-preserving diffeomorphisms of M^3 that fix ∂M^3 pointwise.

Lemma 2.5. *Let X be the 3-manifold with boundary obtained by removing k disjoint open balls from S^3 . Then $\text{Mod}(X)$ is generated by sphere twists about embedded spheres that are parallel to components of ∂X .*

Remark 2.6. Just like Theorem 2.1, Lemma 2.5 does *not* assert that these sphere twists are nontrivial in the mapping class group. In fact, one can show that they are trivial if $k = 1$ and nontrivial if $k \geq 2$, and the twist subgroup of the manifold X in Lemma 2.5 is isomorphic to $(\mathbb{Z}/2)^{k-1}$. Here $(k - 1)$ appears instead of k since the product of all the boundary twists is trivial; see [12, p. 214–215]. We will not need any of this, so we will not prove it.

Proof of Lemma 2.5. The proof will be by induction on k . The base case $k = 0$ simply asserts that $\text{Mod}(S^3) = 1$, which is a theorem of Cerf [2]. We remark that even more is true: the 3-dimensional Smale Conjecture proved by Hatcher [9] says that $\text{Diff}^+(S^3)$ is homotopy equivalent to $\text{SO}(4)$. Assume now that $k > 0$ and that the lemma is true for smaller k . Let X' be the result of gluing a closed 3-ball B to a component of ∂X . We thus have $\text{Diff}^+(X, \partial X) = \text{Diff}^+(X', \partial X' \sqcup B)$. Let $\text{Emb}^+(B, X')$ be the space of orientation-preserving embeddings of B into the interior of X' . Restricting elements of $\text{Diff}^+(X', \partial X')$ to B gives a map

$$\text{Diff}^+(X', \partial X') \rightarrow \text{Emb}^+(B, X')$$

that fits into a fiber bundle

$$\text{Diff}^+(X', \partial X' \sqcup B) \rightarrow \text{Diff}^+(X', \partial X') \rightarrow \text{Emb}^+(B, X').$$

Identifying $\text{Mod}(X)$ and $\text{Mod}(X')$ with π_0 of the relevant diffeomorphism groups, the associated long exact sequence in homotopy groups contains the segment

$$\pi_1(\text{Emb}^+(B, X')) \longrightarrow \text{Mod}(X) \longrightarrow \text{Mod}(X') \longrightarrow \pi_0(\text{Emb}^+(B, X')). \quad (2.2)$$

Fix oriented trivializations of the tangent bundles of B and X' . For an orientation-preserving embedding $\iota: B \rightarrow X'$, these trivializations allow us to identify the derivative $D_0\iota: T_0B \rightarrow T_{\iota(0)}X'$ with a matrix in the subgroup $\text{GL}_3^+(\mathbb{R})$ of $\text{GL}_3(\mathbb{R})$ consisting of matrices whose determinant is positive. The map $\text{Emb}^+(B, X') \rightarrow X' \times \text{GL}_3^+(\mathbb{R})$ taking an embedding $\iota: B \rightarrow X'$ to $(\iota(0), D_0\iota)$ is a homotopy equivalence (see, e.g., [19, Theorem 9.12]). The group $\text{GL}_3^+(\mathbb{R})$ deformation retracts onto its maximal compact subgroup $\text{SO}(3) \cong \mathbb{RP}^3$, so we deduce that

$$\pi_0(\text{Emb}^+(B, X')) = \pi_0(X' \times \text{GL}_3^+(\mathbb{R})) = 0$$

and

$$\pi_1(\text{Emb}^+(B, X')) = \pi_1(X' \times \text{GL}_3^+(\mathbb{R})) = \pi_1(\text{GL}_3^+(\mathbb{R})) = \mathbb{Z}/2.$$

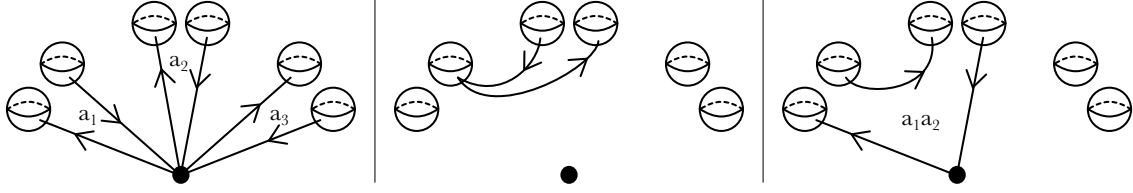


Figure 2: On the left hand side, M_3 is obtained by gluing the 6 boundary components of X together in pairs as indicated. The generators for $\pi_1(M_3) = F_3$ are $\{a_1, a_2, a_3\}$. In the middle, we indicate a diffeomorphism $\phi: M_3 \rightarrow M_3$ that drags one boundary sphere of X along a closed path. As is shown on the right, on $\pi_1(M_3)$ the diffeomorphism ϕ takes a_1 to a_1a_2 .

