Finding a Boundary for an Open Manifold

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FINDING A BOUNDARY FOR AN OPEN MANIFOLD.

By W. Browder, J. Levine and G. R. Livesay.*

Given an open manifold, when is it the interior of a compact manifold with boundary? We consider the case of piecewise linear (combinatorial) or smooth manifolds W of dimension $\geqq 6$ and we give necessary and sufficient conditions on the homology of W and homotopy at ∞ (see § 1) so that W is isomorphic (i. e., combinatorially equivalent or diffeomorphic) to the interior of a compact manifold with 1-connected boundary (Theorem 1). As a consequence we get an h-cobordism theorem for certain types of open manifolds of dimension $\geqq 6$ (Corollary to Theorem 2). We are indebted to E. H. Connell for suggesting the latter theorem and the possibility of deducing it from the first.

All manifolds will be piecewise linear or smooth and isomorphic will mean either combinatorially equivalent or diffeomorphic.

1. Statement of results. A space X is said to be 1-connected at ∞ if for any compact $C \subset X$ there is a compact D, $C \subset D \subset X$ such that $X - D$ is 1-connected.

Theorem 1. *Let W be an open manifold of dimension $\geqq 6$. Then W is isomorphic to the interior of a compact manifold U with* 1*-connected boundary if and only if the homology $H_*(W)$ is finitely generated and W is* 1*-connected at ∞. Further such a U is unique up to isomorphism.*

Actually the proof given here could be modified slightly so that the condition of 1-connected at ∞ could be weakened to W having a finite number of ends, each of which is 1-connected (see [8]).

Theorem 1 can be considered as a partial generalization of the result of Stallings [9] that contractible open manifolds of dimension $\geqq 5$ which are 1-connected at ∞ are isomorphic to R^n (the interior of the n-ball).

Two connected manifolds M_1, M_2 (not necessarily closed) are called h-cobordant if there is a manifold with boundary V, with $\partial V = M_1 \cup (-M_2)$, and such that each component M_i of ∂V is a deformation retract of V.

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THEOREM 2. *Let M_1, M_2 satisfy the hypothesis of Theorem 1, and let V be an h-cobordism between them which is 1-connected at ∞. If $\bar{M}_1$, $\bar{M}_2$ are the compact bounded manifolds produced by Theorem 1, then $\bar{M}_1$ and $\bar{M}_2$ are h-cobordant, i.e., there is a compact manifold with boundary $\bar{V}$ such that $\partial\bar{V} = \bar{M}_1 \cup \bar{U} \cup \bar{M}_2$, where $\bar{U}$ is an h-cobordism between $\partial\bar{M}_1$ and $\partial\bar{M}_2$, and such that the inclusions $\bar{M}_i \subset \bar{V}$ are homotopy equivalence, $i = 1, 2$.*

COROLLARY. *Let M_1, M_2, V be as above, and in addition $\pi_1 M_1 = \pi_1 M_2 = 0$. Then M_1 is isomorphic with M_2.*

2. Uniqueness of the boundary.

THEOREM 3. *Let U_1 and U_2 be compact n-manifolds with 1-connected boundaries, U_1 embedded in interior U_2 and the inclusion of U_1 in U_2 induces homology isomorphism. Suppose also that V is 1-connected, where $V = U_2 -$* interior *U_1. Then V is an h-cobordism between ∂U_1 and ∂U_2.*

Proof. The map $(V, \partial U_1) \subset (U_2, U_1)$ is an excision so that $H_*(V, \partial U_1) \cong H_*(U_2, U_1) = 0$ since the inclusion induces homology isomorphism between U_1 and U_2. Since V, ∂U_1, ∂U_2 are 1-connected it follows that ∂U_1 and V are homotopy equivalent by the theorem of J. H. C. Whitehead and it follows that V and ∂U_2 are homotopy equivalent similarly, using relative Poincaré duality.

COROLLARY. *If $W =$ interior $U_1 =$ interior U_2, ∂U_1, ∂U_2 1-connected, then ∂U_1 is h-cobordant to ∂U_2.*

Actually the corollary holds in more generality without the assumption of 1-connectedness.

3. Proof of Theorem 1.

PROPOSITION 4. *Let W be as in Theorem 1. Then given a compact set C there is a connected manifold U with boundary $U \subset W$, ∂U 1-connected, $C \subset$ interior U, such that the inclusion induces a homology isomorphism.*

Before we prove this, we will indicate how Theorem 1 follows from the proposition.

