Topology Vol. 29, No. 4, pp. 419–440, 1990.
Printed in Great Britain.

0040-9383 90 \$03.00+.00

ON THE HOMOTOPY THEORY OF SIMPLY CONNECTED FOUR MANIFOLDS

TIM D. COCHRAN† and NATHAN HABEGGER‡

(*Received in revised form* 14 *December* 1988)

§0. INTRODUCTION AND STATEMENT OF RESULTS

THIS PAPER is concerned with the homotopy theory of 1-connected 4-manifolds. Our principal results are explicit characterizations and computations of the groups $HE(X)$, $HE^+(X)$, $HE_{id}(X)$ and $\pi_4(X)$ (respectively: homotopy classes of self-homotopy equivalences; those which preserve orientation; those which induce the identity on homology). Along the way we correct several errors in the literature on 4-manifolds [9, 11, 15, 21].

More generally, since our techniques are homotopy-theoretic, we work in the category of Poincaré complexes, usually oriented (however, by the work of Freedman [6], there is no gain in generality.) Let X be a 1-connected, oriented 4-dimensional Poincaré complex, let $(H_2(X), \cdot)$ be its intersection form of rank r, and let

$$w_2(X) \in H^2(X;\mathbb{Z}_2) \equiv \mathrm{Hom}(H_2(X;\mathbb{Z}_2), \mathbb{Z}_2)$$

be its second Stiefel–Whitney class. It is well known that the oriented homotopy type of X is determined (in theory) by $(H_2(X), \cdot)$ and that X has the oriented homotopy type of $\bigvee^r S^2 \cup_A e^4$ for some *attaching map* $A \in \pi_3\left(\bigvee^r S^2\right)$ (e.g. [13]). Some details of the above are included as Appendix I.

Denote by $\mathrm{Aut}(H_2(X), \cdot)$ [respectively $\mathrm{Aut}(H_2(X), \pm\cdot)$] the group of automorphisms of $H_2(X)$ preserving the intersection form (respectively, up to sign). Note that any such automorphism induces an automorphism of $\ker(w_2(X))$. Of course, $\ker(w_2(X))$ is a $\mathbb{Z}_2$-vector space of rank $r-1$ or r; depending on whether $w_2(X) \neq 0$ or $w_2(X) = 0$. Let ϕ:$\mathrm{Aut}(H_2(X), \pm\cdot) \to \mathrm{Aut}[\ker(w_2(X))]$ be this representation. Our principal result is:

THEOREM 3.1. *Suppose that* $r = \mathrm{rank}(H_2(X)) > 0$; *then* $HE(X) \cong \ker(w_2(x)) \rtimes_\phi \mathrm{Aut}(H_2(X), \pm\cdot)$; *and* $HE^+ \cong \ker(w_2(X)) \rtimes_\phi \mathrm{Aut}(H_2(X), +)$.

[Note that if $r=0$, $X \simeq S^4$ and $HE(S^4) \cong \mathbb{Z}_2$]. In other words, there is a short exact sequence

$$1 \to \ker((w_2(X)) \xrightarrow{\tau \circ N} HE(X) \to \mathrm{Aut}(H_2(X), \pm\cdot) \to 1$$

which is split, with $\mathrm{Aut}(H_2(X), \pm\cdot)$ acting on $\ker(w_2)$ in the natural way. The map $HE(X) \to \mathrm{Aut}(H_2(X), \pm\cdot)$ is the natural one, with kernel $HE_{id}(X)$ $(r>0)$. The isomorphism $\ker(w_2(X)) \to HE_{id}(X)$ is derived in §2, and is the restriction of the composite

† Supported by a National Science Foundation Postdoctoral Fellowship.
‡ Supported by a National Science Foundation Postdoctoral Fellowship and a Yale Junior Faculty Fellowship.

map of sets $H_2(X;\mathbb{Z}_2)\equiv\pi_2(X)\otimes\pi_4(S^2)\xrightarrow{N}\pi_4(X)\xrightarrow{\tau}HE_{id}(X)$ where $N(x\otimes 1)=x\circ\eta^2=x\circ\eta\circ\Sigma\eta$ is *not* a homomorphism (η is the Hopf map). The homomorphism τ is the standard operation of $\pi_4(X)$ on $[X, X]$ given by "pinching off" a 4-sphere from the 4-cell of X $(X\to X\vee S^4\xrightarrow{id\vee\beta}X)$ [4]. Let us pause to review the previous work on this subject.

It has been known for some time that

$$1\to HE_{id}(X)\to HE(X)\to \mathrm{Aut}(H_2(X), \pm\cdot)\to 1$$

is exact ($r>0$) [23] [9; Thm. 2]. A splitting seems never to have been derived (in particular see p. 44 of [2; §4]). The action of $\mathrm{Aut}(H_2(X), \pm\cdot)$ on $HE_{id}(X)$ can be deduced from statements in §6 of [1].

The principal result of P. Kahn [9] is a presentation $\pi(\Sigma\vee S^2, X)\xrightarrow{\phi}\pi_4(X)\xrightarrow{\tau}HE_{id}(X)\to 1$ where ϕ is computed (see §2). The details for this seem to have been present already in [1]. However, Kahn incorrectly claims that $\pi_4(X)$ is finite [9, pp. 31, 45]. We are grateful to Bill Massey for pointing out Kahn's paper and its error (see Endnote 1 for a discussion of this). The above presentation is also implicit in J. H. Baues' announcement [2; Thm 4.1], although τ is not explicitly identified there. We extend these results.

THEOREM 2.2. *The map of sets* $H_2(X;\mathbb{Z}_2)\xrightarrow{\tau\circ N}HE(X)$ *is quadratic (see §1) and sends* $\ker(w_2(X))$ *isomorphically onto* $HE_{id}(X)$ *and sends* $(H_2(X;\mathbb{Z}_2)-\ker(w_2))$ *to the identity.*

In improving on Kahn and Baues, we benefited from the more geometric viewpoint taken by F. Quinn in [15; §2]. Recall that a homotopy equivalence, f, can be assigned a *normal invariant* $n(f)$ in $\mathbb{N}^{TOP}(X)\equiv[X, G/TOP]\xleftrightarrow{S}H^2(X;\mathbb{Z}_2)\times H^4(X;\mathbb{Z})$ (see §5). Quinn's viewpoint is that elements of $HE_{id}(X)$ can be "detected by their normal invariants." However, the arguments presented there for his crucial 2.1 are in error (see Endnote 2). We show that

THEOREM 5.1. *Suppose X is a 1-connected, closed, oriented 4-manifold. The map of sets* $H_2(X;\mathbb{Z}_2)\xrightarrow{\tau\circ N}HE_{id}(X)\xrightarrow{S\circ n}H^2(X;\mathbb{Z}_2)$ *is a quadratic map. When restricted to* $\ker(w_2(X))$, *it is an isomorphism (of groups) onto* $w_2^{\perp}=\{a\,|\,a\cup w_2=0\}$ *(by 2.2, the same is true of* $S\circ n$*). Thus, elements of* $HE_{id}(X)$ *are detected by their normal invariants.*

This proves Quinn's 2.1.

As a proof of the surjectivity of $S\circ n$, Quinn references an argument of C.T.C. Wall [19, p. 237]. There, Wall states (for X *P.L.*) that the "surgery sequence"

$$S(X)\xrightarrow{n}\mathbb{N}^{PL}(X)\xrightarrow{\theta}L_4 \tag{0.1}$$

is exact (here $S(X)$ is the PL structure set and L_4 is the Wall group). The actual argument intends to show that the composite

$$H_2(X;\mathbb{Z}_2)\xrightarrow{N}\pi_4(X)\xrightarrow{\tau}HE_{id}(X)\xrightarrow{n}\mathbb{N}^{TOP}(X)$$

is *onto* the kernel of θ (above). However, Wall seems to misapply D. Sullivan's Characteristic Variety Theorem. This same error is reproduced and elaborated in R. Mandlebaum's survey on 4-manifolds [11; p. 94]. Our proof of 5.2 contains the details to correct that argument as well; so that we have

THEOREM 5.2. *$HE_{id}(X)\xrightarrow{n}\mathbb{N}^{PL}(X)$ is an injective map whose image is the kernel of $\mathbb{N}^{PL}(X)\xrightarrow{\theta}L_4$.*

Moreover, Quinn's main Theorem 1.1 relies on the claim that M. Freedman had proved that any element of $\mathrm{Aut}(H_2(X),\cdot)$ is realized by a homeomorphism [6; Theorem 1.5 Addendum]. Therein, Freedman begins by referencing the same mistaken argument of Wall. Our 5.2 then also corrects this historical gap.

Given the aforementioned work of Kahn and Baues, it is not surprising that the proof of 2.2 (hence most of our other results) requires an *explicit* calculation of $\pi_4(X)$ and the homomorphism τ. Baues' has announced a presentation of π_4 of any 1-connected 4-complex, as the push-out of a diagram of groups [2; 2.6]. He does not present it "explicitly" or calculate it as an abelian group. We carry this out in §1 and §2. We find, for example, that $\pi_4(X)$ as an abelian group, depends only on the rank of $(H_2(X),\cdot)$, not on its signature or type (see [12]). (Is this true for the higher homotopy groups?).

THEOREM 1.1. *Let X be an oriented, 1-connected 4-dimensional Poincaré complex with $r>1$. There is a split short exact sequence*

$$0\to\pi_4(S^3)\oplus\pi_2\Big(\bigvee^r S^2\Big)\xrightarrow{\theta}\pi_4\Big(\bigvee^r S^2\Big)\xrightarrow{i_*}\pi_4(X)\to 0$$

where $\theta(a, b)=A\circ a+[A, b]$ where A is the attaching map of the top cell and $[,]$ is a Whitehead product.