Plugging these into (2.2), we get an exact sequence

$$\mathbb{Z}/2 \longrightarrow \text{Mod}(X) \longrightarrow \text{Mod}(X') \longrightarrow 0.$$

The image of $\mathbb{Z}/2$ in $\text{Mod}(X)$ is a sphere twist about a sphere parallel to a component of ∂X , and by induction $\text{Mod}(X')$ is generated by sphere twists about spheres parallel to components of $\partial X'$. The lemma follows. \square

The proof. We now have all the ingredients needed for the proof of Theorem 2.1 above.

Proof of Theorem 2.1. Recall that

$$\rho: \text{Mod}(M_n) \rightarrow \text{Out}(\pi_1(M_n)) = \text{Out}(F_n)$$

is the natural map. We must prove the following two facts.

Claim 1. *The map ρ is surjective.*

Let X be the result of removing $2n$ disjoint open balls from S^3 . As in Figure 2, we can identify M_n with the result of gluing the boundary component of X together in pairs. Let $\{a_1, \dots, a_n\} \in \pi_1(M_n) = F_n$ be the generators indicated in Figure 2. An old theorem of Nielsen ([23]; see [22, §I.4] for a textbook reference) says that the group $\text{Aut}(F_n)$ is generated by the following elements:

- For distinct $1 \leq i, j \leq n$, elements L_{ij} and R_{ij} defined via the formulas

$$L_{ij}(a_k) = \begin{cases} a_j a_k & \text{if } k = i \\ a_k & \text{if } k \neq i \end{cases} \quad \text{and} \quad R_{ij}(a_k) = \begin{cases} a_k a_j & \text{if } k = i \\ a_k & \text{if } k \neq i \end{cases} \quad (1 \leq k \leq n).$$

- For $1 \leq i \leq n$, elements I_i defined via the formula

$$I_i(a_k) = \begin{cases} a_k^{-1} & \text{if } k = i, \\ a_k & \text{if } k \neq i. \end{cases}$$

It is enough to find elements of $\text{Diff}^+(M_n)$ realizing these automorphisms. This is an easy exercise; for instance, as we show in Figure 2 we can realize R_{12} as a diffeomorphism that drags one boundary sphere of X through another. We remark that this surjectivity was originally proved by Whitehead [25, 26], who used the more complicated generating set of “Whitehead transformations”.

Claim 2. *The kernel of ρ is the twist subgroup $\text{Twist}(M_n)$, and $\text{Twist}(M_n)$ is generated by the sphere twists about the core spheres of the n summands of $S^2 \times S^1$ in M_n .*

Clearly $\text{Twist}(M_n) \subset \ker(\rho)$, so it is enough to prove that every element of $\ker(\rho)$ is a product of sphere twists about the core spheres S_1, \dots, S_n of the $S^2 \times S^1$ summands of M_n . Consider some $[f] \in \ker(\rho)$, and let $\iota: \sqcup_{i=1}^n S^2 \rightarrow M_n$ be the embedding of those core spheres. Fix a basepoint $x_0 \in M_n$. Isotoping f , we can assume that $f(x_0) = x_0$ and that $f_*: \pi_1(M_n, x_0) \rightarrow \pi_1(M_n, x_0)$ is the identity. Theorem 2.4 then implies that f also induces the identity on $\pi_2(M_n, x_0)$. It follows that ι is homotopic to $f \circ \iota$, so by Theorem 2.3 we can isotope f such that $\iota = f \circ \iota$. Let X be the result of cutting M_n open along the image of ι , so X is diffeomorphic to the result of removing $2n$ open balls from S^3 . Since $\iota = f \circ \iota$, the mapping class $[f]$ is in the image of the homomorphism $\text{Mod}(X) \rightarrow \text{Mod}(M_n)$ that glues the boundary components back together. Lemma 2.5 says that $\text{Mod}(X)$ is generated by sphere twists about spheres parallel to its boundary components. These map to the $[T_{S_i}]$ in $\text{Mod}(M_n)$, and the desired result follows. \square

3 Crossed homomorphisms and exact sequences

As preparation for proving Theorem A, this section reviews the connection between crossed homomorphisms and split exact sequences. Let G and H be groups such that G acts on H on the right. We will write this action using superscripts: for $g \in G$ and $h \in H$, the image of h under g will be denoted h^g . A *crossed homomorphism* from G to H is a set map $\lambda: G \rightarrow H$ such that

$$\lambda(g_1 g_2) = \lambda(g_1)^{g_2} \lambda(g_2).$$

This implies in particular that

$$\lambda(1) = \lambda(1^2) = \lambda(1)^1 \lambda(1) = \lambda(1)^2,$$

so $\lambda(1) = 1$. If the action of G on H is trivial, then this reduces to the definition of a homomorphism. Just like for an ordinary homomorphism, the kernel $\ker(\lambda) = \{g \in G \mid \lambda(g) = 1\}$ is a subgroup of G ; however, it is not necessarily a normal subgroup.

As the following standard lemma shows, these are closely related to splittings of short exact sequences.

Lemma 3.1. *Let G be a group and let $A \triangleleft G$ be an abelian normal subgroup, so G acts on A on the right via the formula*

$$a^g = g^{-1}ag \quad (a \in A, g \in G).$$

Letting $Q = G/A$, the short exact sequence

$$1 \longrightarrow A \longrightarrow G \longrightarrow Q \longrightarrow 1$$

splits if and only if there exists a crossed homomorphism $\lambda: G \rightarrow A$ that restricts to the identity on A . Moreover, if such a λ exists, then we can choose a splitting $Q \rightarrow G$ whose image is $\ker(\lambda)$, so $G = A \rtimes \ker(\lambda)$.

