

A SHORT SURVEY OF THE WORK OF COCHRAN-ORR-TEICHNER ON KNOT CONCORDANCE

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We present a survey of the advances of Cochran-Orr-Teichner on knot concordance, centred around the results in their main paper [4]. These notes owe a lot to lectures of Kent Orr which I attended in Heidelberg in December 2008 and to lecture notes of Peter Teichner from San Diego in 2001 which I typed up.

1. INTRODUCTION

Definition 1.1. [8] An oriented knot K is a topologically *slice knot* if there is an oriented embedded locally flat disk $D^2 \subseteq D^4$ whose boundary $\partial D^2 \subset \partial D^4 = S^3$ is the knot K . Here locally flat means locally homeomorphic to a standardly embedded $\mathbb{R}^2 \subseteq \mathbb{R}^4$.

Two knots K_1, K_2 are *concordant* if there is an embedded locally flat annulus $S^1 \times I \subset S^3 \times I$ such that $\partial(S^1 \times I) \subseteq S^3 \times I$ is the disjoint union of the knots $K_1 \sqcup -K_2$, where the knot $-K$ arises from K by reversing the orientation of the knot and of the ambient space S^3 : on diagrams this latter means switching under crossings to over crossings and vice versa. The set of concordance classes of knots form a group \mathcal{C} under the operation of connected sum with the identity element given by the class of slice knots. \square

Fox and Milnor gave this definition in 1959 [8]; they were interested in removing singularities of surfaces in a 4-manifold: a singularity is removable if the concordance class of its link vanishes. They gave a condition which a slice knot satisfies, namely that its Alexander polynomial factorises in the form $f(t)f(t^{-1})$ for some f . One can also consider smoothly slice knots and require that embeddings are smooth rather than just locally flat, but we will consider topological manifolds and locally flat embeddings in these notes.

The first major progress in the study of the concordance group was in 1968 when Levine [16], who was studying high-dimensional knots (Definition 1.2), defined an *algebraic concordance group*, namely the Witt group of Seifert forms. The *Seifert form* is the linking form on the first homology $H_1(F)$ of a Seifert surface F , defined by pushing one of a pair of curves off the surface slightly along a normal vector. A form is said to be *algebraically null-concordant* if it is represented by a matrix congruent to one of the form:

$$\begin{pmatrix} 0 & A \\ B & C \end{pmatrix},$$

for block matrices A, B, C such that $C = C^T$ and $A - B^T$ is invertible.

The idea is that if there is a half-basis of curves on F with self linking zero, it might be possible to cut the Seifert surface along these curves and glue in discs, embedded in D^4 , so as to construct a slice disc. This is called *ambient surgery*. This may be problematic as we shall see below, but certainly a slice knot has an algebraically null-concordant Seifert form, so we have an algebraic obstruction. Levine [16] and Stoltzfus [21] calculated the group of Seifert forms to be isomorphic to:

$$\bigoplus_{\infty} \mathbb{Z} \oplus \bigoplus_{\infty} \mathbb{Z}_2 \oplus \bigoplus_{\infty} \mathbb{Z}_4.$$

The infinite cyclic summands are detected by the Levine-Tristram ω -signatures.

Definition 1.2. An oriented m -dimensional knot K is an oriented, locally flat embedding of $S^m \subset S^{m+2}$. An m -knot is topologically slice if there is an oriented, locally flat embedded disk $D^{m+1} \subseteq D^{m+3}$ whose boundary $\partial D^{m+1} \subset \partial D^{m+3} = S^{m+2}$ is the knot K . The group of concordance classes of m -knots is denoted \mathcal{C}_m . \square

Every m -knot has a Seifert $m+1$ -manifold F in S^{m+2} , with boundary the knot, and there is a linking form on the middle dimensional homology of F defined as above which gives us the Seifert form. We push the interior of F into D^{m+3} , and try to perform ambient surgery in D^{m+3} on the Seifert manifold to make it highly connected and therefore, by the h-cobordism theorem, a disk D^{m+1} . In the case of even-dimensional knots there is no obstruction to this, and we can always guarantee by general position that we can glue in embedded rather than immersed discs when we try to do ambient surgery. Kervaire [14] showed that:

$$\mathcal{C}_{2n} \cong 0.$$

For odd dimensional knots $K: S^{2n-1} \subset S^{2n+1}$, the algebraic concordance class of the Seifert form obstructs the possibility of embedding all of the surgery disks. Levine showed [16] for odd high dimensional knots, with $n \geq 2$, that:

$$\mathcal{C}_{2n-1} \xrightarrow{\cong} \bigoplus_{\infty} \mathbb{Z} \oplus \bigoplus_{\infty} \mathbb{Z}_2 \oplus \bigoplus_{\infty} \mathbb{Z}_4.$$

For high-dimensional knots we can always assume by surgery that the fundamental group of the complement of a Seifert $2n$ -manifold pushed into D^{2n+2} is \mathbb{Z} , and we can always guarantee that we can glue in embedded discs, as long as the algebraic obstruction vanishes, using the Whitney trick, when we try to do ambient surgery. An odd-dimensional knot in high dimensions, so when $n > 1$, is slice if and only if it is algebraically null-concordant. However when $n = 1$, our case of interest, the Whitney trick fails, this program does not work and Levine's map is only a surjection. Whenever we try to do surgery to kill an element of the fundamental group of the complement of a Seifert surface in D^4 , we simultaneously create another element of the fundamental group. In dimension four there is no guarantee that disks can be embedded, only immersed, even if the linking form obstruction vanishes, and attempts to remove intersection points create further problems with the fundamental

group. These problems do not disappear in general unless, like Casson and Freedman ([1], [9]), we can push them away to infinity. Obstructing concordance of knots in dimension 3 starts with the high-dimensional obstruction, but in contrast to the high-dimensional case, this is only the first stage.

There is a more intrinsic version of the algebraic concordance obstruction. If we cut the knot exterior

$$X := \text{cl}(S^3 \setminus (K(S^1) \times D^2))$$

open along a Seifert surface, and then glue infinitely many copies of X together along the Seifert surface, we obtain a space X_∞ , the *infinite cyclic* or *universal abelian cover* of the knot exterior, which is independent of the choice of Seifert surface. The $\mathbb{Z}[\mathbb{Z}]$ module $H_1(X_\infty; \mathbb{Z}) \cong H_1(X; \mathbb{Z}[\mathbb{Z}])$, called the *Alexander module*, is therefore an invariant of the knot. It is a torsion module (see [17]), and we can define the Blanchfield homology linking pairing

$$\text{Bl}: H_1(X; \mathbb{Z}[\mathbb{Z}]) \times H_1(X; \mathbb{Z}[\mathbb{Z}]) \rightarrow \mathbb{Q}(t)/\mathbb{Z}[t, t^{-1}]$$

as follows. For $x, y \in C_1(X_\infty; \mathbb{Z})$, find a $z \in C_2(X_\infty; \mathbb{Z})$ such that $\partial z = p(t)x$ for some Laurent polynomial $p(t) \in \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$ where t generates the deck transformation group of X_∞ (taking $p(t) = \Delta_K(t)$, the Alexander polynomial, will always work, for instance). Then define:

$$\text{Bl}(x, y) = \frac{\sum_{i=-\infty}^{\infty} (z, yt^{-i})t^i}{p(t)} \in \mathbb{Q}(t)/\mathbb{Z}[t, t^{-1}]$$

where $(,)$ is the \mathbb{Z} -valued intersection pairing of chains in C_2 and C_1 .

This is equivalent to defining the Blanchfield pairing via the isomorphisms:

$$\begin{aligned} H_1(X; \mathbb{Z}[t, t^{-1}]) &\xrightarrow{\cong} H^2(X; \mathbb{Z}[t, t^{-1}]) \xrightarrow{\cong} H^1(X; \mathbb{Q}(t)/\mathbb{Z}[t, t^{-1}]) \\ &\xrightarrow{\cong} \text{Hom}_{\mathbb{Z}[t, t^{-1}]}(H_1(X; \mathbb{Z}[t, t^{-1}]), \mathbb{Q}(t)/\mathbb{Z}[t, t^{-1}]) \end{aligned}$$

where the isomorphisms come from Poincaré duality, a connecting Bockstein homomorphism, and a Universal Coefficient Spectral Sequence. The Blanchfield form arises from a Seifert matrix V as follows (see Kearton [13]):

$$\text{Bl}(a, b) = a^T(1 - t)(tV - V^T)^{-1}b \pmod{\mathbb{Z}[\mathbb{Z}]}.$$

Note that in order to invert the matrix it is necessary to pass to the field of fractions $\mathbb{Q}(t)$ of $\mathbb{Z}[t, t^{-1}]$. The appearance of the factor $(1 - t)$ corresponds to the duality; it measures the intersection of 2-chains and 1-chains in a certain handle decomposition which begins with the Seifert surface: [12] page 158. For a slice knot, the Blanchfield form is *metabolic*; that is, there is a submodule $P \subset H_1(X; \mathbb{Z}[\mathbb{Z}])$ such that $P = P^\perp$.