Proof of Theorem 1. Let $C_1 \subset C_2 \subset \cdots \subset W$ be a sequence of compact sets such that $W = \bigcup_{i=1}^{\infty} C_i$, and $W - C_i$ is 1-connected. By Proposition 4 we may find compact manifolds with boundary U_i such that $U_i \supset U_{i-1} \cup C_i$, ∂U_i 1-connected and $U_i \subset W$ induces homology isomorphism. Then $\cup U_i \supset \cup C_i = W$, so $\cup U_i = W$. Set $V_i = \overline{U_{i+1} - U_i}$. Now $V_i \subset W - C_i$, which is 1-connected, $\partial V_i = \partial U_{i+1} \cup \partial U_i$, which are 1-connected, so that it follows from van Kampen's theorem that $\pi_1(W - C_i) =$ free product of $\pi_1(U_i - C_i), \pi_1(V_i)$ and $\pi_1(W - U_{i+1})$. Since $\pi_1(W - C_i)$ is trivial, it follows that $\pi_1(V_i)$ is trivial. By Theorem 3, V_i is an h-cobordism between ∂U_i and ∂U_{i+1}, and since ∂U_i is 1-connected and has dimension $\geqq 5$, it follows from the h-cobordism theorem of Smale [7] (which has been proved in the piecewise linear case by Stallings) that $\overline{U_{i+1} - U_i}$ is isomorphic to $\partial U_i \times I$. Hence, it follows that each U_i is isomorphic to U_1 and $W = \cup U_i$ is isomorphic to interior of U_1, which completes the proof of Theorem 1.

Now we outline the proof of Proposition 4.

Since $H_*(W)$ is finitely generated we may find a compact set $C' \subset W$ such that $H_*(C')$ maps onto $H_*(W)$. Hence for any subset O such that $C' \subset O \subset W$, $H_*(O)$ maps onto $H_*(W)$. We shall show (Lemma 6) that we can find a compact manifold with boundary $U \subset W$ such that (1) $C \cup C' \subset$ interior U, (2) ∂U is 1-connected, and (3) $W - U$ is 1-connected. Set $V =$ closure $W - U$.

Consider the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
\to H_{k+1}(W) & \xrightarrow{j} & H_{k+1}(W,U) & \xrightarrow{\partial} & H_k(U) & \xrightarrow{i} & H_k(W) \to \\
\uparrow g_2 & & \uparrow g_3 & & \uparrow g_1 & & \uparrow g_2 \\
\to H_{k+1}(V) & \xrightarrow{j'} & H_{k+1}(V,\partial U) & \xrightarrow{\partial'} & H_k(\partial U) & \xrightarrow{i'} & H_k(V) \to .
\end{array}$$

Now $(V, \partial U) \subset (W, U)$ is an excision so that g_3 is an isomorphism. Hence i onto implies i' onto since $j \equiv 0$ implies $j' \equiv 0$. Similarly i' mono implies i mono, since $\partial' \equiv 0$ implies $\partial \equiv 0$. Since i is onto by construction, i' is onto, and it suffices to make i' mono in order to make i an isomorphism.

Now the manifold U obtained from Lemma 5 may in fact have too much homology above dimension 1, so that ∂U will also, and $\ker i'$ will be non-zero. Therefore we shall show how to enlarge U to a larger U' such that the kernel from $H_*(U')$ to $H_*(W)$ is smaller. One way in which to do this is to add handles to U along ∂U to kill some of the excess homology of U, i.e. find $D^k \times D^{n-k} \subset V$, $D^k \times D^{n-k} \cap U = S^{k-1} \times D^{n-k} \subset \partial U$, $S^{k-1} \times 0$ representing an element $x \in H_{k-1}(\partial U)$ which goes to O in $H_{k-1}(V)$, and take $U' = U \cup D^k \times D^{n-k}$.

Assuming $H_i(\partial U) \to H_i(V)$ is mono (hence isomorphism) for $i < k - 1$, and onto for $i = k - 1$, and since ∂U and V are 1-connected, the relative Hurewicz Theorem implies that $\pi_k(V, \partial U) = H_k(V, \partial U)$ and we get a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \pi_k(V,\partial U) & \longrightarrow & \pi_{k-1}(\partial U) & \longrightarrow & \pi_{k-1}(V) & \to 0 \\ & \downarrow \cong & & \downarrow & & \downarrow & \\ 0 \to & H_k(V,\partial U) & \longrightarrow & H_{k-1}(\partial U) & \xrightarrow{i'} & H_{k-1}(V) & \to 0 \end{array}$$

so that $\ker i'$ is represented by spherical homology classes in the lowest dimension. If $k-1 < \frac{1}{2}(n-1)$, $n = \dim W$, then one may deduce from a general position argument that an element $x \in \ker i' \subset H_{k-1}(\partial U)$ is represented by an embedded $S^{k-1} \subset \partial U$ and using either general position or Whitney's embedding theorem that it bounds an embedded $D^k \subset V$, from which we may get a handle. In case $\frac{1}{2}(n-1) \leqq k-1 < n-3$, we may obtain the handles differently, using Smale's theory of handle bodies (see [7]). At the very last stage, $k-1 = n-3$, still another technique must be used, which is not obviously the same as attaching k dimensional handles.

We give the details in the next sections.