Proof. If the exact sequence splits, then there exists a subgroup \overline{Q} of G projecting isomorphically to Q , so we can uniquely write all $g \in G$ as $g = qa$ with $q \in \overline{Q}$ and $a \in A$. This allows us to define a set map $\lambda: G \rightarrow A$ via the formula

$$\lambda(qa) = a \quad (q \in \overline{Q}, a \in A).$$

This restricts to the identity on A , and is a crossed homomorphism since for $q_1, q_2 \in \overline{Q}$ and $a_1, a_2 \in A$ we have

$$\lambda(q_1a_1q_2a_2) = \lambda(q_1q_2a_1^{q_2}a_2) = \lambda(q_1q_2a_1^{q_2a_2}a_2) = a_1^{q_2a_2}a_2 = \lambda(q_1a_1)^{q_2a_2}\lambda(q_2a_2).$$

We remark that the second equality is where we use the fact that A is abelian.

Conversely, assume that there exists a crossed homomorphism $\lambda: G \rightarrow A$ that restricts to the identity on A . Define $\overline{Q} = \ker(\lambda)$, so $\overline{Q} < G$ satisfies $\overline{Q} \cap A = 1$. To prove the theorem, we must prove that the surjection $\pi: G \rightarrow Q$ restricts to an isomorphism $\pi: \overline{Q} \rightarrow Q$. Since $\overline{Q} \cap A = 1$, the projection $\pi: \overline{Q} \rightarrow Q$ is injective, so we must only prove that it is surjective. Consider $q \in Q$. We can find some $g \in G$ such that $\pi(g) = q$. Since $\lambda(g^{-1}) \in A$, we have $\pi(\lambda(g^{-1})g) = \pi(g) = q$, so it is enough to prove that $\lambda(g^{-1})g \in \overline{Q} = \ker(\lambda)$. For this, we compute

$$\lambda(\lambda(g^{-1})g) = \lambda(\lambda(g^{-1}))^g\lambda(g) = \lambda(g^{-1})^g\lambda(g) = \lambda(g^{-1}g) = 1.$$

Here the second inequality uses the fact that λ restricts to the identity on A . □

4 The derivative crossed homomorphism

By Lemma 3.1, to prove that the exact sequence

$$1 \longrightarrow \text{Twist}(M_n) \longrightarrow \text{Mod}(M_n) \longrightarrow \text{Out}(F_n) \longrightarrow 1$$

splits, we must construct a crossed homomorphism $\text{Mod}(M_n) \rightarrow \text{Twist}(M_n)$ that restricts to the identity on $\text{Twist}(M_n)$. We will do this in two steps, the first in this section and second in the next. As we said in the introduction, the idea of our construction goes back to work of Krylov [18] that was further developed by Crowley [4] and Krannich [17].

Frame bundle. What we do in this section works in complete generality, so let M^3 be any closed oriented 3-manifold. Let TM^3 be the tangent bundle of M^3 and let $\text{Fr}(TM^3)$ be the principle $\text{GL}_3^+(\mathbb{R})$ -bundle of oriented frames of TM^3 . Here recall that $\text{GL}_3^+(\mathbb{R})$ is the subgroup of $\text{GL}_3(\mathbb{R})$ consisting of matrices whose determinant is positive. The points of $\text{Fr}(TM^3)$ thus consist of orientation-preserving linear isomorphisms $\tau: \mathbb{R}^3 \rightarrow T_p M^3$, where $p \in M^3$ is a point. The group $\text{GL}_3^+(\mathbb{R})$ act on $\text{Fr}(TM^3)$ on the right in the usual way: regarding elements of $\text{GL}_3^+(\mathbb{R})$ as isomorphisms $\mathbb{R}^3 \rightarrow \mathbb{R}^3$, we have

$$\tau \cdot M = \tau \circ M \quad \text{for } \tau: \mathbb{R}^3 \rightarrow T_p M^3 \text{ in } \text{Fr}(TM^3) \text{ and } M \in \text{GL}_3^+(\mathbb{R}).$$

This action preserves the fibers of the projection $\text{Fr}(TM^3) \rightarrow M^3$, and its restriction to each fiber is simply transitive.

Trivializations. Since M^3 is oriented, its tangent bundle TM^3 is trivial. An *oriented trivialization* of TM^3 is a continuous section $\sigma: M^3 \rightarrow \text{Fr}(TM^3)$ of the bundle $\text{Fr}(TM^3) \rightarrow M^3$. Let $\text{Triv}(M^3)$ be the set of oriented trivializations of TM^3 and let $C(M^3, \text{GL}_3^+(\mathbb{R}))$ be the space of continuous maps $M^3 \rightarrow \text{GL}_3^+(\mathbb{R})$. The group structure of $\text{GL}_3^+(\mathbb{R})$ endows $C(M^3, \text{GL}_3^+(\mathbb{R}))$ with the structure of a topological group, and $C(M^3, \text{GL}_3^+(\mathbb{R}))$ acts continuously on $\text{Triv}(M^3)$ on the right via the formula

$$\begin{aligned} \sigma \cdot \phi &= \left(p \mapsto \sigma(p) \cdot \phi(p) \right) \quad \text{for } \sigma: M^3 \rightarrow \text{Fr}(TM^3) \text{ in } \text{Triv}(M^3) \\ &\quad \text{and } \phi: M^3 \rightarrow \text{GL}_3^+(\mathbb{R}) \text{ in } C(M^3, \text{GL}_3^+(\mathbb{R})). \end{aligned}$$

The action is also simply transitive.