The next stage was the seminal work of Casson and Gordon [2], who found algebraically null-concordant knots which are not slice; they used the metaboliser of a linking form on a k -fold branched covering of a knot to define representations of the fundamental group of a 4-manifold whose boundary is M_K , the result of zero-framed surgery on K , so that they could calculate the signature of the twisted intersection form of the 4-manifold. They made use

of the key observation that the vanishing of first-order linking information in a 3-manifold controls the representations of the fundamental group which extend over a 4-manifold which has the 3-manifold as boundary, which enables the construction of a second order intersection form on the 4-manifold. For a slice disc complement the signatures of the intersection form which Casson and Gordon defined vanish, yielding an obstruction theory.

In 1999, Cochran-Orr-Teichner defined an infinite filtration of the concordance group. They understood that the Casson-Gordon invariants obstructed sliceness on a second level. Recall the heuristic above that if the Seifert form is algebraically null concordant we can attempt to surger along the curves with zero self-linking and try to create a slice disk. Instead of being able to glue in disks, we can certainly glue in surfaces. We can then ask whether these surfaces have sufficiently many curves with zero self linking: the Casson-Gordon invariants obstruct, roughly speaking, the existence of these curves. The Cochran-Orr-Teichner filtration essentially iterates this idea. It is defined by looking at successive quotients of the derived series¹ of the fundamental group, and constructing so-called higher order Blanchfield forms to control which representations extend over their 4-manifolds. By using the Blanchfield form on the infinite cyclic cover instead of the \mathbb{Q}/\mathbb{Z} valued linking forms on the finite cyclic covers as in the Casson-Gordon type representations Cochran-Orr-Teichner keep greater control on the fundamental group, which significantly improves the power of their obstruction theory. Their representations map into fixed groups which they call universally solvable groups, and the values of the representations depend for their definitions on choices of the way in which the lower level obstructions vanish.

Finally, with this extra control on the fundamental group, extra technology is required to extract invariants of the Witt classes of intersection forms. Cochran-Orr-Teichner use the theory of L^2 -signatures to obtain signatures which capture their obstruction theory and are able to show that their filtration is highly non-trivial.

2. THE GEOMETRIC FILTRATION OF THE KNOT CONCORDANCE GROUP

Definition 2.1. We recall the definition of the zero-framed surgery along K in S^3 , M_K : attach a solid torus to the boundary of the knot exterior $X = \text{cl}(S^3 \setminus (K(S^1) \times D^2))$ in such a way that the longitude of the knot bounds in the solid torus.

$$M_K = X \cup_{S^1 \times S^1} D^2 \times S^1.$$

The homology groups of M_K are given by:

$$H_i(M_K; \mathbb{Z}) \cong \mathbb{Z} \text{ for } i = 0, 1, 2, 3;$$

and are 0 otherwise. $H_1(M_K; \mathbb{Z})$ is generated by a meridian of the knot, and $H_2(M_K; \mathbb{Z})$ is generated by a Seifert surface for K capped off with a disc in $D^2 \times S^1$. The fundamental

¹For a group G , $G^{(0)} := G$; the rest of the derived series is defined inductively by $G^{(n+1)} = [G^{(n)}, G^{(n)}]$

group is given by:

$$\pi_1(M_K) \cong \frac{\pi_1(X)}{\langle l \rangle}$$

where $l \in \pi_1(S^1 \times S^1) \leq \pi_1(X)$ represents the zero-framed longitude of K . □

Cochran-Orr-Teichner [4] defined a geometric filtration of the knot concordance group which revealed the depth of structure in \mathcal{C} . The filtration is based on the following characterisation of slice knots: notice that the exterior of a slice disc for a knot K is a 4-manifold whose boundary is M_K , since the extra $D^2 \times S^1$ which is glued onto the knot exterior X is the boundary of a regular neighbourhood of a slice disc.

Proposition 2.2. K is topologically slice if and only if M_K bounds a topological 4-manifold W such that

- (i): $i_*: H_1(M_K; \mathbb{Z}) \xrightarrow{\cong} H_1(W; \mathbb{Z})$ where $i: M_K \hookrightarrow W$ is the inclusion map;
- (ii): $H_2(W; \mathbb{Z}) \cong 0$;
- (iii): $\pi_1(W)$ is normally generated by the meridian of the knot.

Proof. The exterior of a slice disc D , $W := \text{cl}(D^4 \setminus (D \times D^2))$, satisfies all the conditions of the Proposition, as can be verified using Mayer-Vietoris and Seifert-Van Kampen arguments on the decomposition of D^4 into W and $D \times D^2$. Conversely, suppose we have a manifold W which satisfies all the conditions of the Proposition. Glue in $D^2 \times D^2$ to the $D^2 \times S^1$ part of M_K . This gives us a 4-manifold W' with $H_*(W'; \mathbb{Z}) \cong H_*(D^4; \mathbb{Z})$, $\pi_1(W') \cong 0$ and $\partial W' = S^3$, so K is slice in W' . We can then apply Freedman's topological h-cobordism theorem ([9]) to show that $W' \approx D^4$ and so K is in fact slice in D^4 . □

To filter the condition of sliceness with a geometric obstruction theory, Cochran-Orr-Teichner look for 4-manifolds which could potentially be changed to make a slice disk exterior. We start with a 4-manifold W with $\partial W = M_K$ which satisfies conditions (i) and (iii) of Proposition 2.2 and perform homology surgery with respect to a circle; that is we aim to perform surgery on embedded 2-spheres in W in order to kill $H_2(W; \mathbb{Z})$ and obtain a \mathbb{Z} -homology circle. Typically classes in $H_2(W; \mathbb{Z})$ will be represented by immersed spheres or embedded surfaces of non-zero genus rather than by embedded spheres. We can measure how close we are to being able to kill $H_2(W; \mathbb{Z})$ by surgery by looking at the middle-dimensional equivariant intersection form:

$$\lambda: H_2(W; \mathbb{Z}[\pi_1(W)]) \times H_2(W; \mathbb{Z}[\pi_1(W)]) \rightarrow \mathbb{Z}[\pi_1(W)].$$

Using coefficients in $\pi_1(W)$ allows us to detect surfaces and their intersections. The next problem comes from the fact that there can be a large variation in $\pi_1(W)$ for different choices of W . In order to define an obstruction theory, Cochran-Orr-Teichner take representations which factor through quotients by elements of the derived series to fixed groups Γ_{n-1} :

$$\rho_{n-1}: \pi_1(W) \rightarrow \frac{\pi_1(W)}{\pi_1(W)^{(n)}} \rightarrow \Gamma_{n-1}.$$

If there is an embedded surface $N \subseteq W$ with $\pi_1(N) \leq \pi_1(W)^{(n)}$, called an (n) -*surface*, then as far as the n th level intersection form

$$\lambda_n: H_2(W; \mathbb{Z}[\pi_1(W)/\pi_1(W)^{(n)}) \times H_2(W; \mathbb{Z}[\pi_1(W)/\pi_1(W)^{(n)}) \rightarrow \mathbb{Z}[\pi_1(W)/\pi_1(W)^{(n)]}$$

can see we have an embedded sphere. Of course it may not actually be embedded, but in this way Cochran-Orr-Teichner obtain calculable obstructions. For $n = 1$, this is essentially the Cappell-Shaneson technique for obstructing the concordance of high-dimensional knots $S^m \subseteq S^{m+2}$. We now give the definition of the Cochran-Orr-Teichner filtration:

Definition 2.3 ([4] Definition 1.2). A *Lagrangian* of a symmetric form λ on a free module P is a submodule $L \subseteq P$ of half-rank on which λ vanishes. For $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, let λ_n be the intersection form, and μ_n the self-intersection form, on the middle dimensional homology $H_2(W^{(n)}; \mathbb{Z}) \cong H_2(W; \mathbb{Z}[\pi_1(W)/\pi_1(W)^{(n)})$ of the n th derived cover of a 4-manifold W , that is the regular covering space $W^{(n)}$ corresponding to the subgroup $\pi_1(W)^{(n)} \leq \pi_1(W)$.

$$\lambda_n: H_2(W; \mathbb{Z}[\pi_1(W)/\pi_1(W)^{(n)}) \times H_2(W; \mathbb{Z}[\pi_1(W)/\pi_1(W)^{(n)}) \rightarrow \mathbb{Z}[\pi_1(W)/\pi_1(W)^{(n)]}$$

An (n) -*Lagrangian* is a submodule of $H_2(W; \mathbb{Z}[\pi_1(W)/\pi_1(W)^{(n)})$, on which λ_n and μ_n vanish, which maps via the covering map onto a Lagrangian of λ_0 .