COROLLARY 1.3. *If $r>0$, $\pi_4(X)\cong\mathbb{Z}_2^{(r-1)(r+2)/2}\oplus\mathbb{Z}^{r(r-2)(r+2)/3}$.*

See §1 for details.

In conclusion, all homotopy-theoretic facts in this paper have been given homotopy-theoretic proofs. As such, these results could have been proved by the homotopy theorists of the 1950s and 1960s. But it was the interplay between geometry and homotopy theory which pointed the way to the results contained herein. If one fixes a manifold structure on X then a short proof of the splitting of 3.1 is available, as explained at the end of §5.

§1. π_4 OF A SIMPLY-CONNECTED 4-DIMENSIONAL POINCARÉ COMPLEX

In this section we will give an explicit presentation for $\pi_4(X)$ and use this to compute $\pi_4(X)$ as an abelian group. We find, for example, that the latter depends only on the rank of the intersection form (not on its signature or type). J. H. Baues has announced a "presentation" of π_4 of any 1-connected 4-complex as the push-out of a diagram of groups [2, 2.6]. He does not present it "explicitly" or calculate it as an abelian group. The explicit presentation will be necessary for our study of the self-homotopy-equivalences, and *there* the type (even/odd) of the form will enter. Henceforth set $M=\bigvee_{i=1}^{r} S^2$, which will at times be identified with the 2-skeleton of X.

Throughout we shall be concerned with quadratic maps.

Definition. A function $q\colon A\to B$ between arbitrary abelian groups is *quadratic* if

(i) for each integer m and each $x\in A$, $q(mx)=m^2q(x)$
(ii) the function $A\times A\to B$ given by $q(x+y)-q(x)-q(y)$ is bilinear.

If $B=\mathbb{R}$ this is the definition of a real quadratic form [13; App. I]. For any space Y the maps $G: \pi_2(Y) \to \pi_3(Y)$ and $N: \pi_2(Y) \to \pi_4(Y)$ defined by $Y \to Y \circ \eta$ and $Y \to Y \circ \eta^2$ are quadratic because of the following facts:

(0.2) $(x+y)\circ\eta = x\circ\eta + y\circ\eta + [x, y]$
(0.3) $[y, y] = 2(y\circ\eta)$
(0.4) $[x+y]\circ\eta^2 = x\circ\eta^2 + y\circ\eta^2 + [x, y]\circ\Sigma\,\eta$
(0.5) $[x, y]\circ\Sigma\,\eta$ and $[x, y]$ are bilinear.

((0.2), (0.4), (0.5) are consequences of 5.11 and 5.16 of [22].) It is easy to see that a quadratic map $q: A \to B$ induces a quadratic map $A \otimes \mathbb{Z}_n \to B \otimes \mathbb{Z}_n$ for each n.

THEOREM 1.1. *Let X be an oriented simply-connected 4-dimensional Poincaré complex with $r = \text{rank } H_2(X) > 1$. There is a split short exact sequence*

$$0 \to \pi_4(S^3) \oplus \pi_2(M) \xrightarrow{\theta} \pi_4(M) \xrightarrow{i_*} \pi_4(X) \to 0$$

where θ is given by $\theta(a, b) = A\circ a + [A, b]$ where A is the attaching map of the top cell.

Remark 1.2. The cases $r=0$, 1 must be handled separately. If $r=0$, clearly $X \simeq S^4$ so $\pi_4(X) \cong \mathbb{Z}$. If $r=1$, then $X \simeq \pm CP(2)$, so $\pi_4(X) \cong \pi_4(\mathbb{C}P(\infty)) \cong 0$ and the above sequence becomes $\mathbb{Z}_2 \oplus \mathbb{Z} \xrightarrow{\theta} \mathbb{Z}_2$ where $\theta(1, b) = A\cdot\Sigma\,\eta$ is non-zero (this will be demonstrated after Lemma 1.5) so θ is onto. Of course θ is no longer injective.

COROLLARY 1.3. *If $r > 1$, $\pi_4(X) \cong \mathbb{Z}_2^{(r-1)(r+2)/2} \oplus \mathbb{Z}^{r(r-2)(r+2)/3}$.*

Proof of 1.3. By the Hilton–Milnor theorem [21],

$$\pi_4(M) \cong r\pi_4(S^2) \oplus \tfrac{1}{2}r(r-1)\pi_4(S^3) \oplus \tfrac{1}{3}(r^3-r)\pi_4(S^4)$$

where $r(r-1)/2$ is the cardinality of $\{(i,j)\,|\, i>j,\ 1 \leqslant i,j \leqslant r\}$ and $\frac{1}{3}(r^3-r)$ is the cardinality of $\{(i, j, k)\,|\, i>j \leqslant k,\ 1 \leqslant i, j, k \leqslant r\}$.

Now if $X \simeq M \cup_A e^4$, $x_i \in \pi_2(X)$ are the obvious generators, η is the Hopf map and $\eta^2 \equiv \eta\circ\Sigma\,\eta$, we may apply 1.1 and the Hilton–Milnor theorem to deduce

COROLLARY 1.4. *If $r>0$, $\pi_4(X)$ is generated by $\{x_i\circ\eta^2\}$, $\{[x_i, x_j]\circ\Sigma\eta\,|\, i>j\}$, $\{[[x_i, x_j], x_k]\,|\, i>j \leqslant k\}$ with relations $A\circ\Sigma\eta$ and $[A, x_i]$ $i=1, \ldots, r$.*

Let χ: $(D^4, S^3) \to (X, M)$ be the characteristic map of the 4-cell of X. One has the diagram

$$\begin{array}{ccc} \pi_4(D^4, S^3) & \xrightarrow{\cong} & \pi_3(S^3) \\ \downarrow & & \downarrow \\ \pi_4(X, M) & \longrightarrow & \pi_3(M) \end{array}$$

where $1 \in \pi_3(S^3)$ maps to $A \in \pi_3(M)$ which (if $r \geqslant 1$) is a non-zero element of a free group, hence of infinite order. The vertical isomorphism follows since $\pi_4(X, M) \cong H_4(X, M) \cong \mathbb{Z}$ by the relative Hurewicz theorem. Thus the above maps are injective and the maps i_* and f in the diagram below are surjective.

$$\begin{array}{ccccc} \pi_5(D^4, S^3) & \xrightarrow{\cong} & \pi_4(S^3) & \longrightarrow & 0 \\ \downarrow & & \downarrow A\circ & & \downarrow \\ \pi_5(X, M) & \longrightarrow & \pi_4(M) & \xrightarrow{i_*} & \pi_4(X) \\ \downarrow & & \downarrow f & & \downarrow \\ \pi_5(X, M, D^4) & \xrightarrow{\partial} & \pi_4(M, S^3) & \xrightarrow{j_*} & \pi_4(X, D^4) \end{array}$$

Here, the left-most vertical sequence is part of the homotopy exact sequence of the triad $(X; M, D^4)$ [8; p. 160]. The map j_* is surjective by the Blakers–Massey Excision Theorem [21; p. 366]. The diagram shows that i_* is surjective and that $\pi_4(S^3)$ maps into ker i_*. Hence, in order to check that θ maps onto ker i_*, we need only check that $\pi_2(M) \to \pi_4(M, S^3)$, given by $a \to [a, A]$, and ∂ have the same image.

Now a theorem of Namioka [14; p. 727] gives $\pi_5(X; M, D^4) \cong \pi_2(X, D^4) \otimes \pi_4(X, M) \cong \pi_2(X) \cong \pi_2(M)$. Moreover, tracing through the proof of Namioka's theorem one can check that it is natural with respect to maps of diagrams satisfying the hypotheses of the theorem. Given $a \in \pi_2(M)$, we can construct a map of $Y = D^4 \vee S^2 \to X$, via a on S^2 and the characteristic map on D^4. This yields the diagram:

$$\begin{array}{ccccccc} \pi_2(S^2) \cong \pi_5(Y; D^4, S^3 \vee S^2) & \longrightarrow & \pi_4(S^3 \vee S^2, S^3) & \xrightarrow{j_*} & \pi_4(Y, D^4) & \longrightarrow & 0 \\ \downarrow a \qquad\qquad \downarrow & & & & \downarrow & & \\ \pi_2(M) \cong \quad \pi_5(X; D^4, M) & \xrightarrow{\partial} & \pi_4(M, S^3) & \longrightarrow & \pi_4(X, D^4) & \longrightarrow & 0 \end{array}$$

Examining the Hilton–Milnor decomposition of $\pi_4(S^3 \vee S^2)$ as $\pi_4(S^3) \oplus \pi_4(S^2) \oplus \pi_4(S^4)$, we see that $\pi_4(S^2 \vee S^3, S^3) \cong \pi_4(S^2) \oplus \mathbb{Z}$ where the $\mathbb{Z}$ is generated by the Whitehead product of the inclusion maps $S^2 \to S^3 \vee S^2$, $S^3 \to S^3 \vee S^2$. Thus ker(j_*) is generated by the Whitehed product and hence a generator of $\pi_2(S^2)$ maps to the Whitehead product. Thus $\partial(a) = [a, A]$ (up to sign). This concludes the verification that θ maps onto ker i_*. It also proves that elements $[a, A] \in \pi_4(M)$ are trivial in $\pi_4(X)$ (see the second diagram), and hence that image $(\theta) \subset$ kernel (i_*).

To prove that θ is a split inclusion we make use of the following.