Diffeomorphism actions. For $f \in \text{Diff}^+(M^3)$, the derivative Df of f induces a map $(Df)_*: \text{Fr}(TM^3) \rightarrow \text{Fr}(TM^3)$ defined via the formula

$$(Df)_*(\tau) = \left((D_p f) \circ \tau: \mathbb{R}^3 \rightarrow T_{f(p)} M^3 \right) \quad \text{for } \tau: \mathbb{R}^3 \rightarrow T_p M^3 \text{ in } \text{Fr}(TM^3).$$

Using this, we can define a right action of $\text{Diff}^+(M^3)$ on $\text{Triv}(M^3)$ via the following formula, where we use superscripts to avoid confusing this action with the above action of $C(M^3, \text{GL}_3^+(\mathbb{R}))$:

$$\sigma^f = (Df^{-1})_* \circ \sigma \circ f \quad \text{for } f \in \text{Diff}^+(M^3) \text{ and } \sigma: M^3 \rightarrow \text{Fr}(TM^3) \text{ in } \text{Triv}(M^3).$$

The group $\text{Diff}^+(M^3)$ also has a right action on $C(M^3, \text{GL}_3^+(\mathbb{R}))$ defined via the formula

$$\phi^f = \phi \circ f \quad \text{for } f \in \text{Diff}^+(M^3) \text{ and } \phi: M^3 \rightarrow \text{GL}_3^+(\mathbb{R}) \text{ in } C(M^3, \text{GL}_3^+(\mathbb{R})).$$

These three different actions are related by the formula

$$(\sigma \cdot \phi)^f = \sigma^f \cdot \phi^f \quad \text{for } f \in \text{Diff}^+(M^3) \text{ and } \phi \in C(M^3, \text{GL}_3^+(\mathbb{R})) \text{ and } \sigma \in \text{Triv}(M^3).$$

Both sides of this formula are the element of $\text{Triv}(M^3)$ whose value at a point $p \in M^3$ is the linear isomorphism $\mathbb{R}^3 \rightarrow T_p M^3$ given by

$$(D_{f(p)} f^{-1}) \circ (\sigma(f(p))) \circ (\phi(f(p))).$$

Derivative crossed homomorphism. Our next goal is to construct a crossed homomorphism

$$\mathcal{D}: \text{Diff}^+(M^3) \rightarrow C(M^3, \text{GL}_3^+(\mathbb{R}))$$

that we will call the *derivative crossed homomorphism*. In a suitable sense, it encodes the action of $\text{Diff}^+(M^3)$ on $\text{Triv}(M^3)$. The derivative crossed homomorphism depends on a choice of a base trivialization $\sigma_0 \in \text{Triv}(M^3)$ that we fix once and for all. Now consider $f \in \text{Diff}^+(M^3)$. We have $\sigma_0^f \in \text{Triv}(M^3)$, and as we noted above the topological group $C(M^3, \text{GL}_3^+(\mathbb{R}))$ acts simply transitively on $\text{Triv}(M^3)$. It follows that there exists a unique $\phi_f \in C(M^3, \text{GL}_3^+(\mathbb{R}))$ such that

$$\sigma_0^f = \sigma_0 \cdot \phi_f.$$

We define $\mathcal{D}(f) = \phi_f^{-1}$. Here the inverse refers to the group structure on the space $C(M^3, \text{GL}_3^+(\mathbb{R}))$ induced by the group structure on $\text{GL}_3^+(\mathbb{R})$. The inverse will be needed to make \mathcal{D} a crossed homomorphism – if you examine the formulas below, you will see that without it \mathcal{D} would be a crossed anti-homomorphism.

To check that \mathcal{D} is indeed a crossed homomorphism, note that for $f_1, f_2 \in \text{Diff}^+(M^3)$ we have

$$\sigma_0^{f_1 f_2} = \sigma_0 \cdot \mathcal{D}(f_1 f_2)^{-1}$$

and

$$\begin{aligned} \sigma_0^{f_1 f_2} &= \left(\sigma_0^{f_1} \right)^{f_2} \\ &= \left(\sigma_0 \cdot \mathcal{D}(f_1)^{-1} \right)^{f_2} \\ &= \sigma_0^{f_2} \cdot \left(\mathcal{D}(f_1)^{f_2} \right)^{-1} \\ &= \sigma_0 \cdot \mathcal{D}(f_2)^{-1} \cdot \left(\mathcal{D}(f_1)^{f_2} \right)^{-1}. \end{aligned}$$

Here again the inverses refer to the group structure on the space $C(M^3, \text{GL}_3^+(\mathbb{R}))$ induced by the group structure on $\text{GL}_3^+(\mathbb{R})$. We thus have

$$\mathcal{D}(f_1 f_2) = \mathcal{D}(f_1)^{f_2} \cdot \mathcal{D}(f_2),$$

as desired.