We say that a knot K is (n) -*solvable* if M_K bounds a spin 4-manifold W such that the inclusion induces an isomorphism on first homology and such that W admits two dual (n) -*Lagrangians*. This means that λ_n pairs the two Lagrangians together non-singularly and their images freely generate $H_2(W; \mathbb{Z})$.

We say that K is $(n.5)$ -*solvable* if in addition one of the (n) -Lagrangians is an $(n+1)$ -Lagrangian. \square

Remark 2.4. This filtration of the knot concordance group relates strongly to geometric filtrations using gropes and Whitney towers (see [4] Section 8 for more information), objects which feature prominently in the theory of the classification of 4-manifolds (see e.g. [9]). A slice knot is (n) -solvable for all $n \in \mathbb{N}_0$ by Proposition 2.2, and it is hoped, but not known to be true, that if a knot is (n) -solvable for all n then it is topologically slice.

A knot is (0) -solvable if and only if its Arf invariant vanishes, and (0.5) -solvable if and only if it is algebraically slice i.e. its Seifert form is null-concordant.

We ask for the 4-manifold to be *spin* so that the self-intersection forms μ_n can be well-defined on homology classes of $H_2(W^{(n)}; \mathbb{Z})$ - see Section 7 of [4]. The self-intersection form is crucial in surgery theory for keeping track of the bundle data. The Cochran-Orr-Teichner obstructions do not depend on it but it ought to be considered if one wishes to capture the geometric (n) -solvability criteria as closely as possible.

The size of an (n) -Lagrangian is controlled only by its image under the map induced by the covering map $W^{(n)} \rightarrow W$ in $H_2(W; \mathbb{Z})$; the intersection forms of W are typically singular due to the presence of the boundary M_K . The requirement roughly speaking is that we have a Lagrangian of λ_n on the non-singular part of $H_2(W^{(n)}; \mathbb{Z})$. We can see from the long exact

sequence of a pair that the intersection form is non-singular on the part of $H_2(W^{(n)}; \mathbb{Z})$ which neither lies in the image under inclusion of $H_2(\partial W^{(n)}; \mathbb{Z})$ nor is Poincaré dual to a relative class in $H_2(W^{(n)}, \partial W^{(n)}; \mathbb{Z})$ which has non-zero boundary in $H_1(\partial W^{(n)}; \mathbb{Z})$. The existence of a dual (n) -Lagrangian means that we have a non-singular part of sufficient size. The dual (n) -Lagrangian maps to a dual Lagrangian of λ_0 , implying that the form λ_0 is hyperbolic on $H_2(W; \mathbb{Z})$ (see Remark 7.6 of [4] for the required basis change), which is a necessary condition if we wish to modify W by surgery into a homology circle.

Note that $H_2(W; \mathbb{Z})$ is a free module since if it had torsion this would appear in $H^3(W; \mathbb{Z})$ by universal coefficients. However $H^3(W; \mathbb{Z})$ is isomorphic to $H_1(W, M_K; \mathbb{Z})$ by Poincaré duality, which is zero by the long exact sequence of a pair since the inclusion of the boundary M_K into W induces an isomorphism on first homology.

The dual classes are very important. When looking for a half basis of embedded spheres, or perhaps just of (n) -surfaces, as candidates for surgery, or when looking for embedded gropes, we can use the duals to remove unwanted intersections between surfaces by tubing between an intersection point and the intersection of one of the surfaces with its dual - see Section 8 of [4]. If we achieve a half basis of framed embedded spheres, when doing surgery on a such a 2-sphere $S^2 \times D^2$, and replacing it with $D^3 \times S^1$, without the existence of dual classes we would create new classes in $H_1(W; \mathbb{Z})$. This would ruin the condition that $i_*: H_1(M_K; \mathbb{Z}) \xrightarrow{\cong} H_1(W; \mathbb{Z})$ is an isomorphism, which is necessary for a 4-manifold to be a slice disc complement.

3. THE COCHRAN-ORR-TEICHNER OBSTRUCTION THEORY

We now describe the obstruction theory of Cochran-Orr-Teichner [4] which they use to detect that certain knots are not (1.5) -solvable - and indeed that certain knots are not $(n.5)$ -solvable for any $n \in \mathbb{N}_0$, but we focus on (1.5) -solvability for the exposition. To define their obstructions, Cochran-Orr-Teichner have representations ρ of the fundamental group $\pi_1(M_K)$ of M_K which extend to representations of $\pi_1(W)$ for (1) -solutions W :

$$\begin{array}{ccc} \pi_1(M_K) & \xrightarrow{i_*} & \pi_1(W) \\ & \searrow \rho & \swarrow \tilde{\rho} \\ & \Gamma & \end{array}$$

where $\partial W = M_K$ and

$$\Gamma = \Gamma_1 := \mathbb{Z} \ltimes \frac{\mathbb{Q}(t)}{\mathbb{Q}[t, t^{-1}]},$$

their *universally* (1) -solvable or metabelian group. The representation:

$$\rho: \pi_1(M_K) \rightarrow \pi_1(M_K)/\pi_1(M_K)^{(2)} \rightarrow \mathbb{Z} \ltimes H_1(M_K; \mathbb{Q}[t, t^{-1}]) \rightarrow \mathbb{Z} \ltimes \frac{\mathbb{Q}(t)}{\mathbb{Q}[t, t^{-1}]}$$

is given by:

$$g \mapsto (n := \phi(g), h := gt^{-\phi(g)}) \mapsto (n, \text{Bl}(p, h)),$$

where $\phi: \pi_1(M_K) \rightarrow \mathbb{Z}$ is the abelianisation homomorphism and t is a preferred meridian in $\pi_1(M_K)$. Bl is the Blanchfield form defined fully below in Definition 3.1, and p is an element of $H_1(M_K; \mathbb{Q}[t, t^{-1}])$ chosen to lie in a metaboliser of the Blanchfield form so that the representation extends over the 4-manifold W (see Theorem 3.2).

Definition 3.1. The rational *Blanchfield form* is the non-singular Hermitian pairing

$$\text{Bl}: H_1(M_K; \mathbb{Q}[\mathbb{Z}]) \times H_1(M_K; \mathbb{Q}[\mathbb{Z}]) \rightarrow \frac{\mathbb{Q}(\mathbb{Z})}{\mathbb{Q}[\mathbb{Z}]} = \frac{\mathbb{Q}(t)}{\mathbb{Q}[t, t^{-1}]}$$

which is defined by the sequence of isomorphisms:

$$\overline{H_1(M_K; \mathbb{Q}[\mathbb{Z}])} \xrightarrow{\cong} H^2(M_K; \mathbb{Q}[\mathbb{Z}]) \xrightarrow{\cong} H^1(M_K; \mathbb{Q}(\mathbb{Z})/\mathbb{Q}[\mathbb{Z}]) \xrightarrow{\cong} \text{Hom}_{\mathbb{Q}[\mathbb{Z}]}(H_1(M_K; \mathbb{Q}[\mathbb{Z}]), \mathbb{Q}(\mathbb{Z})/\mathbb{Q}[\mathbb{Z}]).$$

The first isomorphism is Poincaré duality: this involves the involution on the group ring to convert right modules to left modules. The second isomorphism is the inverse of a Bockstein homomorphism: associated to the short exact sequence of coefficient groups:

$$0 \rightarrow \mathbb{Q}[t, t^{-1}] \rightarrow \mathbb{Q}(t) \rightarrow \frac{\mathbb{Q}(t)}{\mathbb{Q}[t, t^{-1}]} \rightarrow 0$$

is a long exact sequence in cohomology

$$H^1(M_K; \mathbb{Q}(t)) \rightarrow H^1(M_K; \mathbb{Q}(t)/\mathbb{Q}[t, t^{-1}]) \xrightarrow{\beta} H^2(M_K; \mathbb{Q}[t, t^{-1}]) \rightarrow H^2(M_K; \mathbb{Q}(t)).$$