4. Proof of Proposition 4; codimensions > 3. In this section, we will prove a weaker form of Proposition 4.

PROPOSITION 5. *Let W^n be an open manifold of dimension $n \geqq 6$, 1-connected at ∞ and with $H_*(W)$ finitely generated. Then given a compact $C \subset W$ and $k \leqq n-3$, there is a compact manifold with boundary U, with ∂U and $W-U$ 1-connected, $C \subset$ interior of U, and if $i\colon U \to W$ is the inclusion, then $i_*\colon H_i(U) \to H_i(W)$ is an isomorphism for $i < k$, and onto for all i.*

Proposition 4 is the same but without restriction on k.

LEMMA 6. *If W is a manifold of dimension $\geqq 5$ which is 1-connected at ∞ and given compact $C \subset W$, then there exists a compact manifold U with 1-connected boundary with $C \subset$ interior of U and such that $W-U$ is 1-connected.*

(Note that this shows that if W is 1-connected at ∞ of dimension $\geqq 5$, then $\pi_1(W)$ is finitely generated, since by Van Kampen's theorem $\pi_1(W) = \pi_1(U)$.)

Proof of Lemma 6. Let D be compact, $C \subset D \subset W$ such that $W-D$ is 1-connected. We may find a compact manifold with boundary U' with $D \subset$ interior U'. This follows in the differentiable case by choosing a proper function f, with $f(D) = 0$, $f \geqq 0$ and letting $U' = f^{-1}([0,\epsilon])$ where ϵ is a regular value of f. In the piecewise linear case, D lies in a finite subcomplex K of W and we take U' to be a regular neighborhood of K in W. By taking connected sums along the boundary of the different components of U', we may assume U' is connected. Then $\partial U'$ divides W into two parts, U' and $W-U'$, and U' is connected. Since W is 0-connected at ∞ it follows that all but one of the components of $W-U'$ are compact. Define U'' to be the union of U' and all the compact components of $W-U'$. Since W is connected each such component meets U', so that U'' is connected and $W-U''$ is connected. Then the components of $\partial U''$ may be joined by disjoint arcs in $W-U''$. We let $U'''\equiv U''\cup$ (closed neighborhood of these arcs) and it follows that U''' and $\partial U'''$ are connected.

Now Lemma 6 follows from:

LEMMA 7. *Let M^n be a closed submanifold of W^{n+1}, $n\geqq 4$, and suppose that $\pi_1(W)=0$ and $\pi_1(M)$ is finitely generated. Then we can do surgery on M inside W to get M' with $\pi_1(M')=0$. In particular we can find 2-disks, $D_1^2,\cdots,D_k^2\subset W$ with $D_i^2\cap M=S_i^1=$ boundary of D_i^2, meeting M transversally, such that $M\cup D_1\cup\cdots\cup D_k$ is simply connected, so that the surgeries corresponding to $S_1^1,\cdots,S_k^1$ produce a simply connected manifold.*

For a proof see [1, Lemma 3.1].

As indicated in § 3, since $H_*(W)$ is finitely generated we may find a compact $C'\subset W$ such that $H_*(C')$ maps onto $H_*(W)$, so that for $C'\subset O\subset W$, $H_*(O)$ maps onto $H_*(W)$. Then applying Lemma 6, we may find $U_1\subset W$ with ∂U_1 1-connected and $C\cup C'\subset$ interior of U_1, and $W-U_1$ 1-connected.

Also as indicated in § 3, we may kill the kernel of $H_i(\partial U_1)\to H_i(V_1)$ where $V_1=$ closure of $W-U_1$, and this will kill the kernel of $H_i(U_1)\to H_i(W)$. Since ∂U_1 and V_1 are 1-connected the Hurewicz theorem tells us that every element $x\in H_2(\partial U_1)$ is represented by a map $f\colon S^2\to\partial U_1$ and if $i_{1*}x=0$ in $H_2(V_1)$, $i_1\colon\partial U_1\to V_1$ then f is homotopic to a constant in V_1. Since dimension $\partial U_1=n-1\geqq 5$, it follows from general position that f is homotopic to an embedding g of $S^2\subset\partial U_1$, and if $n>6$, g extends to an embedding $\bar g\colon D^3\subset V_1$, $\bar g\mid S^2=g$, and $\bar g D^3$ meets ∂U_1 transversally in $g(S^2)$. If $n=6$, the existence of $\bar g\colon D^3\to V_1$ with the required properties follows from Whitney's embedding theorem [11] or from the result of Irwin [4]. Now for a generator of kernel i_{1*} take such a 3-disk D^3, and define $U_1'=$ regular neighborhood of $U_1\cup D^3$ in W. This can be made smooth using a theorem of Hirsch [3], or one can define $U_1'=U_1\cup D^3\times D^{n-3}$ and round the corners in an appropriate way, where $D^3\times D^{n-3}$ is a neighborhood of D^3 in V_1. Now $W-U_1'$ is homotopy equivalent to V_1-D^3, and since D^3 is of codimension $\geqq 3$, it follows that $W-U_1'$ is still 1-connected. Similarly $V_1\cap U_1'\cong\partial U_1\cup D^3$ so $V_1\cap U_1'$ is 1-connected, and since $\partial U_1'\cong V_1\cap U_1'-D^3$, it follows that $\partial U_1'$ is 1-connected. Since ∂D^3 is a generator x of kernel i_{1*}, $i_1\colon\partial U_1\to V_1$, it follows that kernel $i_{1*}'\cong$ kernel $i_{1*}/(x)$, so that continuing in this way we will finally arrive at $U_2 \supset U_1$ such that $i_{2*}\colon H_2(U_2) \to H_2(W)$ is an isomorphism, with ∂U_2 and $V_2 = \text{closure } W - U_2$ still 1-connected.