Homotopy classes. We now pass to homotopy. Let $[\sigma]$ denote the homotopy class of $\sigma \in \text{Triv}(M^3)$ and let

$$\text{HTriv}(M^3) = \{[\sigma] \mid \sigma \in \text{Triv}(M^3)\},$$

and let $[\phi]$ denote the homotopy class of $\phi \in C(M^3, \text{GL}_3^+(\mathbb{R}))$ and let

$$[M^3, \text{GL}_3^+(\mathbb{R})] = \{[\phi] \mid \phi \in C(M^3, \text{GL}_3^+(\mathbb{R}))\}.$$

Finally, let $\text{Mod}(M^3) = \pi_0(\text{Diff}^+(M^3))$ denote the mapping class group of M^3 , and for $f \in \text{Diff}^+(M^3)$ let $[f] \in \text{Mod}(M^3)$ denote its isotopy class. The group structures of $\text{Diff}^+(M^3)$ and $C(M^3, \text{GL}_3^+(\mathbb{R}))$ induce group structures on $\text{Mod}(M^3)$ and $[M^3, \text{GL}_3^+(\mathbb{R})]$, and the right actions of $\text{Diff}^+(M^3)$ and $C(M^3, \text{GL}_3^+(\mathbb{R}))$ on $\text{Triv}(M^3)$ induce right actions of $\text{Mod}(M^3)$ and $[M^3, \text{GL}_3^+(\mathbb{R})]$ on $\text{HTriv}(M^3)$ that we will continue to write with superscripts and \cdot 's, respectively. For $[f] \in \text{Mod}(M^3)$ and $[\phi] \in [M^3, \text{GL}_3^+(\mathbb{R})]$ and $[\sigma] \in \text{HTriv}(M^3)$, we still have the relationship

$$([\sigma] \cdot [\phi])^{[f]} = [\sigma]^{[f]} \cdot [\phi]^{[f]}.$$

Finally, the derivative crossed homomorphism $\mathcal{D}: \text{Diff}^+(M^3) \rightarrow C(M^3, \text{GL}_3^+(\mathbb{R}))$ descends to a derivative crossed homomorphism

$$\mathfrak{D}: \text{Mod}(M^3) \rightarrow [M^3, \text{GL}_3^+(\mathbb{R})]$$

whose characteristic property is that

$$[\sigma_0]^{[f]} = [\sigma_0] \cdot \mathfrak{D}([f])^{-1} \quad \text{for } [f] \in \text{Mod}(M^3).$$

5 The twisting crossed homomorphism

Just like in the last section, let M^3 be a closed oriented 3-manifold. Fix some $\sigma_0 \in \text{Triv}(M^3)$, and let $\mathfrak{D}: \text{Mod}(M^3) \rightarrow [M^3, \text{GL}_3^+(\mathbb{R})]$ be the associated derivative crossed homomorphism.

Twisting crossed homomorphism. The group $\text{GL}_3^+(\mathbb{R})$ deformation retracts to its maximal compact subgroup $\text{SO}(3) \cong \mathbb{RP}^3$. We thus have $\pi_1(\text{GL}_3^+(\mathbb{R})) \cong \mathbb{Z}/2$. The π_1 -functor therefore induces a group homomorphism

$$h: [M^3, \text{GL}_3^+(\mathbb{R})] \longrightarrow \text{Hom}(\pi_1(M^3), \mathbb{Z}/2) = \text{H}^1(M^3; \mathbb{Z}/2).$$

The group $\text{Mod}(M^3)$ acts on the right on both $[M^3, \text{GL}_3^+(\mathbb{R})]$ and $\text{H}^1(M^3; \mathbb{Z}/2)$, and h commutes with this action in the sense that for $[f] \in \text{Mod}(M^3)$ and $[\phi] \in [M^3, \text{GL}_3^+(\mathbb{R})]$ we have

$$h([\phi]^{[f]}) = h([\phi])^{[f]}.$$

This implies that the composition of h with the derivative crossed homomorphism \mathfrak{D} is a crossed homomorphism

$$\mathfrak{T}: \text{Mod}(M^3) \longrightarrow \text{H}^1(M^3; \mathbb{Z}/2)$$

that we will call the *twisting crossed homomorphism*. Since the twist subgroup $\text{Twist}(M^3) < \text{Mod}(M^3)$ acts trivially on $\pi_1(M^3)$, the restriction of \mathfrak{T} to $\text{Twist}(M^3)$ is a homomorphism (not just a crossed homomorphism).