$H_1(M_K; \mathbb{Q}[t, t^{-1}])$ is a torsion $\mathbb{Q}[t, t^{-1}]$ -module, with the Alexander polynomial annihilating the module. $\mathbb{Q}(t)$ is flat over $\mathbb{Q}[t, t^{-1}]$, so

$$H_1(M_K; \mathbb{Q}(t)) \cong \mathbb{Q}(t) \otimes_{\mathbb{Q}[t, t^{-1}]} H_1(M_K; \mathbb{Q}[t, t^{-1}]) \cong 0.$$

Then on the one hand Universal Coefficients, since $\mathbb{Q}(t)$ is a field, and on the other hand Poincaré duality shows that:

$$H^1(M_K; \mathbb{Q}(t)) \cong \text{Hom}_{\mathbb{Q}(t)}(H_1(M_K; \mathbb{Q}(t)), \mathbb{Q}(t)) \cong 0$$

and

$$H^2(M_K; \mathbb{Q}(t)) \cong H_1(M_K; \mathbb{Q}(t)) \cong 0,$$

which together imply that Bockstein homomorphism β is an isomorphism. The final isomorphism:

$$H^1(M_K; \mathbb{Q}(\mathbb{Z})/\mathbb{Q}[\mathbb{Z}]) \xrightarrow{\cong} \text{Hom}_{\mathbb{Q}[\mathbb{Z}]}(H_1(M_K; \mathbb{Q}[\mathbb{Z}]), \mathbb{Q}(\mathbb{Z})/\mathbb{Q}[\mathbb{Z}])$$

is given by the Universal Coefficient theorem. This applies since $\mathbb{Q}[\mathbb{Z}]$ is a Principal Ideal Domain. To see that the map is an isomorphism we need to see that:

$$\text{Ext}_{\mathbb{Q}[\mathbb{Z}]}^1(H_1(M_K; \mathbb{Q}[\mathbb{Z}]), \mathbb{Q}(\mathbb{Z})/\mathbb{Q}[\mathbb{Z}]) \cong 0.$$

$\mathbb{Q}(\mathbb{Z})/\mathbb{Q}[\mathbb{Z}]$ is a divisible $\mathbb{Q}[\mathbb{Z}]$ -module, which implies that it is injective since $\mathbb{Q}[\mathbb{Z}]$ is a PID (see [20] I 6.10). We can calculate the Ext groups using the injective resolution of $\mathbb{Q}(\mathbb{Z})/\mathbb{Q}[\mathbb{Z}]$ of length 0 (see [11] IV.8), which implies that

$$\text{Ext}_{\mathbb{Q}[\mathbb{Z}]}^j(A, \mathbb{Q}(\mathbb{Z})/\mathbb{Q}[\mathbb{Z}]) \cong 0$$

for $j \geq 1$ for any $\mathbb{Q}[\mathbb{Z}]$ -module A ; in particular this holds for $A = H_1(M_K; \mathbb{Q}[\mathbb{Z}])$. So indeed we have an isomorphism from the Universal Coefficient theorem as claimed and the rational Blanchfield form is non-singular. For the improvements necessary to see that the Blanchfield form is also non-singular with $\mathbb{Z}[\mathbb{Z}]$ coefficients see [17].

We say that the Blanchfield pairing is *metabolic* if it has a metaboliser. A *metaboliser* for the Blanchfield form is a submodule $P \subseteq H_1(M_K; \mathbb{Q}[\mathbb{Z}])$ such that:

$$P = P^\perp := \{v \in H_1(M_K; \mathbb{Q}[\mathbb{Z}]) \mid \text{Bl}(v, w) = 0 \text{ for all } w \in P\}.$$

□

One of the key theorems of Cochran-Orr-Teichner in [4] is their Theorem 4.4. They show, using duality, that the kernel of the inclusion induced map:

$$i_*: H_1(M_K; \mathbb{Q}[\mathbb{Z}]) \rightarrow H_1(W; \mathbb{Q}[\mathbb{Z}]),$$

for (1)-solutions W , is a metaboliser for the Blanchfield form. This then implies that choices of $p \in H_1(M_K; \mathbb{Q}[\mathbb{Z}])$ control which representations of the form of ρ extend over the (1)-solution, where p is in the definition of ρ . Choosing $p \in P$, where P is a metaboliser for the Blanchfield form, is necessary for the representation to extend to $\pi_1(W)$. This is very useful for applications, since the Blanchfield form can be calculated explicitly for a given knot; one method calculates the form in terms of a Seifert matrix. The philosophy is that linking information in the 3-manifold controls intersection information in the 4-manifold. Note that we require that the Blanchfield form is metabolic, i.e. that the first order obstruction vanishes, in order for the representation $\tilde{\rho}$ and thence the second order obstruction to be defined. We give the proof of the following lemma in full since it is a crucial argument.

Theorem 3.2 ([4] Theorem 4.4). Suppose M_K is (1)-solvable via W . Then if we define:

$$P := \ker(i_*: H_1(M_K; \mathbb{Q}[\mathbb{Z}]) \rightarrow H_1(W; \mathbb{Q}[\mathbb{Z}])),$$

then the rational Blanchfield form Bl of M_K is metabolic and in fact $P = P^\perp$ with respect to Bl .

Proof. Before proving the Theorem, Cochran-Orr-Teichner state in their Lemma 4.5, whose proof we only sketch, that the sequence:

$$TH_2(W, M_K; \mathbb{Q}[\mathbb{Z}]) \xrightarrow{\partial} H_1(M_K; \mathbb{Q}[\mathbb{Z}]) \xrightarrow{i_*} H_1(W; \mathbb{Q}[\mathbb{Z}])$$

is exact, where TH_2 denotes the torsion part of the second homology. The idea is that the (1)-Lagrangian and its duals generate the free part of $H_2(W; \mathbb{Q}[\mathbb{Z}])$. The existence of the duals is used to show that the intersection form:

$$\lambda_1: H_2(W; \mathbb{Q}[\mathbb{Z}]) \rightarrow \text{Hom}_{\mathbb{Q}[\mathbb{Z}]}(H_2(W; \mathbb{Q}[\mathbb{Z}]), \mathbb{Q}[\mathbb{Z}])$$

is surjective. We consider those classes in $H_2(W, M_K; \mathbb{Q}[\mathbb{Z}])$ which map to zero under the composition of Poincaré duality and universal coefficients:

$$\kappa: H_2(W, M_K; \mathbb{Q}[\mathbb{Z}]) \xrightarrow{\cong} H^2(W; \mathbb{Q}[\mathbb{Z}]) \rightarrow \text{Hom}_{\mathbb{Q}[\mathbb{Z}]}(H_2(W; \mathbb{Q}[\mathbb{Z}]), \mathbb{Q}[\mathbb{Z}]).$$

Since the first map is an isomorphism and the final group is free, of the same rank as the free part of $H^2(W; \mathbb{Q}[\mathbb{Z}])$ (by the Universal coefficient theorem since $\mathbb{Q}[\mathbb{Z}]$ is a PID), the kernel of this composition is torsion. An element p of $\ker(i_*)$ lifts to a relative class $x \in H_2(W; M_K; \mathbb{Q}[\mathbb{Z}])$. We can remove any part of this which comes from $H_2(W; \mathbb{Q}[\mathbb{Z}])$ without affecting the boundary of x . We simply remove the part of x which has non-zero duality according to κ , which comes from $H_2(W; \mathbb{Q}[\mathbb{Z}])$ since λ_1 is surjective. We are then left with an element in the kernel of κ which is therefore torsion and whose boundary is still p .