LEMMA 1.5. *A homomorphism $A \to B$ of finitely-generated abelian groups is a split inclusion provided that:*

(a) *The induced map Torsion(A) $\to$ Torsion(B) is a split inclusion, and*

(b) *for each prime p, the induced map $(A/TorA \otimes \mathbb{Z}/p\mathbb{Z}) \to (B/TorB) \otimes (\mathbb{Z}/p\mathbb{Z})$ is an inclusion.*

Applying the lemma to our map θ, note that $\mathrm{Tor}(\pi_4(S^3) \oplus \pi_2(M)) \cong \pi_4(S^3) \cong \mathbb{Z}_2$ and $\mathrm{Tor}(\pi_4(M))$ is a $\mathbb{Z}_2$-module according to the Hilton–Milnor theorem. Hence (a) would follow from the fact that $A \circ \Sigma\eta \neq 0$. To see this, note that composition induces a homomorphism $\pi_3(M) \otimes \pi_4(S^3) \to \mathrm{Tor}(\pi_4(M))$ (this uses 0.5). This is easily seen to be an isomorphism by the Hilton–Milnor decomposition. Thus $A \circ \Sigma\eta$ is non-zero precisely if $A \neq 0 \bmod 2$. But A is a primitive element of $\pi_3(M)$ (c.f. Appendix).

To check (b), consider the homomorphism $\pi_3(M) \xrightarrow{\psi} \pi_2(M) \otimes \pi_2(M)$ defined by $x \circ \eta \to x \otimes x$ and $[x, y] \to x \otimes y + y \otimes x$. (This is well-defined by 0.2.) We obtain a diagram

$$\begin{array}{ccc} \pi_3(M)\otimes\pi_2(M) & \xrightarrow{[\,,\,]} & \pi_4(M) \\ \Big\downarrow \theta\otimes 1 & & \searrow \\ & & \dfrac{\pi_4(M)}{\text{Torsion}} \\ & & \nearrow \\ \pi_2(M)\otimes\pi_2(M)\otimes\pi_2(M) & \xrightarrow{\sigma} & \pi_4(M) \end{array}$$

where $\sigma(x \otimes y \otimes z) = -[[x, z], y]$. That this is a homomorphism follows from linearity of Whitehead products [21]. Commutativity of the diagram follows from the Barcus–Barratt formula [7, Theorem 8.3]

$$[x \circ \eta, z] = [x, z] \circ \Sigma\eta - [[x, z], x] \tag{1.6}$$

and the Jacobi identity.

Now suppose $x \not\equiv 0 \bmod p$. Choose a basis $\{x_i\}$ of $\pi_2(M) \otimes \mathbb{Z}/p\mathbb{Z}$ with $x = x_1$. Then $\pi_3(M) \otimes \mathbb{Z}/p\mathbb{Z}$ has a basis $\{x_i \circ \eta \mid i = 1, \ldots, r\} \cup \{[x_i, x_k] \mid i > k\}$ in which

$$A = \sum_{i=1}^{r} \alpha_{ii} x_i \circ \eta + \sum_{i>k} \alpha_{ik}[x_i, x_k]$$

where $\alpha_{ij} = \langle x_i^* \cup x_k^*, [X] \rangle$ for $\{x_i^*\}$ the Hom-dual basis of $H^2(X; \mathbb{Z}/p\mathbb{Z})$ (cf. Appendix). Furthermore $(\pi_4(M)/\text{torsion}) \otimes \mathbb{Z}/p\mathbb{Z}$ has a basis $\{[[x_i, x_j], x_k] \mid i > j \leqslant k\}$ (cf. Appendix). Then, modulo torsion,

$$[A, x] \equiv \sigma \circ (\psi \otimes 1)(A \otimes x) \equiv -\sum_{\substack{1 \leqslant i \leqslant r \\ 1 \leqslant k \leqslant r}} \alpha_{ik}[[x_i, x_1], x_k].$$

We wish to express this in terms of the basis (above) of $(\pi_4(M)/\text{torsion})$. For $i > 1$, $[[x_i, x_1], x_k]$ is a basis element. For $i = 1$, rewrite $-\alpha_{1k}[[x_i, x_1]x_k]$ as $\alpha_{1k}[[x_1, x_k], x_1] + \alpha_{1k}[[x_k, x_1], x_1]$ or $2\alpha_{ik}[[x_k, x_1]x_1]$. Thus

$$[A, x] \equiv -\sum_{\substack{j \neq 1 \\ k \neq 1}} \alpha_{ik}[[x_i, x_1], x_k] - \sum_{i \neq 1} \alpha_{i1}[[x_i, x_1], x_1] + 2\sum_{i \neq 1} \alpha_{1k}[[x_k, x_1], x_1].$$

Finally, $[A, x] \equiv \sum_{i \neq 1} a_{ik}[[x_i, x_1], x_k]$ where $a_{ik} = -\alpha_{ik}$ if $k \neq 1$ and $a_{ik} = +\alpha_{ik}$ if $k = 1$. Thus $[A, x]$ is a sum of distinct basis elements whose coefficients (up to sign) are the elements of the matrix $\langle x_i^* \cup x_k^*, [X] \rangle$, excluding the first row. Since $r > 1$, this sum is non-empty. At least one of these coefficients must be non-zero $\bmod p$, since otherwise $\det(\alpha_{ik}) \equiv 0 \bmod p$ contradicting the unimodularity of the cup-product form. Thus $[A, x] \not\equiv 0$. □

§2. THE ISOMORPHISM $\ker(w_2(X)) \cong HE_{id}(X)$

In [9, Thm. 1] one finds an exact sequence

$$\pi(\Sigma M, X) \xrightarrow{\Phi} \pi_4(X) \xrightarrow{\tau} HE(X) \to HE(M)$$

where the homomorphism Φ is given by the formula:

$$\Phi(b) = b \circ \Sigma A + \Sigma \alpha_{ij}[b \circ \Sigma x_i, x_j], \qquad \alpha_{ij} = \langle x_i^* \cup x_j^*, [X] \rangle$$

and τ is the "twist map" as in §0. Note that this is not valid for $r=0$. Hence P. Kahn gives a presentation for $HE(X)$ which, for the most part, is contained in §6 of [1] (see also [4]). Now take b to be $z \circ p_i$ where $p: \Sigma M \to \Sigma S^2$ is the ith projection and $z \in \pi_3(X)$. This yields

$$\Phi(z \circ p_i) = \alpha_{ii}(z \circ \Sigma \eta) + \sum_j \alpha_{ij}[z, x_j].$$

The above exact sequence (with Φ as immediately above) is implicit in Baues' [2, Thm. 4.1] where he announces an exact sequence

$$H^2(X; \pi_3(X)) \xrightarrow{\Phi} \pi_4(X) \xrightarrow{g} HE_{id}(X) \to 1.$$

He does not explicitly identify g. Our Theorem 2.2 will carry this much farther, but for now we have:

THEOREM 2.1. *If $r > 0$, there is an exact sequence*

$$\pi_2(X) \otimes \pi_3(X) \xrightarrow{\Phi} \pi_4(X) \xrightarrow{\tau} HE_{id}(X) \to 1$$

where Φ is given by $a \otimes b \to [a, b] + w_2(a)(b \circ \Sigma \eta)$.

Proof of 2.1. We merely modify Kahn's sequence. Note that if $y_i = \widehat{x_i^*}$, the Poincaré dual of x_i^*, then $y_i \cdot x_j = \delta_{ij}$ and $y_i = \Sigma \alpha_{ij} x_j$ so that $y_i \cdot y_j = \alpha_{ij}$ (cf. Appendix). Our previously derived expression for Φ becomes

$$\Phi(z \circ p_i) = w_2(y_i)(z \circ \Sigma \eta) + [z, y_i].$$

Now observe that $\pi(\Sigma M, X) \equiv H^3(\Sigma M; \pi_3(X)) \cong H^2(X) \otimes \pi_3(X)$ where the identifications respect the group structures. Under this isomorphism $z \circ p_i$ clearly corresponds to $x_i^* \otimes z$. This, under $H^2(X) \otimes \pi_3(X) \cong \pi_2(X) \otimes \pi_3(X)$, corresponds to $\widehat{x_i^*} \otimes z$. Thus, under the above identifications, $\Phi(y \otimes z) = w_2(y)(z \circ \Sigma \eta) + [z, y]$ as claimed. □

The following result was announced by F. Quinn [15]. The errors in his proof are discussed on §0 and endnote 2. Recall the map of sets $\tau \circ N$: $H_2(X: \mathbb{Z}_2) \cong \pi_2(X) \otimes \pi_4(S^2) \xrightarrow{N} \pi_4(X) \xrightarrow{\tau} HE_{id}(X)$ where $N(x_i \otimes 1) = x_i \circ \eta^2$ as in §0.

THEOREM 2.2. *The quadratic mapping $H_2(X; \mathbb{Z}_2) \xrightarrow{\tau \circ N} HE_{id}(X)$ sends $\ker(w_2(X))$ isomorphically onto $HE_{id}(X)$ and sends $(H_2(X; \mathbb{Z}_2) - \ker(w_2))$ to the identity.*

Proof. If $r = \text{rank}(H_2(X)) = 0$ then both groups are trivial (because we have defined $HE_{id}(X)$ to act as the identity on $H_4(X)$ as well as on H_2). If $r = 1$, then $X \simeq \pm CP(2)$ so $\ker(w_2) \cong \pi_4(\pm CP(2)) \cong HE_{id}(X) \cong 0$ by 2.1.