Effect on sphere twists. The following lemma shows how to calculate \mathfrak{T} on a sphere twist:

Lemma 5.1. *Let S be an embedded 2-sphere in M^3 . Then $\mathfrak{T}(T_S) \in H^1(M^3; \mathbb{Z}/2)$ is the cohomology class that is Poincaré dual to $[S] \in H_2(M^3; \mathbb{Z}/2)$.*

Proof. Identify S with $S^2 \subset \mathbb{R}^3$. Recall that T_S is constructed from a loop $\ell: [0, 1] \rightarrow \text{SO}(3)$ with $\ell(0) = \ell(1) = \text{id}$ that generates $\pi_1(\text{SO}(3), \text{id}) \cong \mathbb{Z}/2$. This generator rotates S by a full twist about an axis, and T_S is represented by a diffeomorphism τ that is the identity outside a tubular neighborhood $U \cong S \times [0, 1]$ of S , and on U is defined by $\tau(s, t) = (\ell(t) \cdot s, t)$. Let $p_0 \in S$ be one of the two intersection points of the axis of rotation defining ℓ with S .

Consider a smoothly embedded closed curve $\gamma: S^1 \rightarrow M^3$. Homotoping γ , we can assume that it only intersects U in segments of the form $p_0 \times [0, 1]$ (which it might traverse in either direction). It follows that τ fixes γ pointwise. Let $\mathcal{D}: \text{Diff}^+(M^3) \rightarrow C(M^3, \text{GL}_3^+(\mathbb{R}))$ be the derivative crossed homomorphism that descends to $\mathfrak{D}: \text{Mod}(M^3) \rightarrow [M^3, \text{GL}_3^+(\mathbb{R})]$ when we pass to homotopy. The composition

$$[0, 1] \xrightarrow{\gamma} M^3 \xrightarrow{\mathcal{D}(\tau)} \text{GL}_3^+(\mathbb{R})$$

is a loop whose image in $\pi_1(\text{GL}_3^+(\mathbb{R})) \cong \mathbb{Z}/2$ represents $\mathfrak{T}(T_S)([\gamma])$. Examining the definitions, we see that this element of $\pi_1(\text{GL}_3^+(\mathbb{R})) \cong \mathbb{Z}/2$ simply counts the number of times γ traverses $p_0 \times [0, 1]$, which equals the $\mathbb{Z}/2$ -algebraic intersection number of γ with S . The lemma follows. \square

Connect sums of $S^2 \times S^1$. We now specialize this to the connect sum M_n of n copies of $S^2 \times S^1$. Recall from Theorem 2.1 that $\text{Twist}(M_n)$ is generated by the sphere twists about the core spheres $S^2 \times *$ of the n summands $S^2 \times S^1$ of M_n . These clearly commute with each other and are Poincaré dual to a basis for $H^1(M^3; \mathbb{Z}/2)$, so Lemma 5.1 implies the following:

Corollary 5.2. *The twisting crossed homomorphism $\mathfrak{T}: \text{Mod}(M_n) \rightarrow H^1(M_n; \mathbb{Z}/2)$ restricts to an isomorphism $\text{Twist}(M_n) \cong H^1(M_n; \mathbb{Z}/2)$.*

In particular, we recover Laudenbach's theorem [20] saying that $\text{Twist}(M_n) \cong (\mathbb{Z}/2)^n$. We actually get more: since $\mathfrak{T}: \text{Mod}(M_n) \rightarrow H^1(M_n; \mathbb{Z}/2)$ is a crossed homomorphism, the isomorphism $\text{Twist}(M_n) \cong H^1(M_n; \mathbb{Z}/2)$ in Corollary 5.2 is an isomorphism of $\text{Mod}(M_n)$ -modules, where $\text{Mod}(M_n)$ acts on its normal subgroup $\text{Twist}(M_n)$ by conjugation.

Summary. Combining Corollary 5.2 with Lemma 3.1 and the exact sequence

$$1 \longrightarrow \text{Twist}(M_n) \longrightarrow \text{Mod}(M_n) \longrightarrow \text{Out}(F_n) \longrightarrow 1$$

from Theorem 2.1, we conclude the following:

Theorem 5.3. *Let $[\sigma_0]$ be the homotopy class of a trivialization σ_0 of the tangent bundle of M_n and let $\mathfrak{T}: \text{Mod}(M_n) \rightarrow \text{H}^1(M_n; \mathbb{Z}/2)$ be the associated twisting crossed homomorphism. The following then hold:*

- $\text{Mod}(M_n) = \text{Twist}(M_n) \rtimes \ker(\mathfrak{T})$.
- $\text{Twist}(M_n) \cong \text{H}^1(M_n; \mathbb{Z}/2)$ as a $\text{Mod}(M_n)$ -module.
- $\ker(\mathfrak{T}) \cong \text{Out}(F_n)$.

This is almost Theorem A. All that is missing is the fact that $\ker(\mathfrak{T}) \cong \text{Out}(F_n)$ is the $\text{Mod}(M_n)$ -stabilizer of $[\sigma_0]$, which we will prove in the next section (see Corollary 6.2).

6 $\text{Out}(F_n)$ acts trivially on homotopy classes of trivializations

In this section, we prove the following.