Using this, we construct a non-singular *relative linking pairing* on the 4-manifold with boundary W .

$$\beta_{rel}: \overline{TH_2(W, M_K; \mathbb{Q}[\mathbb{Z}])} \times H_1(W; \mathbb{Q}[\mathbb{Z}]) \rightarrow \mathbb{Q}(\mathbb{Z})/\mathbb{Q}[\mathbb{Z}].$$

This is defined in a similar manner to the Blanchfield pairing on M_K ; we use the composition of isomorphisms:

$$\overline{TH_2(W, M_K; \mathbb{Q}[\mathbb{Z}])} \xrightarrow{\cong} TH^2(W; \mathbb{Q}[\mathbb{Z}]) \xrightarrow{\cong} H^1(W, \mathbb{Q}(\mathbb{Z})/\mathbb{Q}[\mathbb{Z}]) \xrightarrow{\cong} \text{Hom}_{\mathbb{Q}[\mathbb{Z}]}(H_1(W; \mathbb{Q}[\mathbb{Z}]), \frac{\mathbb{Q}(\mathbb{Z})}{\mathbb{Q}[\mathbb{Z}]})$$

where as before these are given by Poincaré duality, a Bockstein homomorphism, and the Universal Coefficient theorem. To see that the second map is an isomorphism, as before we have a long exact sequence with connecting homomorphism given by the Bockstein:

$$H^1(W; \mathbb{Q}(\mathbb{Z})) \rightarrow H^1(W; \mathbb{Q}(\mathbb{Z})/\mathbb{Q}[\mathbb{Z}]) \xrightarrow{b} H^2(W; \mathbb{Q}[\mathbb{Z}]) \rightarrow H^2(W; \mathbb{Q}(\mathbb{Z})).$$

Proposition 2.11 of [4] says here that $H^1(W; \mathbb{Q}(\mathbb{Z})) \cong 0$, while $H^2(W; \mathbb{Q}(\mathbb{Z}))$ is $\mathbb{Q}[\mathbb{Z}]$ -torsion free, so it follows that we have an isomorphism:

$$b^{-1}: TH^2(W; \mathbb{Q}[\mathbb{Z}]) \xrightarrow{\cong} H^1(W, \mathbb{Q}(\mathbb{Z})/\mathbb{Q}[\mathbb{Z}]).$$

The universal coefficients argument for the final map runs parallel to the corresponding argument for M_K in Definition 3.1.

We now make use of our non-singular pairings Bl and β_{rel} in the following diagram: all coefficients are taken to be $\mathbb{Q}[\mathbb{Z}]$ and the functor \bullet^\wedge is the Pontryagin dual:

$$\bullet^\wedge := \overline{\text{Hom}_{\mathbb{Q}[\mathbb{Z}]}(\bullet, \mathbb{Q}(\mathbb{Z})/\mathbb{Q}[\mathbb{Z}])}.$$

We have:

$$\begin{array}{ccccc}
TH_2(W, M_K) & \xrightarrow{\partial_*} & H_1(M_K) & \xrightarrow{i_*} & H_1(W) \\
\downarrow \beta_{rel} & & \downarrow \text{Bl} & & \downarrow \beta_{rel} \\
H_1(W)^\wedge & \xrightarrow{i^\wedge} & H_1(M_K)^\wedge & \xrightarrow{\partial^\wedge} & TH_2(W, M_K)^\wedge.
\end{array}$$

The vertical maps are isomorphisms and the squares commute. We show that $P := \ker(i_*) \subseteq P^\perp$. Let $x, y \in P$. Then

$$i_*(x) = 0$$

so there is a $w \in TH_2(W, M_K; \mathbb{Q}[\mathbb{Z}])$ such that $\partial(w) = x$. By commutativity of the diagram above have that:

$$i^\wedge \circ \beta_{rel}(w) = \text{Bl}(\partial(w)) = \text{Bl}(x).$$

But also:

$$i^\wedge \circ \beta_{rel}(w) = \beta_{rel}(w) \circ i_*$$

which implies that:

$$\text{Bl}(x) = \beta_{rel}(w) \circ i_*,$$

so

$$\text{Bl}(x, y) = \text{Bl}(x)(y) = \beta_{rel}(w)(i_*(y)) = \beta_{rel}(w)(0) = 0$$

since also $y \in P$. Therefore $x \in P^\perp$ and $P \subseteq P^\perp$.

We now show that $P^\perp \subseteq P$. Since $P = \ker(i_*)$ we have an induced monomorphism:

$$i_*: H_1(M_K; \mathbb{Q}[\mathbb{Z}])/P \rightarrow H_1(W; \mathbb{Q}[\mathbb{Z}])$$

As in Definition 3.1, $\mathbb{Q}(\mathbb{Z})/\mathbb{Q}[\mathbb{Z}]$ is a divisible $\mathbb{Q}[\mathbb{Z}]$ -module, so it is injective since $\mathbb{Q}[\mathbb{Z}]$ is a PID (see [20] I 6.10). This means that taking duals, we have that:

$$i^\wedge: H_1(W; \mathbb{Q}[\mathbb{Z}])^\wedge \rightarrow (H_1(M_K; \mathbb{Q}[\mathbb{Z}])/P)^\wedge$$

is surjective. Let $x \in P^\perp$, so by definition $\text{Bl}(x, y) = 0$ for all $y \in P$. Therefore we can lift $\text{Bl}(x) \in H_1(M_K; \mathbb{Q}[\mathbb{Z}])^\wedge$ to an element of $(H_1(M_K; \mathbb{Q}[\mathbb{Z}])/P)^\wedge$ and therefore to an element of $H_1(W; \mathbb{Q}[\mathbb{Z}])^\wedge$ since i^\wedge is surjective. If $\text{Bl}(x) \in \text{im}(i^\wedge)$ then, since the vertical maps of our diagram above are isomorphisms we see that $x \in \text{im}(\partial)$ which means $x \in P$ by exactness of the top row, so $P^\perp \subseteq P$ as claimed. \square

Theorem 3.6 of [4] then shows that the representations $\rho: \pi_1(M_K) \rightarrow \Gamma$ with $p \in P$ extend over W .

Remark 3.3. Note that we deliberately work over the PID $\mathbb{Q}[\mathbb{Z}]$ in the above argument, and that this is vital for the deductions in several instances. There is always the problem in knot concordance that we do not know that $i_*: \pi_1(M) \rightarrow \pi_1(W)$ is surjective. For ribbon knot exteriors W , this is the case, but otherwise we cannot guarantee a surjection.

In the case that i_* were surjective for (1)-solutions W , $\pi_1(W)$ would be simply a quotient of $\pi_1(M)$, and then there would be no need to localise coefficients; we would have that:

$$P := \ker(i_*: H_1(M_K; \mathbb{Z}[\mathbb{Z}]) \rightarrow H_1(W; \mathbb{Z}[\mathbb{Z}]))$$

is a metaboliser for the Blanchfield form on $H_1(M_K; \mathbb{Z}[\mathbb{Z}])$. However, one main reason for considering coefficients in $\mathbb{Q}[\mathbb{Z}]$ is that there could conceivably be \mathbb{Z} -torsion in $H_1(W; \mathbb{Z}[\mathbb{Z}])$, in which case the best we could hope to show is that $P \subset P^\perp$. In order to get a metaboliser we need to introduce:

$$Q := \{x \in H_1(M_K; \mathbb{Z}[\mathbb{Z}]) \mid nx \in P \text{ for some } n \in \mathbb{Z}\}.$$

Then $Q = Q^\perp$ with respect to the Blanchfield form ([10] Proposition 2.7, see also the work of Letsche [15]). For the proof of Theorem 3.2 above however, this means we lose control on the *size* of P ; the zero submodule also satisfies $P \subset P^\perp$. If $i_*: \pi_1(M) \rightarrow \pi_1(W)$ is onto, then as in [10] 6.3, there is no \mathbb{Z} -torsion in $H_1(W; \mathbb{Z}[\mathbb{Z}])$. Since we only know this to be the case for ribbon knots, Cochran-Orr-Teichner localise coefficients in order to get a Principal Ideal Domain $\mathbb{Q}[\mathbb{Z}]$. Since it is intimately related to the ribbon-slice problem this problem of \mathbb{Z} -torsion is often also referred to as a ribbon-slice problem.

Now suppose that there is (1)-solution W . Then for each $p \in P = \ker(i_*)$ we have a representation

$$\tilde{\rho}: \pi_1(W) \rightarrow \Gamma = \mathbb{Z} \ltimes \frac{\mathbb{Q}(\mathbb{Z})}{\mathbb{Q}[\mathbb{Z}]}$$

which enables us to define the intersection form:

$$\lambda_1: H_2(W; \mathbb{Q}\Gamma) \times H_2(W; \mathbb{Q}\Gamma) \rightarrow \mathbb{Q}\Gamma.$$

W is a manifold with boundary, so in general this will be a singular intersection form. To define a non-singular form we localise coefficients: Cochran-Orr-Teichner use the non-commutative *Ore localisation* to formally invert all the non-zero elements in $\mathbb{Q}\Gamma$ to obtain a skew-field \mathcal{K} .