We will need the following (cf. [6; Lemma 1]).

LEMMA 8. *Let X be an n-manifold with boundary $\partial X = M \cup N$, M, N, X all 1-connected, $n \geqq 6$. Suppose $\pi_k(X, M) = 0$ for $2 \leqq k \leqq r - 1 < n - 4$. Then any element $w \in H_{r+1}(X, M)$ can be represented by an embedded handle $D^{r+1} \times D^{n-r-1} \subset X$, meeting ∂X normally in $S^r \times D^{n-r-1} \subset M$.*

Proof. We give the proof in the smooth case using the handle body theory of Smale and for the combinatorial situation we confine ourselves to remarking that the analogous facts are true in that case, as has been shown by Stallings.

Now a theorem of Smale [7; 6.5] says that under our hypotheses, X has a handle decomposition $X = \bigcup_{i=r-1}^{n} X_i$, where $X_{r-1} = M \times I$ and

$$X_j = X_{j-1} \cup H_1{}^j \cup \cdots \cup H_q{}^j,\ \ H_i{}^j = D^j \times D^{n-j},\ \ H_i{}^j \cap H_k{}^j = 0,\quad i \neq k,$$
$$H_i{}^j \cap X_{j-1} = S^{j-1} \times D^{n-j} \subset \partial X_{j-1} - (M \times 0).$$

Since X_j is the homotopy type of X_{j-1} with some j-dimensional disks attached, it follows that $h_j\colon H_k(X_j) \to H_k(X)$ is an isomorphism for $k < j$ and onto for $k = j$, so that $\bar{h}_j\colon H_k(X_j, M) \to H_k(X, M)$ is iso for $k < j$ and onto for $k = j$. Hence there is a $w' \in H_{r+1}(X_{r+1}, M)$ such that $\bar{h}_{r+1} w' = w$.

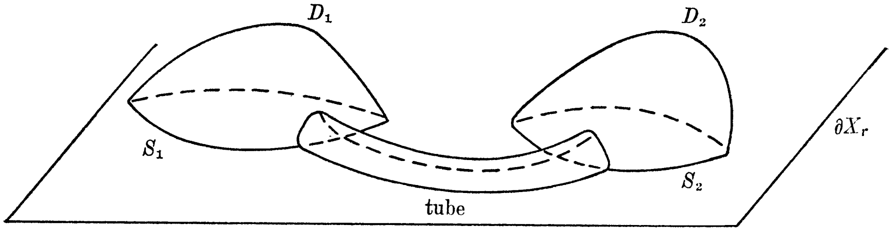
Consider the exact sequence

$$\cdots \to H_{r+1}(X_r, M) \to H_{r+1}(X_{r+1}, M) \xrightarrow{k_*} H_{r+1}(X_{r+1}, X_r)$$
$$\xrightarrow{\partial} H_r(X_r, M) \to \cdots.$$

Since $X_r \cong M \bigcup \cup D^r$, $H_{r+1}(X_r, M) = 0$, so $H_{r+1}(X_{r+1}, M)$ is mapped isomorphically by k_* onto the kernel $\partial \subset H_{r+1}(X_{r+1}, X_r)$. Let $y = k_* w'$.

Now $H_{r+1}(X_{r+1}, X_r)$ is a free abelian group with generators represented by the relative homology classes of the "core" of each handle, i. e. $(D^{r+1} \times 0, S^{r+1} \times 0) \subset (H^{r+1}, \partial H^{r+1})$. Smale [7] has shown (see also Wallace [10]) that if we are given any basis for $H_{r+1}(X_{r+1}, X_r)$ that we may find a set of handles in X_{r+1} attached to X_r so that $X_{r+1} = X_r \cup$ (these handles) and the cores of the new handles yield the given basis for $H_{r+1}(X_{r+1}, X_r)$. Hence we may assume that there is a handle such that $y = mz$, z representing the core of one of the handles of X_{r+1}. But if the core is of codimension > 1, which is the case here, any multiple of its homology is represented by a handle also. If the handle is $D^{r+1} \times D^{n-r-1}$, then we may embed in it m disjoint disks of the form $D_\alpha = D^{r+1} \times t_\alpha \subset D^{r+1} \times D^{n-r-1}$, where t_α are m disjoint points of D^{n-r-1}. Since the codimension is > 1, $\partial D_\alpha{}^{r+1}$ do not separate

$S^r \times D^{n-r-1}$, so that we may join the $S_\alpha{}^r = \partial D_\alpha{}^{r+1}$ by tubes $S^{r-1} \times I$ in $S^r \times D^{n-r-1}$ so as to form the connected sum of the $S_\alpha{}^r$, and the D_α's can be connected by tubes $D^r \times I$ in $D^{r+1} \times D^{n-r-1}$ to form the connected sum along the boundaries of the D_α, with the proper orientation. The picture gives an indication of the process:



(Two igloos connected by a tunnel might be an appropriate title.)