Now assume $r > 1$. Combining 1.1 and 2.1 gives a diagram

$$\begin{array}{ccccc} \pi_2(M) \otimes \pi_3(M) & \xrightarrow{\tilde\Phi} & \pi_4(M) & & \\ \downarrow & & \downarrow i_* & & \\ \pi_2(X) \otimes \pi_3(X) & \xrightarrow{\Phi} & \pi_4(X) & \xrightarrow{\tau} & HE_{id}(X) \to 0 \end{array}$$

where $\tilde\Phi(a \otimes b) = w_2(a)(b \circ \Sigma \eta) + [a, b]$. This yields

$$HE_{id}(X) \cong \pi_4(\vee S^2)/(im(\tilde\Phi) + im(\theta)).$$

Claim. $im(\theta) \subset im(\tilde{\Phi})$.

Proof of claim. We must show that the relations $A \circ \Sigma\eta$ and $[y, A]$ are consequences of the relations $w_2(a)(b \circ \Sigma\eta) + [a, b]$. Since $[y, A] \equiv w_2(y)(A \circ \Sigma\eta)$ modulo the image of $\tilde{\Phi}$, it suffices to check that $A \circ \Sigma\eta$ is in $im(\tilde{\Phi})$.

Now $w_2(y)[x, y] \circ \Sigma\eta + [y, [x, y]]$ lies in $im(\tilde{\Phi})$ as does $w_2(x)(y \circ \eta^2) - [y, [x, y]] + [x, y] \circ \Sigma\eta$ since it equals $w_2(x)(y \circ \eta^2) + [x, y \circ \eta]$. Thus

$$(1 + w_2(y))([x, y] \circ \Sigma\eta) + w_2(x)(y \circ \eta^2) \in im\,\tilde{\Phi}. \tag{2.3}$$

Now if $w_2 \equiv 0$, the intersection form is even. Since A is the inverse of this even matrix, $\alpha_{ii} \equiv 0 \mod 2$ (see Appendix). Thus we have $A \circ \Sigma\eta = \sum_{i>1} \alpha_{ij}([x_i, x_j] \circ \Sigma\eta)$ (0.5) so that $A \circ \Sigma\eta$ lies in $im\,\tilde{\Phi}$ by (2.3). If $w_2 \not\equiv 0$, choose a basis $\{x_i\}$ of $H_2(X)$ so that, if $i > 1$, $x_i \in \ker(w_2)$. Then $A \circ \Sigma\eta \equiv \sum_{i>1} (\alpha_{ii} - \alpha_{i1})\, x_i \circ \eta^2$ modulo $im(\tilde{\Phi})$ by (2.3). On the other hand, Lemma 2.4 below shows that $\alpha_{ii} \equiv \alpha_{i1} \mod 2$ so that $A \circ \Sigma\eta$ lies in $im\,\tilde{\Phi}$, completing the proof of our claim.

LEMMA 2.4. *Suppose $w_2 \not\equiv 0$ and x_i is a basis for $H_2(X; \mathbb{Z}_2)$ with $w_2(x_i) = 0$ for $i > 1$. If y_i is the "dual" basis defined by $y_i \cdot x_j = \delta_{ij}$, then $w_2 \cap [X] = y_1$, that is to say, w_2 is the Poincaré dual of y_1. Thus $y_i \cdot y_i \equiv y_1 \cdot y_i$.*

Proof of 2.4. $x_i \cdot y_1 \equiv \delta_{i1} = w_2(x_i) = x_i \cdot x_i$, so w_2 is the Poincaré dual of y_1. Recalling, from the Appendix, that $\alpha_{ij} = y_i \cdot y_j$, we have $\alpha_{ii} = y_i \cdot y_i \equiv w_2(y_i) \equiv y_1 \cdot y_i = \alpha_{i1}$ modulo 2 as desired. □

Thus we have shown that $HE_{id}(X) \cong \pi_4(M)/im\tilde{\Phi}$.

Now consider the composite map

$$H_2(X; \mathbb{Z}_2) \cong \pi_2(X) \otimes \pi_4(S^2) \xrightarrow{N} \pi_4(X) \xrightarrow{\tau} HE_{id}(X).$$

From (0.4), we see that $(x + 2y) \circ \eta^2 = x \circ \eta^2 + 2(y \circ \eta^2) + [x, 2y] \circ \Sigma\eta + [y, y] \circ \Sigma\eta$ which is equal to $x \cdot \eta^2$ since $y \circ \eta^2$, $[x, y] \circ \Sigma\eta$ are order 2 and $[y, y] \circ \Sigma\eta = (2(y \circ \eta)) \circ \Sigma\eta = 2(y \circ \eta^2)$. Thus, although composition with η^2 is not bilinear, it still induces a map on the tensor product $\pi_2(X) \otimes \pi_4(S^2) \xrightarrow{N} \pi_4(X)$. The composite mapping $H_2(X; \mathbb{Z}_2) \xrightarrow{N} \pi_4(X)$ is thus "quadratic". Similarly, there is a quadratic mapping $H_2(M; \mathbb{Z}_2) \xrightarrow{N} \pi_4(M)$. We have a commutative diagram:

$$\begin{array}{ccccc} H_2(M; \mathbb{Z}_2) & \rightarrow & \pi_4(M) & \rightarrow & \pi_4(M)/im\tilde{\Phi} \\ \| & & \downarrow & & \tau\downarrow \\ H_2(X; \mathbb{Z}_2) & \rightarrow & \pi_4(X) & \xrightarrow{\tau} & HE_{id}(X) \end{array}$$

In order to prove Theorem 2.2, we must show that the composite map $H_2(X; \mathbb{Z}_2) \equiv H_2(M; \mathbb{Z}_2) \rightarrow \pi_4(M) \rightarrow \pi_4(M)/im\tilde{\Phi}$ is a bijective homomorphism onto the kernel of w_2 and takes $H_2(X; \mathbb{Z}_2) - \ker w_2$ to zero. By (2.3), $[x, y] \circ \Sigma\eta$ lies in $im\tilde{\Phi}$ if $x, y \in \ker w_2$, so for $x, y \in \ker w_2$, $(x + y) \circ \eta^2 \equiv x \circ \eta^2 + y \circ \eta^2$ modulo the image of $\tilde{\Phi}$ (0.4). This shows that the map is a homomorphism on $\ker w_2$. On the other hand, if $y \notin \ker w_2$, (2.3) yields $y \circ \eta^2 \in im\tilde{\Phi}$. It remains only to show bijectivity.

We shall now compute $HE_{id}(X)$ explicitly and see that it is *abstractly* isomorphic to $\ker w_2$ and that our map above is surjective. This will complete the proof.

The composite $\pi_2(M) \otimes \pi_3(M) \xrightarrow{\Phi} \pi_4(M) \to \pi_4(M)/\text{Torsion}$ is simply $a \otimes b \mapsto [a, b]$, which is surjective with kernel K, say. Thus $\tilde{\Phi}$ maps onto $\pi_4(M)/\text{Torsion}$. Examining the diagram below,

$$\begin{array}{ccccccc} K & \xrightarrow{\Phi} & \text{Tor}\,\pi_4(M) & \to & (\text{Tor}\,\pi_4(M))/\tilde{\Phi}(K) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \pi_2(M) \otimes \pi_3(M) & \xrightarrow{\Phi} & \pi_4(M) & \twoheadrightarrow & HE_{id}(X) & \to & 0 \\ & & \downarrow & & & & \\ & & \pi_4(M)/\text{Torsion} & & & & \end{array}$$

we see that $HE_{id}(X) \cong \pi_4/im\tilde{\Phi} \equiv \text{Tor}\,\pi_4(M)/\tilde{\Phi}(K)$ is a module over $\mathbb{Z}_2$. Now K is a free abelian group of rank $r \cdot \left(\frac{r^2+r}{2}\right) - \left(\frac{r^3-r}{3}\right) = \binom{r}{3} + r^2$ $\left(\text{where the } \frac{r^3-r}{3} \text{ term is the cardinality of } \{(i, j, k) \mid 1 \leqslant i, j, k \leqslant r, i > j \leqslant k\}\right)$ generated by elements of the form:

(1) $a \otimes [b, c] + b \otimes [c, a] + c \otimes [a, b]$ and
(2) $a \otimes [a, b] + b \otimes (a \circ \eta)$.

To see this, note that if x_i is a basis for π_2, then the $\binom{r}{3} + r^2$ elements

(1) $x_i \otimes [x_j, x_k] + x_j \otimes [x_k, x_i] + x_k \otimes [x_i, x_j]$, $i \neq j \neq k$ and
(2) $x_i \otimes [x_i, x_j] + x_j \otimes (x_i \circ \eta)$

are linearly independent and primitive, hence span K.

Thus $\tilde{\Phi}(K)$ is generated by

(1) $\alpha(a, b, c) = w_2(a)([b, c] \circ \Sigma\eta) + w_2(b)([c, a] \circ \Sigma\eta) + w_2(c)([a, b] \circ \Sigma\eta)$ and
(2) $\beta(a, b) = (1 + w_2(a))([a, b] \circ \Sigma\eta) + w_2(b)(a \circ \eta^2)$

where we have used formula (1.6). Note that $\beta(a + b, c) = \beta(a, c) + \beta(b, c) + \alpha(a, b, c)$ where we use (0.4) twice and (0.5). Since β is linear in the second variable and α is trilinear, it suffices to take a, b, c belonging to a basis for π_2.

If $w_2 \equiv 0$ then $\alpha \equiv 0$ and $\beta(a, b) = [a, b] \circ \Sigma\eta$. Hence $\tilde{\Phi}(K)$ is generated by $[x_i, x_j] \circ \Sigma\eta$ which, in $\text{Tor}\,\pi_4(\vee S^2)$, is complementary to the span of $x_i \circ \eta^2$ (from the Hilton–Milnor decomposition).