Definition 3.4. A ring A satisfies the *Ore condition*, which defines when a multiplicative subset S of a non-commutative ring without zero-divisors can be formally inverted, if, given $s \in S$ and $a \in A$, there exists $t \in S$ and $b \in A$ such that $at = sb$. Then the Ore localisation $S^{-1}A$ exists. If $S = A - \{0\}$ then $S^{-1}A$ is a skew-field which we denote by $\mathcal{K}(A)$, or sometimes just \mathcal{K} if A is understood. \square

See Chapter 2 of [20] for more details on the Ore condition. Ore localisation is flat so

$$H_2(W; \mathcal{K}) \cong \mathcal{K} \otimes_{\mathbb{Q}\Gamma} H_2(W; \mathbb{Q}\Gamma).$$

The idea is that if the homology of the boundary M_K vanishes with \mathcal{K} coefficients, as is proved in Section 2 of [4], then the intersection form on the middle homology of W becomes non-singular, and we have defined an element in the Witt group of non-singular symmetric forms over \mathcal{K} . To explain how this gives us an obstruction, which does not depend on the choice of 4-manifold, and how this obstruction lives in a group, we define L -groups and the localisation exact sequence in L -theory.

A symmetric Poincaré complex (Definition 3.5) is an algebraic version of a closed manifold. The symmetric structure is the chain level version of the Poincaré duality isomorphisms. A symmetric Poincaré pair (Definition 3.7) is an algebraic version of a manifold with boundary, such as a cobordism (Definition 3.8). The structure in a symmetric pair is the chain level version of Poincaré-Lefschetz duality. A symmetric complex which is not Poincaré is another equivalent way of considering a manifold with boundary - the boundary is represented by the failure of the complex to be Poincaré. Given any manifold M^n with a choice of orientation class $[M] \in H_n(M; \mathbb{Z})$, or $[M, \partial M] \in H_n(M, \partial M; \mathbb{Z})$ if the boundary is non-empty, there is a natural way of extracting this algebraic data.

Definition 3.5 ([18] I). Let W be the standard free $\mathbb{Z}[\mathbb{Z}_2]$ resolution of \mathbb{Z} , but without the \mathbb{Z} at the end, shown below. Geometrically it arises as the augmented chain complex of the universal cover S^∞ of the model for $K(\mathbb{Z}_2, 1)$, namely $\mathbb{R}\mathbb{P}^\infty$, constructed as a CW complex with a cell decomposition which has one cell in each dimension $0, 1, 2, \dots$

$$W : \quad \dots \rightarrow \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2]$$

Given a f.g. projective chain complex C_* over A define the symmetric Q -groups to be:

$$Q^n(C, \varepsilon) := H_n(\text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C^t \otimes_A C)) \cong H_n(\text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \text{Hom}_A(C^{-*}, C_*)))$$

since $C^t \otimes_A C \cong \text{Hom}_A(C^{-*}, C_*)$ via the *slant chain isomorphism*:

$$x \otimes y \mapsto (f \mapsto \overline{f(x)y});$$

we consider the two points of view to be interchangeable. An element φ of $Q^n(C, \varepsilon)$ can be represented by a collection of A -module homomorphisms

$$\{\varphi_s \in \text{Hom}_A(C^{n-r+s}, C_r) \mid r \in \mathbb{Z}, s \geq 0\}$$

such that:

$$d_C \varphi_s + (-1)^r \varphi_s d_C^* + (-1)^{n+s-1} (\varphi_{s-1} + (-1)^s T_\varepsilon \varphi_{s-1}) = 0: C^{n-r+s-1} \rightarrow C_r$$

where $\varphi_{-1} = 0$. The signs which appear here arise from a choice of convention on the boundary maps. If we omit ε from the notation we take $\varepsilon = 1$, so that $Q^n(C) := Q^n(C, 1)$.

A pair (C_*, φ) , with $\varphi \in Q^n(C)$, is called a n -dimensional symmetric A -module chain complex. It is called an n -dimensional symmetric Poincaré complex if the maps $\varphi_0: C^{n-r} \rightarrow$

C_r form a chain equivalence. In particular this implies that they induce isomorphisms (the cap products) on homology:

$$\varphi_0: H^{n-r}(C) \xrightarrow{\cong} H_r(C).$$

The symmetric structure is covariantly functorial with respect to chain maps. A chain map $f: C \rightarrow C'$ induces a map² $f^\%: Q^n(C) \rightarrow Q^n(C')$ given by

$$\begin{aligned} f^\%(\varphi)_s &= (f^t \otimes_A f)(\varphi_s) \in C'^t \otimes_A C'; \text{ or} \\ \varphi_s &\mapsto f\varphi_s f^*. \end{aligned}$$

A homotopy equivalence of n -dimensional symmetric complexes $f: (C, \varphi) \rightarrow (C', \varphi')$ is a chain equivalence $f: C \rightarrow C'$ such that $f^\%(\varphi) = \varphi'$. \square

Definition 3.6 ([18] I). The *algebraic mapping cone* $\mathcal{C}(g)$ of a chain map $g: C \rightarrow D$ is the chain complex given by:

$$d_{\mathcal{C}(g)} = \begin{pmatrix} d_D & (-1)^{r-1}g \\ 0 & d_C \end{pmatrix} : \mathcal{C}(g)_r = D_r \oplus C_{r-1} \rightarrow \mathcal{C}(g)_{r-1} = D_{r-1} \oplus C_{r-2}.$$

\square

Definition 3.7 ([18] I). The *relative Q -groups* of an A -module chain map $f: C \rightarrow D$ are defined to be:

$$Q^{n+1}(f) := H_{n+1}(\text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathcal{C}(f^t \otimes_A f))).$$

An element $(\delta\varphi, \varphi) \in Q^{n+1}(f)$ can be represented by a collection:

$$\{(\delta\varphi_s, \varphi_s) \in (D^t \otimes_A D)_{n+s+1} \oplus (C^t \otimes_A C)_{n+s} \mid s \geq 0\}$$

such that:

$$\begin{aligned} (d_{\otimes}(\delta\varphi_s) + (-1)^{n+s}(\delta\varphi_{s-1} + (-1)^s T_\varepsilon \delta\varphi_{s-1}) + (-1)^n f\varphi_s f^*, \\ d_{\otimes}(\varphi_s) + (-1)^{n+s-1}(\varphi_{s-1} + (-1)^s T_\varepsilon \varphi_{s-1})) = 0 \\ \in (D^t \otimes_A D)_{n+s} \oplus (C^t \otimes_A C)_{n+s-1} \end{aligned}$$

where as before $\delta\varphi_{-1} = 0 = \varphi_{-1}$. A chain map $f: C \rightarrow D$ together with an element $(\delta\varphi, \varphi) \in Q^{n+1}(f)$ is called a $(n+1)$ -dimensional *symmetric pair*. A chain map f together with an element of $Q^{n+1}(f)$ is called an $(n+1)$ -dimensional *symmetric Poincaré pair* if the relative homology class³ $(\delta\varphi_0, \varphi_0) \in H_{n+1}(f^t \otimes_A f)$ induces isomorphisms

$$H^{n+1-r}(D, C) := H^{n+1-r}(f) \xrightarrow{\cong} H_r(D) \quad (0 \leq r \leq n+1).$$

For a symmetric Poincaré pair corresponding to an $(n+1)$ -dimensional manifold with boundary, these are the isomorphisms of Poincaré-Lefschetz duality. \square

²The upper indices here *do not* indicate contravariance; they are used to distinguish from the quadratic structure, which is dual to the symmetric structure in a different way.

³The (co)homology groups of a chain map are defined to be the homology groups of the (dual of) the algebraic mapping cone.