Then this disc and its normal bundle is $\bar{D}^{r+1} \times D^{n-r-1}$ and its core has the desired homology class $y \in H_{r+1}(X_{r+1}, X_r)$, (see [6; Lemma 1]). However, it is attached to ∂X_r rather than $M \times 1 \subset \partial X_{r-1}$, so it remains to show that it can be chosen to miss the handles of X_r and thus be attached to $M \times 1$.

Now if the attaching sphere $\bar{S}^r \times 0$ of this handle does not meet the transverse spheres $s \times S^{n-r-1}$ of the handles of index r in ∂X_r, then it may be moved off these handles down to $M \times 1$ by an isotopy of X_{r+1} so that the image would be the handle we need. For $D^r \times S^{n-r-1} - s \times S^{n-r-1}$ may be deformed by an isotopy into a neighborhood of $S^{r-1} \times S^{n-r-1}$ in $\partial X_r - S^{r-1} \times D^{n-r}$.

If the intersection number of $\bar{S}^r \times 0$ and $s \times S^{n-r-1}$ is 0 then since $r > 2$ and $n - r - 1 < n - 3$, it follows from Whitney's theorem [11] that we may change $\bar{S}^r \times 0$ by an isotopy to miss $s \times S^{n-r-1}$. But if $y \in H_{r+1}(X_{r+1}, X_r)$ is the homotopy class of the core of the handle $\bar{D}^{r+1} \times D^{n-r-1}$, then $\partial y = \sum \alpha_j h_j$, where h_j is the homology class of the r handles which generate $H_r(X_r, M)$ and α_j is the intersection number of $\bar{S}^r \times 0$ and the transverse sphere $s \times S_j{}^{n-r-1}$ of the j-th handle of X_r. Since $\partial y = 0$, all the intersection numbers are zero and there is an isotopy of X_{r+1} which takes $\bar{D}^{r+1} \times D^{n-r-1}$ into a handle attached to ∂X_r. This completes the proof of Lemma 8.

LEMMA 9. *Assume Proposition 5 for $k \leqq n - 3$, so that given compact C one can find $U_k \subset W$, U_k compact manifold with boundary, ∂U_k, and $V_k =$ closure $(W - U_k)$ 1-connected, $C \subset$ interior U_k and $i_{k*}: H_j(U_k) \to H_j(W)$ isomorphism for $j < k$, onto for all j. Then if $x \in (\text{kernel } i_{k*})_k$ there is U_k' containing U_k in its interior with all the above properties, such that $j_*'(x) = 0$ in $H_k(U_k')$, $j': U_k \subset U_k'$. Hence, if $y \in H_{k+1}(W, U_k)$, $\partial y = x \in H_k(U_k)$, there is $U_k' \supset U_k$ as above with $l_*'y = 0$ in $H_{k+1}(W, U_k')$,*

$$l': (W, U_k) \subset (W, U_k').$$

Proof. Given $x \in (\text{kernel } i_{k*})_k$, there is a compact set $D \supset U_k$ such that $j_* x = 0$, $j: U_k \subset D$. By Proposition 5, for $k \leqq n-3$ we can find U_k' with all the required properties and $D \subset$ interior U_k'. Then since $j_* x = 0$ in $H_*(D)$, $j_*' x = i_{D*} j_* x = 0$ in $H_*(U_k')$.

Let $X =$ closure of $U_k' - U_k$, $X \subset V$ and consider the diagram ($U = U_k$, $U' = U_k'$)

$$\begin{tikzcd}
H_k(\partial U) \arrow[d, "h_*"] & H_{k+1}(V, \partial U) \arrow[l, "\bar{\partial}"'] \arrow[r, "\cong"] \arrow[d] \arrow[dr, phantom, "q_*"] & H_{k+1}(W, U) \arrow[r, "\partial"] \arrow[d, "l_*'"] & H_k(U) \arrow[d, "j_*'"] \\
H_k(X) & H_{k+1}(V, X) \arrow[l, "\bar{\partial}'"'] \arrow[r, "\cong"] & H_{k+1}(W, U') \arrow[r, "\partial'"] & H_k(U').
\end{tikzcd}$$

Note that ∂, $\bar{\partial}$, ∂', $\bar{\partial}'$ are mono. Hence if $\partial y = x \in H_k(U)$ and $j_*'(x) = 0$ it follows that $k_*' y = 0$ proving the lemma.