If $w_2 \not\equiv 0$, choose a basis $\{x_i\}$ with $w_2(x_i) = 0$ for $i > 1$. Then our relations are:

$$\begin{array}{ll} \alpha(x_i, x_j, x_k) = 0 & i, j, k > 1 \\ \alpha(x_1, x_j, x_k) = [x_j, x_k] \circ \Sigma\eta & j, k > 1 \\ \alpha(x_1, x_1, x_k) = 0 & k > 1 \\ \alpha(x_1, x_1, x_1) = 0 & \\ \beta(x_i, x_j) = [x_i, x_j] \circ \Sigma\eta & i, j > 1 \\ \beta(x_1, x_i) = 0 & i > 1 \\ \beta(x_i, x_1) = [x_i, x_1] \circ \Sigma\eta + (x_1 \circ \eta^2) & i > 1 \\ \beta(x_1, x_1) = x_1 \circ \eta^2 & \end{array}$$

where we have used (0.5) and the fact that $[x_i, x_i] = 2(x_i \circ \eta)$. These span a $\mathbb{Z}_2$-vector space in $\text{Tor}\,\pi_4(\vee S^2)$ complementary to the span of $x_i \circ \eta^2$, $i > 1$.

Thus, in either case, $HE_{id}(X)$ is a free $\mathbb{Z}_2$-module of rank equal to the rank of $\ker(w_2)$, and, for an appropriately chosen basis of $H_2(X)$, has $\{\tau(x_i \circ \eta^2) \mid w_2(x_i) = 0\}$ as a basis. □

§3. SPLITTING THE MAP $HE(X) \to \mathrm{Aut}(H_2(X), \pm\cdot)$: THE CASE $w_2 = 0$

In this section we identify $HE(X)$ with a group of automorphisms of a certain algebraic object, enabling us to prove our main theorem in the case $w_2(X)$ is zero.

MAIN THEOREM 3.1. *Suppose* rank $H_2(X) = r$ *is non-zero; then* $HE(X) \cong (\ker(w_2(X)) \rtimes_{\Phi} \mathrm{Aut}(H_2(X), \pm\cdot)$ *where* $Aut(H_2(X), \pm\cdot)$ *acts naturally on the kernel of* w_2 *(see* §0 *and* 2.2 *for the map* $\ker(w_2(X)) \xrightarrow{N\circ\tau} HE(X))$.

The case $w_2(X) \neq 0$ is technically more complicated and postponed to §4. Assume throughout §3 that $w_2(X)$ is zero.

Recall that stable homotopy $\sigma_*()$ yields an (Eilenberg–MacLane) unbased homology theory π_*^S by setting $\pi_*^S(X, A) \equiv \sigma_*(X^+/A^+)$ where X^+ means "adjoin a basepoint." This leads to $\tilde{\pi}_*^S \equiv \pi_*^S(X, x_0)$, *reduced stable homotopy* [21; pp. 552, 572, 579, 584]. However $\pi_*^S(X, x_0)$ is $\sigma_*(X^+/x_0^+)$ which is merely $\sigma_*(X)$ where in the latter expression we have *fixed* (arbitrarily) a basepoint of X. Thus $\tilde{\pi}_*^S$ is a reduced homology theory and may be analyzed via the Atiyah–Hirzebruch spectral sequence [21; p. 360].

PROPOSITION 3.2. *Suppose X is a* 1*-connected, oriented,* 4*-dimensional Poincaré complex with* $w_2(X) = 0$. *There is an exact sequence*:

$$0 \to H_2(X; Z_2) \xrightarrow{N} \tilde{\pi}_4^S(X) \xrightarrow{d} H_4(X) \to 0$$

where $HE(X)$ acts on this sequence restricting to the natural actions on each term. Furthermore, $N(x_i)$ *is represented by* $x_i \circ \eta^2 \in \pi_4(X)$ *and d is "degree," i.e. the degree of the map* $S^{4+k} \to \Sigma^k X$.

Remark. 3.2 fails if $w_2(X) \neq 0$ (see proof below).

Proof of 3.2. Consider the Atiyah–Hirzebruch spectral sequence $\tilde{H}_p(X; \pi_q^S) \to \tilde{\pi}_{p+q}^S(X)$ where $\pi_*^S = \pi_*^S(\mathrm{pt.}) \equiv \sigma_*(S^0)$ is "stable homotopy". We, *claim*, provided $w_2(X) = 0$, that $d^2\colon E_{4,0}^2 \to E_{2,1}^2$ and $d^2\colon E_{4,1}^2 \to E_{2,2}^2$ are zero. This will imply directly that $E_{2,2}^2 \cong E_{2,2}^\infty$, and $E_{4,0}^2 \cong E_{4,0}^\infty$, yielding the exact sequence

$$0 \to E_{2,2}^2 \xrightarrow{N} \tilde{\pi}_4^S(X) \xrightarrow{d} E_{4,0}^2 \to 0.$$

Since this spectral sequence employs a filtration by skeleta, any (cellular) map of spaces will act naturally on this sequence. Furthermore, N is the composite $H_2(M) \otimes \pi_2^S \cong E_{2,2}^\infty \equiv$ image $(\tilde{\pi}_4^S(M) \xrightarrow{i_*} \tilde{\pi}_4^S(X))$ which is clearly the claimed map $x_i \otimes 1 \to x_i \circ \eta^2$. Moreover, the map d factors through $\tilde{\pi}_4^S(X) \xrightarrow{f_*} \tilde{\pi}_4^S(S^4)$, where $f\colon X \to S^4$ has degree 1, and $\tilde{\pi}_4^S(S^4) \cong E_{4,0}^2 \cong H_4(S^4) \overset{f_*^{-1}}{\cong} H_4(X)$. Thus the map d is given by "degree" since the corresponding map for $\tilde{\pi}_4^S(S^4)$ obviously is.

Recall that if $w_2(X)=0$, then the attaching map is represented by a matrix with an even diagonal so ΣA is trivial. Thus, stably, $X \simeq M \vee S^4$ so the spectral sequence collapses entirely as claimed. In fact, the sequence of 3.2 is merely part of the long exact stable homotopy sequence for the pair (X, M). Moreover, in the cases at hand, the Atiyah–Hirzebruch spectral sequence will always reduce to the long exact sequence of the pair, since X has only two non-trivial skeleta. However, we shall give another proof of the claim, valid for *any* Y, which will be used in §4 as well.

First note that $\tilde{\pi}_*^S(Y)$ and $E^2_{p,*}$ are modules over π_*^S and that d^2 is a module map [21, p. 628]. Since $E^2_{4,1} \cong E^2_{4,0} \otimes \pi_1^S$, this shows that d^2: $E^2_{4,1} \to E^2_{2,2}$ is determined by d^2: $E^2_{4,0} \to E^2_{2,1}$. Hence it suffices to compute the latter.

Claim. For any y, d^2: $E^2_{4,0} \to E^2_{2,1}$ is given by $d^2(x) = (Sq^2)_* r_*(x)$, where $r_*: H_4(Y) \to H_4(Y; Z_2)$ is reduction modulo two, and $(Sq^2)_*: H_4(Y; Z_2) \to H_2(Y; Z_2)$ is the dual of Sq^2: $H^2(Y; Z_2) \to H^4(Y; Z_2)$. Since the latter is $a \to a \cup a \equiv a \cup w_2(Y)$, the present claim implies our earlier claim that d^2 vanishes if $w_2(X)$ is zero.

Clearly it suffices to check this for $Y = K(Z_2, 2)$, since if $\theta(x) = d^2(x) - (Sq^2)_* r_*(x)$ is non-zero, we can find f: $Y \to K(Z_2, 2)$ with $f_*(\theta(x)) \neq 0$. But, as θ is natural, $f_*(\theta(x)) = \theta(f_*(x))$ implying that θ is non-zero for $K(Z_2, 2)$.

Moreover, if f: $CP(2) \to K(Z_2, 2)$ is the non-trivial map, then f_* is an isomorphism on $H_2(\ ; Z_2)$ and an epimorphism on $H_4(\ ; Z_2)$ so it suffices to check the claim for $CP(2)$. Since $(Sq^2)_* r_*$ is the only non-trivial map $H_4(CP(2)) \to H_2(CP(2); Z_2)$, it suffices to show that d^2 is non-trivial. But since $H_4(CP(2)) \xrightarrow{d^2} H_2(CP(2); Z_2) \to \tilde{\pi}_3^S(CP(2)) \to 0$ is exact, it suffices to show $\tilde{\pi}_3^S(CP(2)) = 0$. This follows from Corollary 4.17 [21; §12] as extended by [24] where it is shown that $\tilde{\pi}_3^S(CP(2)) \cong \operatorname{cok}((Sq^2)_* r_*)$ which is trivial as noted above. □

To prove 3.1 we shall need some algebraic facts. An *automorphism* of an exact sequence (A_i, ∂_i) is a collection $\{\Psi_i\}$ of automorphisms of $\{A_i\}$ such that $\Psi_i \circ \partial_i = \partial_i \circ \Psi_i$.

LEMMA 3.4. *Suppose K is an abelian group and*

$$\begin{array}{ccc} C & \to & K \rtimes_\Phi E \\ \downarrow g & & \downarrow \\ B & \xrightarrow{f} & E \end{array}$$

is a pull-back diagram. Then $C \cong K \rtimes_{\Phi \circ f} B$ where the action of B on K is $\Phi \circ f$, where Φ is the action of E on K. Moreover the composition $K \to K \rtimes B \cong C$ is the inclusion $K \to (K \rtimes E) \times B$ followed by the natural projection to the pull-back; and the composition $C \approx K \rtimes B \to B$ is g.