Definition 3.8 ([18] I.3). Two n -dimensional ε -symmetric Poincaré f.g. projective A -module chain complexes (C, φ) and (C', φ') are *cobordant* if there is a $(n+1)$ -dimensional ε -symmetric Poincaré pair:

$$(f, f'): C \oplus C' \rightarrow D, (\delta\varphi, \varphi \oplus -\varphi').$$

The equivalence classes of symmetric Poincaré chain complexes under the cobordism relation form a group $L^n(A, \varepsilon)$, with

$$(C, \varphi) + (C', \varphi') = (C \oplus C', \varphi \oplus \varphi'); \quad -(C, \varphi) = (C, -\varphi).$$

As usual if we omit ε from the notation we assume that $\varepsilon = 1$. In the case $n = 0$, $L^0(A)$ coincides with the Witt group of non-singular Hermitian forms over A . \square

Note that an element of an L -group is in particular a symmetric *Poincaré* chain complex. This means that the intersection forms of our 4-manifolds W typically give elements of $L^0(\mathcal{K})$ but not of $L^0(\mathbb{Q}\Gamma)$.

Definition 3.9 ([19] Chapter 3). The *Localisation Exact Sequence in L -theory* is given, for a ring A and a multiplicative subset S which satisfies the Ore condition, as follows:

$$\rightarrow L^n(A) \rightarrow L_S^n(S^{-1}A) \rightarrow L^n(A, S) \rightarrow L^{n-1}(A) \rightarrow \dots$$

The relative L -groups $L^n(A, S)$ are defined to be the cobordism classes of $(n-1)$ -dimensional symmetric Poincaré chain complexes over A which become contractible over $S^{-1}A$, where the cobordisms are also required to be contractible over $S^{-1}A$. For $n = 1$ this is equivalent to the Witt group of $S^{-1}A/A$ valued linking forms on the torsion part of H_0 of the chain complex.

The decoration S on $L_S^n(S^{-1}A)$ refers to a restriction on the class of modules involved in the chain complex. Recall that, for a ring A , $K_0(A)$ is the Grothendieck group of isomorphism classes of finitely generated projective modules over A . A ring homomorphism $g: A \rightarrow B$ induces a morphism $g: K_0(A) \rightarrow K_0(B)$ via $[P] \mapsto [B \otimes_A P]$. The reduced K_0 -groups are given by $\tilde{K}_0(A) := K_0(A)/\text{im}(K_0(\mathbb{Z}))$. We define the subset

$$S := \text{im}(g: \tilde{K}_0(A) \rightarrow \tilde{K}_0(S^{-1}A)) \subset \tilde{K}_0(S^{-1}A).$$

We require that a chain complex $(C, \varphi) \in L_S^n(S^{-1}A)$ satisfies:

$$\sum_i (-1)^i [C_i] \in S \subset \tilde{K}_0(S^{-1}A),$$

so that element $(C, \varphi) \in L_S^n(S^{-1}A)$ is chain homotopic to $S^{-1}(D, \phi) := (S^{-1}A \otimes_A D, \text{Id} \otimes \phi)$ for a chain complex D over A : D is symmetric over A but may not be Poincaré over A , so may not lift to an element of $L^n(A)$, as we shall see below.

The first map in the localisation sequence is given by considering a chain complex over the ring A as a chain complex over $S^{-1}A$, by tensoring up using the inclusion $A \rightarrow S^{-1}A$. The effect of this is that some maps become invertible which previously were not; when

$n = 4$, which is our primary case of interest, the boundary 3-dimensional chain complex, if it has torsion homology modules, becomes contractible and the middle-dimensional intersection form of the 4-dimensional chain complex $C_*(W, M_K; \mathcal{K})$ becomes non-singular, so that we have a 4-dimensional symmetric Poincaré complex. The symmetric chain complex $C_*(W, M_K; \mathbb{Q}\Gamma)$ does not typically lie in the image of this map, since its intersection form only becomes non-singular after localisation.

The second map is the boundary construction: express all the maps of (C_*, φ) in terms of A - this requires a choice of lattice in order to clear denominators (in a non-commutative setting we require the Ore condition to hold to be sure we can clear denominators), so that we have a symmetric but typically not Poincaré complex (C_*, φ) over A , and take the mapping cone $\mathcal{C}(\varphi_0: C^{4-*} \rightarrow C_*)$. This gives a 3-dimensional symmetric Poincaré chain complex over A which becomes contractible over $S^{-1}A$ since φ_0 is a chain equivalence over $S^{-1}A$, i.e. we have an element of $L^4(A, S)$. In our case we consider $C_* = C_*(W, M_K; \mathbb{Q}\Gamma)$ and the boundary construction yields a complex which is chain equivalent to $C_*(M_K; \mathbb{Q}\Gamma)$.

On the level of Witt groups, this map sends a Hermitian S -non-singular intersection form $(L, \lambda: L \rightarrow L^*)$ to the linking form on $\text{coker } \lambda: L \rightarrow L^*$ given by:

$$(x, y) \mapsto \frac{x(z)}{s}$$

where $x, y \in L^*, z \in L, sy = \lambda(z)$ ([19] pp. 242-243).

The third map is the forgetful map on the equivalence relation; it forgets the requirement that the cobordisms be contractible over $S^{-1}A$, simply asking for algebraic cobordisms over A . \square

It turns out that the obstruction theory of Cochran-Orr-Teichner detects the class of $C_*(M_K; \mathbb{Q}\Gamma)$ in $L^4(\mathbb{Q}\Gamma, S)$, where $S := \mathbb{Q}\Gamma - \{0\}$, so we have an invariant of the 3-manifold M_K which does not depend on the choice of 4-manifold. The chain complex $C_*(M_K; \mathcal{K})$ is contractible by the results of [4] Section 2. The first question we ask, corresponding to (1)-solvability, is whether the chain complex of M_K bounds over $\mathbb{Q}\Gamma$. Supposing that it does, i.e. supposing that we have a (1)-solvable knot, we have a chain complex in:

$$\ker(L^4(\mathbb{Q}\Gamma, S) \rightarrow L^3(\mathbb{Q}\Gamma)),$$

so we can express the group detecting that there is no \mathcal{K} -contractible null-cobordism of $C_*(M_K; \mathbb{Q}\Gamma)$ as

$$L_S^4(\mathcal{K}) / \text{im}(L^4(\mathbb{Q}\Gamma));$$

as noted above the intersection information of a 4-manifold is intimately related to the linking information of its boundary 3-manifold.

The (1)-solution W defines an element of $L_S^4(\mathcal{K})$ by taking the symmetric \mathcal{K} -Poincaré chain complex

$$C_*(W, M_K; \mathcal{K}) = \mathcal{K} \otimes_{\mathbb{Q}\Gamma} C_*(W, M_K; \mathbb{Q}\Gamma).$$

The image of $L^4(\mathbb{Q}\Gamma)$ represents the change corresponding to a different choice of 4-manifold W : the obstruction defined must be independent of this choice. Since 2 is invertible in the rings \mathcal{K} and $\mathbb{Q}\Gamma$, we can do surgery below the middle dimension ([18] I 3.3 and 4.3) to see that our invariant lives in

$$\frac{L_S^0(\mathcal{K})}{\text{im}(L^0(\mathbb{Q}\Gamma))}.$$

Taking two choices of 4-manifold W, W' with boundary M_K and gluing to form:

$$V := W \cup_{M_K} -W'$$

we obtain a 4-manifold whose image in $L^4(\mathbb{Q}\Gamma) \cong L^0(\mathbb{Q}\Gamma)$ gives the difference between the Witt classes of the intersection forms of W and W' , showing that the invariant in $L_S^0(\mathcal{K})/\text{im}(L^0(\mathbb{Q}\Gamma))$ is well-defined. If this invariant vanishes then we can hope that the knot is slice or perhaps just (1.5)-solvable; more importantly if our class in $L_S^0(\mathcal{K})/\text{im}(L^0(\mathbb{Q}\Gamma))$ does not vanish then it obstructs (1.5)-solvability and therefore in particular that the knot can be slice.

The argument in [4], page 458, to show that the invariant is independent of the choice of W , uses $\mathbb{Z}\Gamma$ here instead of $\mathbb{Q}\Gamma$. In this case it is unclear that surgery below the middle dimension is possible on the chain complexes in symmetric L -theory: the argument could be mended without taking rational coefficients by taking the quadratic complex of V in $L_4(\mathbb{Z}\Gamma)$. Surgery below the middle dimension is possible in quadratic L -theory ([18] I 4.3), so that $L_4(\mathbb{Z}\Gamma) \cong L_0(\mathbb{Z}\Gamma)$ and we can then take a map $L_0(\mathbb{Z}\Gamma) \rightarrow L^0(\mathbb{Z}\Gamma)$ which forgets the quadratic structure to obtain an element of $L^0(\mathbb{Z}\Gamma)$ as claimed.