Note that also $h_* \bar{\partial} q_*^{-1}(x) = 0$ in $H_k(X)$. Hence we get:

LEMMA 10. *With the hypothesis of Lemma 9, for and $z \in H_{k+1}(V, \partial U)$ there is compact manifold with boundary $X \subset V$, $\partial X = \partial U \cup N$ such that $h_*(\bar{\partial} z) = 0$ in $H_k(X)$.*

Now we prove Proposition 5 by induction, having proved it for $k = 3$. Let U_k be a manifold with the required properties for $k < n-3$, we would like to produce one with the properties for $k+1$. By Lemma 10 for any $x \in (\text{kernel } i_{k*})_k \subset H_k(U_k)$ we may find $y \in H_k(\partial U_k)$ with $l_* y = x$, $l: \partial U_k \subset U_k$ and a $U_k' = U_k \cup X$ so that $h_* y = 0$ in $H_k(X)$, and $y = \bar{=} w$, $w \in H_{k+1}(X, \partial U_k)$. Then by Lemma 8 we can find a handle $\bar{D}^{k+1} \times D^{n-k+1} \subset X$ attached to ∂U_k which represents w, so that y goes to 0 in $\partial U_k \cup \bar{D}^{k+1} \times D^{n-k+1}$. Let $\bar{U}_k = U_k \cup \bar{D}^{k+1} \times D^{n-k-1}$. Then if $\bar{\imath}_k: \bar{U}_k \subset W$,

$$(\text{kernel } \bar{\imath}_{k*})_k \cong (\text{kernel } i_{k*})_k / (x)$$

and we continue until we have made the kernel 0. This completes the proof of Proposition 5.

5. Proof of Proposition 4; conclusion. By Proposition 5 we may assume that given C we can find a compact manifold $U \subset W$ with ∂U 1-connected, $V =$ closure $(W - U)$ 1-connected, $C \subset$ interior U and $i_*: H_i(U) \to H_i(W)$ an isomorphism for $i < n-3$, onto for all i. From the diagram

$$\begin{tikzcd}
0 \arrow[r] & H_{k+1}(W, U) \arrow[r, "\partial"] & H_k(U) \arrow[r, "i_*"] & H_k(W) \arrow[r] & 0 \\
0 \arrow[r] & H_{k+1}(V, \partial U) \arrow[r, "\partial'"] \arrow[u, "q_*", "\cong"'] & H_k(\partial U) \arrow[r, "j_*"] \arrow[u, "q_*"] & H_k(V) \arrow[r] \arrow[u, "q_*"] & 0
\end{tikzcd}$$

where $q\colon (V, \partial U) \subset (W, U)$ is an excision, we see that kernel $i_* \cong$ kernel j_*. Since ∂U is 1-connected $H_{n-2}(\partial U) \cong H^1(\partial U) = 0$, so that

$$H_{k+1}(W, U) = H_{k+1}(V, \partial U) = 0$$

for $k \neq n-3$ and $H_{n-2}(W, U)$ is free.

We can find a compact set D, $U \subset D \subset W$ such that $i_{D*}(\ker i_*) = 0$, $i_D\colon U \to D$ is inclusion. Let U' be a compact manifold with boundary with $U \cup D \subset$ interior U', and satisfying all the conditions of Proposition 5, as above with U. Then $h_*(\ker i_*) = 0$, $h\colon U \to U'$ is inclusion. It follows from the diagram

$$\begin{tikzcd}
0 \arrow[r] & H_{n-2}(W, U) \arrow[r] \arrow[d, "h_*"] & H_{n-3}(U) \arrow[r, "i_*"] \arrow[d, "h_*"] & H_{n-3}(W) \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & H_{n-2}(W, U') \arrow[r] & H_{n-3}(U') \arrow[r, "i_*'"] & H_{n-3}(W) \arrow[r] & 0
\end{tikzcd}$$

that $h_*(H_{n-2}(W, U)) = 0$. Set $V' =$ closure $(W - U')$, $M = \partial U$, $N = \partial U'$, $X =$ closure $(U' - U)$ so that $\partial X = M \cup N$; let $l_M\colon M \to X$, $l_N\colon N \to X$, $r\colon X \to V$ be inclusions. Then, since

$$q\colon (V, M) \to (W, U) \text{ and } \bar{q}\colon (V, X) \to (W, U')$$

are excisions, the diagram

$$\begin{tikzcd}
H_{n-2}(V, M) \arrow[r, "\cong"] \arrow[d, "\bar{h}_*"] & H_{n-2}(W, U) \arrow[d, "h_*"] \\
H_{n-2}(V, X) \arrow[r, "\cong"] & H_{n-2}(W, U')
\end{tikzcd}$$

shows that $\bar{h}_* = 0$. Since $H_{n-2}(V, X)$ and $H_{n-2}(V, M)$ are free, and $H_i(V, X) = H_i(V, M) = 0$ for $i \neq n-2$, this implies that $\bar{h}^* = 0$, $\bar{h}^*\colon H^{n-2}(V, X) \to H^{n-2}(V, M)$.