Lemma 6.1. *Let $[\sigma_0]$ be the homotopy class of a trivialization σ_0 of the tangent bundle of M_n and let $\mathfrak{T}: \text{Mod}(M_n) \rightarrow \text{H}^1(M_n; \mathbb{Z}/2)$ be the associated twisting crossed homomorphism. Then $\ker(\mathfrak{T})$ fixes $[\sigma_0]$.*

Since the $\text{Mod}(M_n)$ -stabilizer of $[\sigma_0]$ is clearly contained in $\ker(\mathfrak{T})$, this implies the following:

Corollary 6.2. *Let $[\sigma_0]$ be the homotopy class of a trivialization σ_0 of the tangent bundle of M_n and let $\mathfrak{T}: \text{Mod}(M_n) \rightarrow \text{H}^1(M_n; \mathbb{Z}/2)$ be the associated twisting crossed homomorphism. Then $\ker(\mathfrak{T})$ is the $\text{Mod}(M_n)$ -stabilizer of $[\sigma_0]$.*

As we noted at the end of §5, Corollary 6.2 together with Theorem 5.3 implies Theorem A from the introduction.

Proof of Lemma 6.1. Let $G = \ker(\mathfrak{T})$, so by Theorem 5.3 we have $G \cong \text{Out}(F_n)$. Let $\mathfrak{D}: \text{Mod}(M_n) \rightarrow [M_n, \text{GL}_3^+(\mathbb{R})]$ be the derivative crossed homomorphism associated to $[\sigma_0]$. Recall that $\text{GL}_3^+(\mathbb{R})$ is homotopy equivalent to $\mathbb{R}\mathbb{P}^3$. Let $\widetilde{\text{GL}}_3^+(\mathbb{R})$ be the universal cover of $\text{GL}_3^+(\mathbb{R})$, so $\widetilde{\text{GL}}_3^+(\mathbb{R})$ is homotopy equivalent to S^3 . For $[f] \in G$, we know that $\mathfrak{D}([f]) \in [M_n, \text{GL}_3^+(\mathbb{R})]$ induces the trivial map on π_1 , so we can lift $\mathfrak{D}([f])$ to an element of $[M_n, \widetilde{\text{GL}}_3^+(\mathbb{R})]$. Though there are two distinct lifts to $\widetilde{\text{GL}}_3^+(\mathbb{R})$ of a map $M_n \rightarrow \text{GL}_3^+(\mathbb{R})$ that induces the trivial map on π_1 (the two lifts correspond to a choice of a lift of a basepoint), these two lifts are homotopic via a homotopy

corresponding to right-multiplication by a path in $\widetilde{\mathrm{GL}}_3^+(\mathbb{R})$ from the identity to the other element of $\widetilde{\mathrm{GL}}_3^+(\mathbb{R})$ projecting to the identity in $\mathrm{GL}_3^+(\mathbb{R})$. From this, we see that in fact \mathfrak{D} lifts to a crossed homomorphism $\widetilde{\mathfrak{D}}: G \rightarrow [M_n, \widetilde{\mathrm{GL}}_3^+(\mathbb{R})]$.

Since $\widetilde{\mathrm{GL}}_3^+(\mathbb{R})$ is homotopy equivalent to S^3 , elements of $[M_n, \widetilde{\mathrm{GL}}_3^+(\mathbb{R})]$ are classified by their degree, i.e., as a group we have $[M_n, \widetilde{\mathrm{GL}}_3^+(\mathbb{R})] \cong \mathbb{Z}$. What is more, since orientation-preserving diffeomorphisms of M_n act by degree 1, the action of $\mathrm{Mod}(M_n)$ on $[M_n, \widetilde{\mathrm{GL}}_3^+(\mathbb{R})] \cong \mathbb{Z}$ is trivial. It follows that the crossed homomorphism

$$\widetilde{\mathfrak{D}}: G \rightarrow [M_n, \widetilde{\mathrm{GL}}_3^+(\mathbb{R})] \cong \mathbb{Z}$$

is an actual homomorphism (not just a crossed homomorphism). Since the abelianization of $G \cong \mathrm{Out}(F_n)$ is torsion,¹ we deduce that in fact $\widetilde{\mathfrak{D}}$ is trivial. This implies that $\mathfrak{D}|_G$ is also trivial. Since \mathfrak{D} encodes the action of $\mathrm{Mod}(M_n)$ on $[\sigma_0] \in \mathrm{HTriv}(M_n)$ (see §4), this implies that G acts trivially on $[\sigma_0]$, as desired. \square

References

- [1] J. Aramayona and J. Souto, Automorphisms of the graph of free splittings, *Michigan Math. J.* 60 (2011), no. 3, 483–493.
- [2] J. Cerf, *Sur les difféomorphismes de la sphère de dimension trois* ($\Gamma_4 = 0$), *Lecture Notes in Mathematics*, No. 53, Springer-Verlag, Berlin, 1968.
- [3] M. Clay, Y. Qing and K. Rafi, Uniform fellow traveling between surgery paths in the sphere graph, *Algebr. Geom. Topol.* 17 (2017), no. 6, 3751–3778.
- [4] D. J. Crowley, On the mapping class groups of $\#_r(S^p \times S^p)$ for $p = 3, 7$, *Math. Z.* 269 (2011), no. 3-4, 1189–1199.
- [5] J. L. Friedman and D. M. Witt, Homotopy is not isotopy for homeomorphisms of 3-manifolds, *Topology* 25 (1986), no. 1, 35–44.
- [6] H. Gluck, The embedding of two-spheres in the four-sphere, *Trans. Amer. Math. Soc.* 104 (1962), 308–333.
- [7] U. Hamenstädt and S. Hensel, Spheres and projections for $\mathrm{Out}(F_n)$, *J. Topol.* 8 (2015), no. 1, 65–92.
- [8] M. Handel and L. Mosher, The free splitting complex of a free group, I: hyperbolicity, *Geom. Topol.* 17 (2013), no. 3, 1581–1672.