LEMMA 3.5. *Let $0 \to K \xrightarrow{i} \pi \xrightarrow{d} C \to 0$ be an exact sequence of abelian groups with C infinite cyclic. Then* $\operatorname{Aut}(0 \to K \xrightarrow{i} \pi \to C \to 0) \cong K \rtimes_\Phi (\operatorname{Aut}(K) \times \operatorname{Aut}(C))$ *where* $\operatorname{Aut}(K)$ *acts on K in the natural fashion and* $\operatorname{Aut}(C)$ *acts by $\pm$ the identity. Moreover, under this identification, K is included in* $\operatorname{Aut}(0 \to K \xrightarrow{i} \pi \to C \to 0)$ *by $k \overset{\Psi}{\mapsto} \Psi_k$ where Ψ_k maps u (an element of π) to $u + ni(k)$ where $d(u) = n \cdot 1$ for some fixed generator $1 \in C$.*

Proof of 3.5. Since C is free, $\pi \cong K \times C$, so the projection mapping

$$\Phi\colon \operatorname{Aut}(0 \to K \xrightarrow{i} \pi \to C \to 0) \to \operatorname{Aut}(K) \times \operatorname{Aut}(C)$$

can be shown to split. We shall describe a map Γ: $\mathrm{Ker}(\Phi) \to K$. Choose a generator 1 of C. Given $\theta \in \mathrm{Ker}\Phi$, choose $p \in \pi$ so that $d(p) = 1$. Then $\theta(p) - p$ is $i(k)$ for a unique k. This may be checked to be independent of p and to be an isomorphism. Note that $\Gamma^{-1} = \Psi$ as given in the statement of the theorem.

It only remains to check the actions. Recall that $f \in \mathrm{Aut}(0 \to K \to \pi \to C \to 0)$ acts on $\theta \in \mathrm{Ker}\Phi$ by conjugation. Thus f acts on $k \in K$ by $k \mapsto \Gamma(f \circ \Psi(k) \circ f^{-1})$. But if $d(p) = 1$ and $p' = f^{-1}(p)$ then $\Gamma(f \circ \Psi(k) \circ f^{-1}) = i^{-1}(f \circ \Psi(k) \circ f^{-1}(p) - p) = i^{-1}(f \circ \Psi(k)(p') - f(p'))$ which is $i^{-1}f(\Psi(k)(p') - p')$. If sign $f = \pm 1$ is the action of f on C then this expression becomes $\mathrm{sign}(f) i^{-1} f(\Psi(k)(p) - p)$, since $d(p') = \mathrm{sign}(f)$, and this is $\mathrm{sign}(f) \cdot f \circ \Gamma(\Psi(k))$. Thus f acts on K by $k \mapsto j\ \mathrm{sign}(f) f(k)$ as claimed. □

PROPOSITION 3.6. *Suppose X is an oriented, 1-connected 4-dimensional Poincaré complex with $w_2(X) = 0$. Then* $\mathrm{Aut}(0 \to H_2(X;\ Z_2) \xrightarrow{N} \tilde{\pi}_4^S(X) \xrightarrow{d} H_4(X) \to 0) \cong H_2(X;\ Z_2) \rtimes_\Phi (\mathrm{Aut}H_2(X) \times \mathrm{Aut}H_4(X))$ *where* $\mathrm{Aut}H_2(X)$ *acts in the natural fashion on* $H_2(X;\ Z_2)$ *and* $\mathrm{Aut}H_4(X)$ *acts trivially. Moreover, the image of $x_i \otimes 1$ in the automorphism group is* $\Psi_i \in \mathrm{Aut}(\tilde{\pi}_4^S(X))$ *where* $\Psi_i(u) = u + d(u)[x_i \circ \eta^2]$.

Proof of 3.6. By 3.2 and 3.5, $\mathrm{Aut}(0 \to H_2(X;\ Z_2) \xrightarrow{N} \tilde{\pi}_4^S(X) \xrightarrow{d} H_4(X) \to 0) \cong H_2(X;\ Z_2) \rtimes_\Phi (\mathrm{Aut}(H_2(X;\ Z_2)) \times \mathrm{Aut}\, H_4(X))$ where $\mathrm{Aut}(H_4(X))$ acts trivially since $H_2(X;\ Z_2)$ is 2-torsion. Moreover the image of $x_i \otimes 1$ in the automorphism group is represented by $\Psi_i \in \mathrm{Aut}(\tilde{\pi}_4^S(X))$ where $\Psi_i(u)$ is $u + d(u) \cdot N(x_i \otimes 1)) = u + d(u)[x_i \circ \eta^2]$. □

THEOREM 3.7. *Suppose X is a 1-connected, oriented 4-dimensional Poincaré complex with $w_2(X) = 0$ and* $\mathrm{rank}(\mathrm{H}_2(X)) > 0$. *There is a pull-back diagram*

$$\begin{array}{ccc} HE(X) \to & \mathrm{Aut}(0 \to H_2(X;Z_2) \to \tilde{\pi}_4^S(X) \xrightarrow{d} H_4(X) \to 0) \\ \downarrow \alpha & \downarrow \Phi \\ \mathrm{Aut}(H_2(X), \pm \cdot) & \xrightarrow{\beta} \mathrm{Aut}(H_2(X)) \times \mathrm{Aut}(H_4(X)) \end{array}$$

where $\beta(u)$ is $(u, \pm 1)$ according as u acts on the intersection form.

Proof of 3.7. It is easy to see that $f \in HE(X)$ is orientation-reversing if and only if $f_* \in \mathrm{Aut}(H_2(X), -)$ (as long as $X \not\simeq S^4$). Thus $\beta \circ \alpha$ is the natural map.

Since Φ is onto by 3.6, if P denotes the pull-back we have an exact sequence $1 \to K \xrightarrow{i} P \to \mathrm{Aut}(H_2(X), \pm \cdot) \to 1$ where $K = \ker\Phi$. Since $1 \to HE_{id}(X) \to HE(X) \to \mathrm{Aut}(H_2(X), \pm \cdot) \to 1$ is exact if $r > 0$ [23], [9, Th. 2], it suffices to see that the induced map J: $HE_{id}(X) \to K$ is an isomorphism. Now 2.2 identifies $HE_{id}(X)$ with $H_2(X;\ Z_2)$ via $\tau \circ N$ and 3.6 identifies $K = \ker\Phi$ with $H_2(X;\ Z_2)$ via Ψ: $H_2(X;\ Z_2) \to \mathrm{Aut}(0 \to H_2(X; Z_2) \to \tilde{\pi}_4^S(X) \to H_4(X) \to 0)$. The induced map J: $H_2(X; Z_2) \to H_2(X; Z_2)$ is given by $J(x_i \otimes 1) = \Psi^{-1}(x_i^\#)$ where $x^\# \in \mathrm{Aut}(\tilde{\pi}_4^S(X))$ is induced by $\tau(x_i \circ \eta^2)$, which is the composite $X \to X \vee S^4 \xrightarrow{id \vee \eta^2} X \vee S^2 \xrightarrow{id \vee x_i} X$. This induces $\tilde{\pi}_4^S(X) \to \tilde{\pi}_4^S(X) \oplus \tilde{\pi}_4^S(S^4) \to \tilde{\pi}_4^S(X) \oplus \tilde{\pi}_4^S(S^2) \to \tilde{\pi}_4^S(X)$ where the first map is $b \mapsto (b, d(b))$ (d as in 3.2). Thus $x_i^\#(b) = b + d(b)[x_i \circ \eta^2]$, which is exactly Ψ_i (see 3.6), so $J(x_i \otimes 1) = \Psi^{-1}\Psi_i = x_i \otimes 1$. This shows that $HE(X)$ is a pull-back under the natural maps. □

We can now prove our Main Theorem 3.1 in the case $w_2(X) = 0$.

Proof of 3.1. Apply 3.4 to 3.7 to conclude that $HE(X) \cong H_2(X; Z_2) \rtimes_{\Phi} \mathrm{Aut}(H_2(X), \pm\cdot)$ where the projection to $\mathrm{Aut}(H_2(X), \pm\cdot)$ is the natural one, and the inclusion of $H_2(X; Z_2)$ is $\tau \circ N$. Furthermore the action of $\mathrm{Aut}(H_2(X), \pm\cdot)$ on $H_2(X; Z_2)$ factors through that of 3.6, which is the natural one. □

§4. THE CASE $w_2(X) \neq 0$

In this section we prove Theorem 3.1 in the case that $w_2(X) \neq 0$. Since 3.2 fails here, we need a different theory, one which can capture the class w_2. Note that $w_2: X \to K(\mathbb{Z}_2, 2)$ and that any homotopy equivalence preserves w_2 up to homotopy. Thus X is a "space over $K(\mathbb{Z}_2, 2)$" in the sense of A. Dold, for example [3, §2]. An object in the category of spaces over B is a pair (X, f) where X is in the category of spaces which have the homotopy type of CW-complexes, and $f: X \to B$ is continuous. Furthermore, if h_* is any *homology theory for spaces over* $K(\mathbb{Z}_2, 2)$ [3, §2], then $HE(X)$ operates on $h_*(X, w_2)$. Since X is 1-connected, for any two representatives of w_2, the corresponding groups $h_*(X, w_2)$ are canonically isomorphic [3; 2.3–2.4]. We shall now define such a homology theory.