Consider the diagram with exact rows:

$$\begin{tikzcd}
0 & H^{n-2}(V, M) \arrow[l] & H^{n-3}(M) \arrow[l] & H^{n-3}(V) \arrow[l] & 0 \arrow[l] \\
0 & H^{n-2}(V, X) \arrow[l] \arrow[u, "\bar{h}^*"] \arrow[d, "h'^*"] & H^{n-3}(X) \arrow[l, "\delta"'] \arrow[u, "l_M{}^*"] \arrow[d, "l_N{}^*"] & H^{n-3}(V) \arrow[l] \arrow[u, "1"] \arrow[d] & 0 \arrow[l] \\
0 & H^{n-2}(V', N) \arrow[l] & H^{n-3}(N) \arrow[l, "\delta'"'] & H^{n-3}(V') \arrow[l] & 0. \arrow[l]
\end{tikzcd}$$

Since $\bar{h}^*=0$ and $H^{n-2}(V,X)$ is free, we can find $\alpha\colon H^{n-2}(V,X)\to H^{n-3}(X)$ such that $l_M{}^*\circ\alpha=0$ and $\delta\circ\alpha=1$ on $H^{n-2}(V,X)$. Since the inclusion $h'\colon(V',N)\to(V,X)$ is an excision,

$$\beta=l_N{}^*\circ\alpha\circ h'^{*-1}\colon H^{n-2}(V',N)\to H^{n-3}(N)$$

is defined, and $\delta'\circ\beta=1$ on $H^{n-2}(V',N)$. Further image $\beta\subset l_N{}^*(\text{kernel } l_M{}^*)$.

LEMMA 11. *$\cap\,\mu_N$ send $l_N{}^*$ (kernel $l_M{}^*)^{n-k-1}$ isomorphically onto (kernel $l_{N*})_k$.*

Proof. We have the commutative diagram with exact rows (see [2]):

$$\begin{tikzcd}
\to H^{n-k-1}(X) \arrow[r, "l^*"] \arrow[d, "\cap\,\nu"] & H^{n-k-1}(\partial X) \arrow[r, "\delta"] \arrow[d, "\cap\,\bar{\mu}"] & H^{n-k}(X,\partial X)\to \\
\to H_{k+1}(X,\partial X) \arrow[r, "\partial"] & H_k(\partial X) \arrow[r, "l_*"] & H_k(X)\to \arrow[u, "\cap\,\nu"']
\end{tikzcd}$$

where $l\colon\partial X\subset X$, $\nu\in H_n(X,\partial X)$ is the fundamental class,

$H_*(\partial X)=H_*(N)+H_*(M)$, $\bar{\mu}=\partial\nu=\mu_N-\mu_M$, $l_*=l_{N*}-l_{M*}$, etc. Then $(l^*H^{n-k-1}(X))\cap\bar{\mu}=(\text{kernel } l_*)_k$. Since $\cap\,\bar{\mu}\mid H^*(N)=\cap\,\mu_N$ and $\cap\,\bar{\mu}\mid H^*(M)=\cap\,\mu_M$, it follows that $(l^*H^{n-k-1}(X))\cap H^*(N)$ is mapped isomorphically by $\cap\,\mu_N$ onto $(\text{kernel } l_*)\cap H_k(N)$. But since $l^*=l_N{}^*-l_M{}^*$, $l^*H^{n-k-1}(X)\cap H_k(N)=l_N{}^*(\text{kernel } l_M{}^*)^{n-k-1}$, and

$$(\text{kernel } l_*)\cap H^*(N)=(\text{kernel } l_{N*})_k,$$

similarly, which proves the lemma.

It follows that $A=(\text{image }\beta)\cap\mu_N$ is a free direct summand of $H_2(N)$, contained in $(\text{kernel } l_{N*})_2$. Now it follows from van Kampen's Theorem that X is 1-connected, for $V=X\cup V'$, $X\cap V'=N$ and V, V', and N are all 1-connected. Then it follows from the Hurewicz Theorem that A consists of spherical cycles in N which are null homotopic in X. By an argument already used in §4 we may find 3-handles $D_i{}^3\times D^{n-3}$ in X whose boundaries $S_i{}^2\times D^{n-3}\subset N$ are such that their homology classes are a basis for $A\subset H_2(N)\cong\pi_2(N)$. If we exchange these handles from X to V' (i.e. add them to V'), we will obtain new manifolds $\bar{X}=X-\text{interior of the handles}$, $\bar{V}=V'\cup(\text{handles})$, $\bar{N}=\bar{V}\cap\bar{X}$. $V=\bar{X}\cup\bar{V}$. Now, since A is a free direct summand of $H_2(N)$, and $H_k(N)\cong H_k(V')$ for $k<n-3$, if $n>6$, we find easily that $H_j(\bar{N})=H_j(N)$ for $j\neq 2$ or $n-3$, $H_j(\bar{V})=H_j(V')$ for $j\neq 2$, and $H_2(\bar{N})=H_2(N)/A=H_2(V)/r_*A=H_2(\bar{V})$, and $\bar{r}_*\colon H_j(\bar{N})\to H_j(\bar{V})$ is still an isomorphism for $j<n-3$, onto for all j. For $n=6$,

the argument that $H_2(\bar{N}) = H_2(N)/A$ is slightly more difficult, but it follows easily from [5; Lemma 5.6]. Now by the same arguments which applied to $(V, \partial U)$, we have that $H^{n-2}(\bar{V}, \bar{N})$ is free, $H_i(\bar{V}, \bar{N}) = 0$ for $i \neq n-2$, and we have the exact sequence

$$0 \leftarrow H^{n-2}(\bar{V}, \bar{N}) \leftarrow H^{n-3}(\bar{N}) \leftarrow H^{n-3}(\bar{V}) \leftarrow 0.$$