¹Here is a quick proof of this classical fact. Since $\mathrm{Aut}(F_n)$ surjects onto $\mathrm{Out}(F_n)$, it is enough to prove that $(\mathrm{Aut}(F_n))^{\mathrm{ab}}$ is torsion. Recall the generators L_{ij} and R_{ij} and I_i for $\mathrm{Aut}(F_n)$ from Claim 1 of the proof of Theorem 2.1. It is enough to prove that each of these generators maps to an element of $(\mathrm{Aut}(F_n))^{\mathrm{ab}}$ of finite order. This is trivial for the order-2 elements I_i , so we must just deal with the L_{ij} and R_{ij} . The key observation is that

$$I_j L_{ij} I_j^{-1} = L_{ij}^{-1} \quad \text{and} \quad I_j R_{ij} I_j^{-1} = R_{ij}^{-1}.$$

The elements L_{ij} and $I_j^{-1} L_{ij} I_j = L_{ij}^{-1}$ map to the same element of $(\mathrm{Aut}(F_n))^{\mathrm{ab}}$, so L_{ij}^2 must map to 0. Similarly, R_{ij}^2 maps to 0 in $(\mathrm{Aut}(F_n))^{\mathrm{ab}}$.

- [9] A. E. Hatcher, A proof of the Smale conjecture, *Diff(S^3) \simeq $O(4)$* , *Ann. of Math. (2)* 117 (1983), no. 3, 553–607.
- [10] A. Hatcher, Homological stability for automorphism groups of free groups, *Comment. Math. Helv.* 70 (1995), no. 1, 39–62.
- [11] A. Hatcher and K. Vogtmann, The complex of free factors of a free group, *Quart. J. Math. Oxford Ser. (2)* 49 (1998), no. 196, 459–468.
- [12] A. Hatcher and N. Wahl, Stabilization for mapping class groups of 3-manifolds, *Duke Math. J.* 155 (2010), no. 2, 205–269.
- [13] H. Hendriks, Applications de la théorie d’obstruction en dimension 3, *Bull. Soc. Math. France Mém.* 1977, no. 53, 81–196.
- [14] A. Hilion and C. Horbez, The hyperbolicity of the sphere complex via surgery paths, *J. Reine Angew. Math.* 730 (2017), 135–161.
- [15] C. Horbez, Sphere paths in outer space, *Algebr. Geom. Topol.* 12 (2012), no. 4, 2493–2517.
- [16] I. Kapovich and M. Lustig, Geometric intersection number and analogues of the curve complex for free groups, *Geom. Topol.* 13 (2009), no. 3, 1805–1833.
- [17] M. Krannich, Mapping class groups of highly connected $(4k + 2)$ -manifolds, to appear in *Selecta Math.*
- [18] N. A. Krylov, On the Jacobi group and the mapping class group of $S^3 \times S^3$, *Trans. Amer. Math. Soc.* 355 (2003), no. 1, 99–117.
- [19] A. Kupers, Lectures on diffeomorphism groups of manifolds, preprint.
<https://people.math.harvard.edu/~kupers/teaching/272x/book.pdf>
- [20] F. Laudenbach, Sur les 2-sphères d’une variété de dimension 3, *Ann. of Math. (2)* 97 (1973), 57–81.
- [21] F. Laudenbach, *Topologie de la dimension trois: homotopie et isotopie*, Société Mathématique de France, Paris, 1974.
- [22] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, reprint of the 1977 edition, *Classics in Mathematics*, Springer-Verlag, Berlin, 2001.
- [23] J. Nielsen, Die Isomorphismengruppe der freien Gruppen, *Math. Ann.* 91 (1924), no. 3-4, 169–209.
- [24] L. Pontrjagin, A classification of mappings of the three-dimensional complex into the two-dimensional sphere, *Rec. Math. [Mat. Sbornik] N. S.* 9 (51) (1941), 331–363.
- [25] J. H. C. Whitehead, On Certain Sets of Elements in a Free Group, *Proc. London Math. Soc. (2)* 41 (1936), no. 1, 48–56.
- [26] J. H. C. Whitehead, On equivalent sets of elements in a free group, *Ann. of Math. (2)* 37 (1936), no. 4, 782–800.

Tara Brendle
 School of Mathematics & Statistics
 University of Glasgow
 University Place
 Glasgow G12 8QQ
 UK
 tara.brendle@glasgow.ac.uk

Nathan Broaddus
 Department of Mathematics
 Ohio State University
 231 W. 18th Ave.
 Columbus, OH 43210
 USA
 broaddus.9@osu.edu

Andrew Putman
 Department of
 Mathematics
 University of Notre Dame
 255 Hurley Hall
 Notre Dame, IN 46556
 USA
 andyp@nd.edu