4. L^2 -SIGNATURES

There remains the not insignificant task of detecting non-zero elements in the Witt group of Hermitian forms in $L^0(\mathcal{K})$. Cochran-Orr-Teichner use an L^2 -signature (see [4] Section 5 for details) to define a homomorphism

$$\sigma^{(2)}: L^0(\mathcal{K}) \rightarrow \mathbb{R}$$

which detects the Witt class of the intersection form and therefore obstructs (1.5)-solvability and therefore sliceness. The L^2 -signature agrees with the ordinary signature on \mathbb{Q} -homology on the image of $L^0(\mathbb{Q}\Gamma)$ so that we can have a well defined obstruction, the reduced L^2 -signature:

$$\sigma^{(2)}(W) - \sigma(W),$$

where $\sigma(W) \in \mathbb{Z}$ is the ordinary signature, for a (1)-solution W . We'll attempt an outline of the beautiful theory of L^2 -signatures next.

The L^2 -signature is a way of taking a signature when the coefficients are in a group ring. We first make the inclusion:

$$\mathbb{Q}\Gamma \hookrightarrow \mathbb{C}\Gamma.$$

We then consider $\mathbb{C}\Gamma$ as a subset of $\mathcal{B}(l^2\Gamma)$, the Hilbert space of square-summable sequences indexed by the elements $g \in \Gamma$. We complete $\mathbb{C}\Gamma$ inside $\mathcal{B}(l^2\Gamma)$ using pointwise convergence and obtain the *Von Neumann Algebra* $\mathcal{N}\Gamma$. We shall later include $\mathcal{N}\Gamma$ into the space $\mathcal{U}\Gamma$ of unbounded operators affiliated to $\mathcal{N}\Gamma$, the equivalent of Ore localisation for Von Neumann algebras. For a (1)-solution W the intersection form

$$\lambda_1: H_2(W; \mathcal{K}) \times H_2(W; \mathcal{K}) \rightarrow \mathcal{K}$$

yields a Hermitian operator on the Hilbert space $(\mathcal{U}\Gamma)^m$.

We define the signature for Hermitian operators λ on $(\mathcal{N}\Gamma)^m$ which will then extend to $(\mathcal{U}\Gamma)^m$. To define a signature we need notions of the dimensions of the positive and negative eigenspaces of λ .

The *functional calculus* yields a correspondence between bounded measurable functions on the spectrum $\text{spec}(\lambda)$ of an operator λ :

$$f: \text{spec}(\lambda) \rightarrow \mathbb{C}$$

and bounded operators $f(\lambda)$ on $(\mathcal{N}\Gamma)^m$, represented as $m \times m$ matrices with elements in $\mathcal{N}\Gamma$. Choosing the characteristic functions

$$p_+, p_-: \text{spec}(\lambda) \subseteq \mathbb{R} \rightarrow \{0, 1\} \subset \mathbb{C}$$

of $(0, \infty)$ and $(-\infty, 0)$ ($\text{spec}(\lambda) \subseteq \mathbb{R}$ since λ is Hermitian) we obtain two Hermitian projection operators $p_+(\lambda), p_-(\lambda)$. The completion to the Von Neumann algebra is necessary for the functional calculus to be well defined on such Heaviside-type functions as p_+, p_- ; they are limits of polynomials so the fact that limits commute with the functional calculus correspondence in Von Neumann algebras is crucial. For example, let p_i be a sequence of polynomials such that $\lim(p_i) = p_+$. We have:

$$p_+(\lambda) = (\lim p_i)(\lambda) = \lim(p_i(\lambda)) \in M_m(\mathcal{N}\Gamma).$$

where $p_i(\lambda)$ makes immediate sense as we can evaluate polynomials on operators which live in a C^* -algebra.

We can then use the *Von Neumann Γ -trace* to define the dimension of the \pm eigenspaces of λ . The Γ -trace of an operator a is defined to be:

$$\text{tr}_\Gamma(a) := \langle e(a), e \rangle_{l^2\Gamma} \in \mathbb{C},$$

using the $l^2\Gamma$ inner product, where $e \in \Gamma \subseteq l^2\Gamma$ is the identity element. This extends to $m \times m$ matrices by taking the matrix trace, that is by summing over the Γ -traces of the diagonal entries. Recall that for projection operators on finite dimensional vector spaces, their trace is equal to the dimension of their image; the Γ -trace is a generalisation of this concept.

We can now define the L^2 -signature of a Hermitian operator λ to be:

$$\sigma^{(2)}(\lambda) := \text{tr}_\Gamma(p_+(\lambda)) - \text{tr}_\Gamma(p_-(\lambda)) \in \mathbb{R}.$$

Hermitian projection operators $a = a^2 = a^*$ have real traces since:

$$\langle ea, e \rangle_{l^2\Gamma} = \langle ea^2, e \rangle_{l^2\Gamma} = \langle ea, ea^* \rangle_{l^2\Gamma} = \langle ea, ea \rangle_{l^2\Gamma} \in \mathbb{R}.$$

Furthermore we can include $\mathcal{N}\Gamma \subset \mathcal{U}\Gamma$, where $\mathcal{U}\Gamma$ is the space of unbounded operators affiliated to $\mathcal{N}\Gamma$. See Lemma 5.6 and the preamble to it of [4] for more details. The functional calculus can be extended to unbounded operators, and it is a theorem that $\mathcal{N}\Gamma$ satisfies the Ore condition, with S as the set of all nonzero divisors, and that this Ore localisation yields $\mathcal{U}\Gamma$.

The introduction of $\mathcal{U}\Gamma$ enables the definition of the L^2 -signature to be extended from Hermitian forms over $\mathbb{Q}\Gamma^n$ to those on \mathcal{K}^n ; $H_2(W; \mathbb{Q}\Gamma)$ may not be free but $H_2(W; \mathcal{K})$ is a module over a skew-field so is a free module, so we can express

$$\lambda_1: H_2(W; \mathcal{K}) \times H_2(W; \mathcal{K}) \rightarrow \mathcal{K}$$

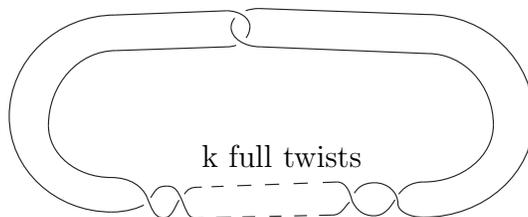
as a matrix with entries in \mathcal{K} , and use this to obtain the L^2 -signature. The reduced L^2 -signature:

$$\tilde{\sigma}^{(2)}(W) := \sigma^{(2)}(W) - \sigma(W) \in \mathbb{R}$$

gives a real number which is independent of the choice of W , so detects $L^0(\mathcal{K})/\text{im}(L^0(\mathbb{Q}\Gamma))$ and obstructs the existence of a (1.5)-solution, provided we check all the metabolisers P of the Blanchfield form and all of the representations which can arise from choices of $p \in P$. Cochran-Orr-Teichner and Cochran-Harvey-Leidy ([4] Section 6, and [5], [6], [3]) are able to use this obstruction and a satellite construction to find knots which are (n) -solvable but which are not $(n.5)$ -solvable for all $n \in \mathbb{N}$. The beauty is that the L^2 -signature of these knots can be calculated in a simple way by integrating the classical Levine-Tristram ω -signatures of the *companion or infection knot* of the satellite construction as ω varies around the circle (see [4] Lemma 5.4, [5] for more on this).

Theorem 4.1 ([5] Theorem 5.2). Suppose K is a 1.5 solvable knot whose Alexander polynomial is not 1 and which admits a Seifert surface F of genus 1. Then there is a homologically essential simple closed curve J , on F , that has self linking number zero, so corresponds to a metaboliser of the Blanchfield form, and such that the integral of the Levine-Tristram signature function of J vanishes.

As a great example, we can use this to recreate the original Casson-Gordon result that the twist knots of Figure 1 are not slice. The zero-linking curves on Seifert surfaces for the algebraically slice twist knots, which are those with $4k + 1 = n^2$ for some $n \in \mathbb{N}$, are torus knots. There exists a closed formula for the integral of the ω -signatures of the torus knots, integrating over $\omega \in S^1$, written about by several people: see [7] for an excellent exposition. The relevant L^2 -signatures of the torus knots are non-zero, proving once again that the only twist knots which are slice are for $k = 0, 2$.

FIGURE 1. The k th Twist Knot

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