Consider the fibration $BSO(n) \xrightarrow{\psi} K(\mathbb{Z}_2, 2)$ with fiber BSpin(n). If (X, f) is any space over $K(\mathbb{Z}_2, 2)$, define $P(X, f, n)$ or merely P_n as the total space of the pull-back of ψ via f. Let γ_n be the universal n-plane bundle over $BSO(n)$ and set $\xi_n = f^*(\gamma_n)$, the pull-back over P_n. Let $T(\xi_n)$ be its Thom space (base-point at ∞), and define $h_*(X, f) = \lim_{n \to \infty} \pi_*(T(\xi_n))$. It then follows that h_* is a homology theory for spaces over $K(\mathbb{Z}_2, 2)$ which we shall denote by $\Omega_*^{w_2}(X, f)$. For any map $Y \to X$ over $K(\mathbb{Z}_2, 2)$, $\Omega_*^{w_2}(X, Y, f)$ is defined similarly so that the sequence $\to \Omega_n^{w_2}(Y) \to \Omega_n^{w_2}(X) \to \Omega_n^{w_2}(X, Y) \to \Omega_{n-1}^{w_2}(Y) \to \ldots$ is exact. It is elementary to check that $\Omega_*^{w_2}$ satisfies Dold's "CYLINDER" axiom. Verification of his "EXCISION" axiom relies on the Thom isomorphism theorem, the relative Hurewicz theorem and excision for ordinary homology.

Remark 4.1. If f is understood, we abbreviate $\Omega_*^{w_2}(X, f)$ as $\Omega_*^{w_2}(X)$. If f is null-homotopic then $\Omega_*^{w_2}$ is isomorphic to $\Omega_*^{\mathrm{Spin}}(X)$ (canonically if X is 1-connected [3; 2.3–2.4]). In particular, $\Omega_*^{w_2}(\mathrm{pt.}) \cong \Omega_*^{\mathrm{Spin}}$. On the other hand, if f: $K(\mathbb{Z}_2, 2) \to K(\mathbb{Z}_2, 2)$ is the identity then $\Omega_*^{w_2}(K, f)$ is Ω_*^{SO}, ordinary oriented cobordism. Let $\tilde{\Omega}_*^{w_2}(X, f)$ stand for $\Omega_*^{w_2}(X, x_0, f)$ where $x_0 \in X$. Beware that $\Omega_*^{w_2}(x_0) \to \Omega_*^{w_2}(X, f)$ is not necessarily a monomorphism.

Following Dold [3; p. 394] one may set up an "Atiyah–Hirzebruch" spectral sequence converging to $\Omega_*^{w_2}(X, x_0, f)$ whose E^2 terms are $\tilde{H}_p(X; \Omega_q^{w_2}(\mathrm{pt.}))$. The local coefficient system acts trivially in our case since $\pi_1(\mathrm{Maps}(X, K(\mathbb{Z}_2, 2))$ is trivial when X is 1-connected [3; 2.4]. In this case, Whitney-sum makes $E_{p,q}^2$ (and $\Omega_*^{w_2}$) a module over the ring $\Omega_*^{w_2}(\mathrm{pt.}) \equiv \Omega_*^{\mathrm{Spin}}$ in such a way that d^2 is a module map.

We need the analogue of 3.2.

PROPOSITION 4.2. *Suppose X is a 1-connected, oriented 4-dimensional Poincaré complex. There is an exact sequence*

$$0 \to H_2(X; \mathbb{Z}_2) \xrightarrow{\tilde{N}} \tilde{\Omega}_4^{w_2}(X) \xrightarrow{d} H_4(X) \to 0$$

(where $\tilde{\Omega}_4^{w_2}(X)$ is $\tilde{\Omega}_4^{w_2}(X, w_2)$) on which $HE(X)$ acts, restricting to the natural actions on H_2 and H_4. Moreover, the map $\tilde{N}$ sends $x_i \otimes 1$ to $i_*([x_i \circ \eta^2])$ where $x_i \circ \eta^2 \in \tilde{\pi}_4^S(M) \cong \tilde{\Omega}_4^{w_2}(M, w_2 \circ i)$ where i: $M \to X$ identifies M with the 3-skeleton of X.

Proof of 4.2. Consider the spectral sequence $\tilde{H}_p(X\colon \tilde{\Omega}_q^{w_2}) \Rightarrow \Omega_{p+q}^{w_2}(X)$ discussed above. Since $\Omega_*^{w_2} \cong \Omega_*^{\text{Spin}}$, this looks like the spectral sequence for $\tilde{\Omega}_*^{\text{Spin}}(X)$, but the differentials are twisted. Note that $\Omega_*^{\text{Spin}} \cong \pi_*^S$ for $* = 0, 1, 2$. Here also Ω_*^{Spin} is a ring, $\Omega_*^{w_2}(X)$ and $E_{p,q}^2$ are modules and d^2 is a module map. Thus, as in the proof of 3.2, it suffices to compute $d^2\colon E_{4,0}^2 \to E_{2,1}^2$ and see that it is zero.

Claim. For any 1-connected space (X,f) over $K(\mathbb{Z}_2, 2)$, $d^2\colon E_{4,0}^2 \to E_{2,1}^2$ is given by the formula $d^2(x) = (Sq^2)_* r_*(x) - f^*(i) \cap r_*(x)$. Here r_* is reduction modulo two, $(Sq^2)^*$ the dual of Sq^2, and $i \in H^2(K(\mathbb{Z}_2, 2), \mathbb{Z}_2)$ the generator.

This claim will suffice to prove 4.2 since, for our Poincaré space (X, w_2), $Sq^2(\alpha) \cup \alpha$, $\alpha \in H^2(X, \mathbb{Z}_2)$, is $\langle a \cup w_2, [X] \rangle$, so $\langle \alpha, (Sq^2)_* r_*([X]) \rangle = \langle \alpha \cup w_2, r_*([X]) \rangle = \langle \alpha, w_2 \cap r_*([X]) \rangle$ implying that $(Sq)_* r_* + w_2 \cup r_*$ is trivial.

It may be easily checked that it suffices to prove the claim for $K(\mathbb{Z}_2, 2) \times K(\mathbb{Z}_2, 2)$ with f being projection onto the second factor. Furthermore, it suffices to prove the claim for $K(\mathbb{Z}_2, 2)$ and $f =$ identity and $f =$ constant map, since $H_4(K(\mathbb{Z}_2, 2) \times K(\mathbb{Z}_2, 2))$ is generated by the images of $H_4(K(\mathbb{Z}_2, 2))$ under the two inclusions and the diagonal map.

For the case $(K(\mathbb{Z}_2, 2)$, identity), consider the sequence

$$\Omega_3^{w_2}(K(\mathbb{Z}_2, 2)) \to \tilde{\Omega}_3^{w_2}(K(\mathbb{Z}_2, 2)) \to \Omega_2^{w_2} \to \Omega_2^{w_2}(K(\mathbb{Z}_2, 2)).$$

Since $\Omega_2^{SO} \cong \Omega_3^{SO} \cong 0$ and $\Omega_2^{\text{Spin}} \cong \mathbb{Z}_2$, 4.1 implies that $\tilde{\Omega}_3^{w_2}(K(\mathbb{Z}_2, 2)) \cong \mathbb{Z}_2$. Thus $E_{2,1}^2 \cong E_{2,1}^\infty \cong \mathbb{Z}_2$ and so $d^2\colon E_{4,0}^2 \to E_{2,1}^2$ is zero. On the other hand, $(Sq^2)_* r_* + i \cap r_*$ is clearly also zero (see our earlier argument and note that $H_4(K(\mathbb{Z}_2, 2))$ is cyclic).

In the case $X = K(\mathbb{Z}_2, 2)$, $f =$ constant, $\tilde{\Omega}_*^{w_2}(X) = \tilde{\Omega}_*^{\text{Spin}}(X)$. Moreover, the map $\{e\} \to \text{Spin}$ induces a 3-connected map $B(\{e\}) \to B\text{Spin}$. Hence the natural transformation $\pi_*^S \to \Omega_*^{\text{Spin}}$ is an isomorphism for $* \leqslant 2$. This implies that the maps $d^2\colon E_{4,0}^2 \to E_{2,1}^2$ (involving fiber dimension $\leqslant 2$), in the spectral sequences for $\tilde{\pi}_*^S(X)$ and $\tilde{\Omega}_*^{\text{Spin}}(X)$ are the same. By the proof of 3.2, $d^2 = (Sq^2)_* r_*$ which is our formula since $f^* = 0$. □

Now consider the sequence

$$\begin{array}{ccccccc}
\Omega_4^{w_2} & \xrightarrow{i} & \Omega_4^{w_2}(K, id) & \to & \tilde{\Omega}_4^{w_2}(K, id) & \to & \Omega_3^{w_2} \\
\downarrow \cong & & \downarrow \cong & & & & \downarrow \cong \\
\Omega_4^{\text{Spin}} & \xrightarrow{i} & \Omega_4^{SO} & & & & \Omega_3^{\text{Spin}}
\end{array}$$

According to [17], the homomorphism i has degree 16. Hence $\tilde{\Omega}_4^{w_2}(K(\mathbb{Z}_2, 2), id) \cong \mathbb{Z}_{16}$. It is well known that $H_4(K(\mathbb{Z}_2, 2))$ is cyclic of order 4 [5]. On the other hand, in the spectral sequence for $\tilde{\Omega}_4^{w_2}(K, id)$, $E_{3,1}^2 \cong \mathbb{Z}_2$, $E_{2,2}^2 \cong \mathbb{Z}_2$ and $E_{4,0}^2 \cong H_4(K)$, so each term must persist to E^∞. Thus $\mathbb{Z}_2 \cong E_{2,2}^2 \to \tilde{\Omega}_4^{w_2}(K(\mathbb{Z}_2, 2), id)$ is the inclusion $\mathbb{Z}_2 \subsetneq \mathbb{Z}_{16}$. In particular, the map $(X, w_2) \xrightarrow{w_2} (K, id)$ induces a map of spectral sequences giving the diagram below.

$$\begin{array}{ccccccccc}
0 & \to & H_2(X, \mathbb{Z}_2) & \xrightarrow{\tilde{N}} & \tilde{\Omega}_4^{w_2}(X) & \to & H_4(X) & \to & 0 \\
& & \downarrow w_2 & & \downarrow \alpha & & \downarrow \beta & & \\
0 & \to & \mathbb{Z}_2 & \to & \mathbb{Z}_{16} & \to & \mathbb{Z}_8 & \to & 0
\end{array} \tag{4.3}$$

Here $HE(X)$ acts as automorphisms of this diagram which act trivially on the bottom row.