By Poincaré duality $H^{n-3}(\bar{N}) \cong H^{n-3}(N)/A'$, where $A' = \beta H^{n-2}(V', N)$. But $H^{n-3}(\bar{V}) \cong H^{n-3}(V')$ ($n \geqq 6$, so $n-3 > 2$), so that it follows that $H^{n-3}(\bar{N})$ and $H^{n-3}(\bar{V})$ are isomorphic groups. Since they are finitely generated and $H^{n-2}(\bar{V}, \bar{N})$ is free, it follows that $H^{n-2}(\bar{V}, \bar{N}) = 0$, so that $H_i(\bar{V}, \bar{N}) = 0$ for all i. It follows by excision, if $\bar{U} = U \cup \bar{X}$, $H_i(W, \bar{U}) = 0$ for all i, $\bar{U} \supset U \supset C$, so that Proposition 4 is proved.

6. The *h*-cobordism theorem.

We now proceed to prove Theorem 2.

Recall V is 1-connected at ∞, $\partial V = M_1 \cup M_2$, M_1, M_2 1-connected at ∞. M_1 and M_2 are deformation retracts of V, and $H_*(M_1)$ (and hence $H_*(M_2)$ and $H_*(V)$) is finitely generated.

By Theorem 1, M_1 and M_2 are isomorphic to interiors of $\bar{M}_1$, $\bar{M}_2$, compact manifolds with boundary. By taking a contraction of $\bar{M}_i$ which embeds $\bar{M}_i$ in $M_i =$ interior $\bar{M}_i$, we may consider $\bar{M}_i \subset M_i \subset V$. Using the product structure in a neighborhood of ∂V, we get embeddings of $\bar{M}_i \times [0,1] \subset V$, with $\bar{M}_i \times [0,1] \cap \partial V = \bar{M}_i \times 0 = \bar{M}_i$. Joining $\bar{M}_1 \times 1$ to $\bar{M}_2 \times 1$ by an arc in interior $V - \bigcup_{i=1,2} \bar{M}_i \times [0,1]$, and thickening the arc, we get a compact manifold U with $\partial V = \bar{M}_1 \cup W \cup \bar{M}_2$ where $\partial W = \partial \bar{M}_1 \cup \partial \bar{M}_2$, $\partial U \cap \partial V = \bar{M}_1 \cup \bar{M}_2$. Then using the same techniques which proved Theorem 1, we may enlarge U to $\bar{V} \subset V$, with V isomorphic to interior $\bar{V}$, adding things only along interior $W \subset \partial U$, so that $\partial \bar{V} = \bar{M}_1 \cup \bar{W} \cup \bar{M}_2$, $\partial \bar{W} = \partial \bar{M}_1 \cup \partial \bar{M}_2$, $\bar{W} \cap \partial V = \partial \bar{M}_1 \cup \partial \bar{M}_2$. From the diagram

$$\begin{tikzcd} \bar{M}_i \arrow[r] \arrow[d] & \bar{V} \arrow[d] \\ M_i \arrow[r] & V \end{tikzcd}$$

it follows that the inclusions $\bar{M}_i \to \bar{V}$ are homotopy equivalences, since the other maps are. It remains to show that $\bar{W}$ is an h-cobordism between $\partial \bar{M}_1$ and $\partial \bar{M}_2$.

Now by the Poincaré duality theorem, with two pieces of boundary (see [2]), we have

$$H^*(\bar{V}, \bar{M}_i) \cong H_*(\bar{V}, \bar{M}_i \cup \bar{W})$$

which is 0 since $\bar{M}_i \to \bar{V}$ is a homotopy equivalence. Hence if

$$H_*(\bar{M}_i) \xrightarrow{i_*} H_*(\bar{M}_i \cup \bar{W}) \xrightarrow{j_*} H_*(\bar{V}),$$

$j_* i_*$ is an isomorphism, j_* is an isomorphism, so that i_* is an isomorphism. Hence $0 = H_*(\bar{M}_i \cup \bar{W}, \bar{M}_i) \cong H_*(\bar{W}, \partial \bar{M}_i)$, by excision, and since $\bar{W}$ and $\partial \bar{M}_i$ are 1-connected, $\bar{W}$ is an h-cobordism.

The corollary follows by results of Smale [7].

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