LEMMA 4.4. *Let D be the diagram*

$$\begin{array}{ccccccccc} 0 & \to & K & \xrightarrow{i} & \pi & \xrightarrow{d} & \mathbb{Z} & \to & 0 \\ & & \downarrow w & & \downarrow \alpha & & \downarrow \beta & & \\ 0 & \to & \mathbb{Z}_2 & \to & \mathbb{Z}_{16} & \to & \mathbb{Z}_8 & \to & 0 \end{array}$$

with horizontal rows exact, K and π abelian. Let $\mathrm{Aut}(D)$ be the group of automorphisms of the diagram which fix the bottom row. Then:

$$\mathrm{Aut}(D) \cong \mathrm{Ker}(w) \rtimes_\phi \left(\mathrm{Aut}\begin{pmatrix} K \\ \downarrow w \\ \mathbb{Z} \end{pmatrix} \times \mathrm{Aut}\begin{pmatrix} \mathbb{Z} \\ \downarrow \beta \\ \mathbb{Z}_8 \end{pmatrix} \right)$$

where $\mathrm{Aut}\begin{pmatrix} K \\ \downarrow \\ \mathbb{Z}_2 \end{pmatrix}$ acts naturally on $\ker(w)$ and $\mathrm{Aut}\begin{pmatrix} \mathbb{Z} \\ \downarrow \beta \\ \mathbb{Z}_8 \end{pmatrix}$ acts by $\pm id$. Moreover, the inclusion ψ of $\ker(w)$ into $\mathrm{Aut}(D)$ is $k \mapsto \psi_k \in \mathrm{Aut}\begin{pmatrix} \pi \\ \downarrow \alpha \\ \mathbb{Z}_{16} \end{pmatrix}$ where $\psi_k(u) = u + ni(k)$ for $d(u) = n \cdot 1$ for some fixed generator $1 \in \mathbb{Z}$.

Remark. $\mathrm{Aut}(\mathbb{Z} \to \mathbb{Z}_8)$ is trivial or $\mathbb{Z}_2$, according to β.

Proof of 4.4. As in 3.5, the projection Φ from $\mathrm{Aut}(D)$ to $\mathrm{Aut}(w) \times \mathrm{Aut}(\beta)$ can be shown to be a split epimorphism (using a splitting of $K \to \pi \twoheadrightarrow \mathbb{Z}$). Once again, describe a map Γ: $\ker\Phi \to \ker(w)$ so that $\Gamma^{-1} = \psi$. The verification of actions is similar. □

Of course 4.4 may be applied immediately to diagram 4.3 to see that $\mathrm{Aut}(4.3) \cong \ker(w_2) \rtimes_\Phi (\mathrm{Aut}(w_2) \times \mathrm{Aut}(\beta))$ where $\mathrm{Aut}(w_2)$ acts naturally and $\mathrm{Aut}(\beta)$ acts trivially. Moreover, $x_i \otimes 1 \in \ker(w_2)$ includes to $\psi_{x_i} \in \mathrm{Aut}(\tilde{\Omega}_4^{w_2}(X))$ which is the map $u \mapsto u + d(u)\,[i_*(x_i \circ \eta^2)]$ (see 4.2).

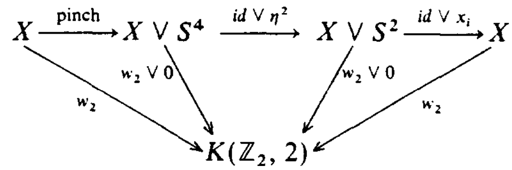
THEOREM 4.5. *Suppose X is a 1-connected, oriented 4-dimensional Poincaré complex and rank $(H_2(X)) > 0$. There is a pull-back diagram*

$$\begin{array}{ccc} HE(X) & \longrightarrow & \mathrm{Aut}\begin{pmatrix} 0 \to H_2(X;\mathbb{Z}_2) \xrightarrow{\tilde{N}} \tilde{\Omega}^{w_2}(X) \xrightarrow{d} H_4(X) \to 0 \\ \downarrow w_2 \qquad\qquad \downarrow \qquad\qquad \downarrow \beta \\ 0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_{16} \longrightarrow \mathbb{Z}_8 \longrightarrow 0 \end{pmatrix} \\ \downarrow p & & \downarrow \Phi \\ \mathrm{Aut}(H_2(X), \pm) \xrightarrow{\gamma} & & \mathrm{Aut}\begin{pmatrix} H_2(X;\mathbb{Z}_2) \\ \downarrow w_2 \\ \mathbb{Z}_2 \end{pmatrix} \times \mathrm{Aut}\begin{pmatrix} H_4(X) \\ \downarrow \beta \\ \mathbb{Z}_8 \end{pmatrix} \end{array}$$

Proof of 4.5. We first show that the natural map $HE(X) \to \mathrm{Aut}(w_2) \times \mathrm{Aut}(\beta)$ factors through p. Any element of $\mathrm{Aut}(H_2(X), \pm\cdot)$ preserves w_2 since $w_2(x) = x \cdot x$ modulo two. Now, as remarked in the proof of 3.7, $f \in HE(X)$ reverses orientation if and only if $f_* \in \mathrm{Aut}(H_2(X), -\cdot)$. Hence if $\mathrm{Aut}(H_2(X), -\cdot)$ is non-trivial, then $\mathrm{Aut}(H_4(X)) \cong \mathrm{Aut}(\beta) \cong \mathbb{Z}_2$ given by $\pm id$. Thus $\mathrm{Aut}(H_2(X), \pm\cdot) \to \mathrm{Aut}(\beta)$ defined by $f_* \to (\text{degree} f)\,(id)$ completes the desired factorization.

Now, proceeding as in the proof of 3.7, since Φ is onto by 4.4, there is an exact sequence $1 \to K \xrightarrow{i} P \to \mathrm{Aut}(H_2(X), \pm\cdot) \to 1$ where P is the pull-back and $K \cong \ker\Phi$. It suffices to show that the induced map J: $HE_{id}(K) \to K$ is an isomorphism. Recall that 2.2 identifies $HE_{id}(X)$ with $\ker(w_2(X))$ via $\tau \circ N$, and the remark following the proof of 4.4 identifies

$\ker(w_2(X))$ via ψ. The induced map J: $\ker(w_2(X))\circlearrowleft$ is $J(x_i \otimes 1) = \psi^{-1}(x_i^\#)$ where $x_i^\# \in \mathrm{Aut}(\tilde{\Omega}_4^{w_2}(X))$ is induced by $\tau(x_i \circ \eta^2) \in HE_{id}(X)$. The following diagram commutes,



up to homotopy, since $x \in \ker(w_2)$. This induces $\tilde{\Omega}_4^{w_2}(X) \to \tilde{\Omega}_4^{w_2}(X) \oplus \tilde{\Omega}_4^{\mathrm{Spin}}(S^4) \to \tilde{\Omega}_4^{w_2}(X) \oplus \tilde{\Omega}_4^{\mathrm{Spin}}(S^2) \to \tilde{\Omega}_4^{w_2}(X)$. Since $\tilde{\Omega}_4^{\mathrm{Spin}}(S^4) \cong H_4(S^4) \cong \pi_4(S^4)$ and $\tilde{\Omega}_4^{\mathrm{Spin}}(S^2) \cong H_2(S^2;\ \mathbb{Z}_2) \cong \tilde{\pi}_4^S(S^2)$, the maps above are $b \to (b, d(b)) \to (b, d(b)\eta^2) \to b + d(b)i_*\ ([x \circ \eta^2])$ where d and i_* are as in 4.2. Note that it is not necessary to know what the map d is. Hence $x_i^\#$ is exactly $\psi(x_i)$ so J is the identity map. It follows that $HE(X)$ is the desired pull-back under the natural maps. □

Proof of 3.1 *in general case.* Apply 3.4 to 4.5.

§5. NORMAL INVARIANTS AND SURGERY

We wish to provide a proof for certain claims made in [19; p. 237] and [15, Prop. 2.1]

Let X be a closed, oriented, 1-connected topological 4-manifold. Proposition 2.1 of [15] asserts that there is a bijection $HE_{id}(X) \xrightarrow{S \circ n} w_2^\perp$ $(w_2^\perp = \{a \in H^2(X;\ \mathbb{Z}_2) \mid a^2 = 0\})$ given by calculating the "normal invariants" in $\mathbb{N}(X) \equiv [X,\ G/TOP] \equiv H^2(X,\ \mathbb{Z}_2) \times H^4(X)$. To show surjectivity, Quinn references Wall's argument of [19; p. 237]. There, Wall states (for X *P.L.*) that $S(X) \xrightarrow{n} \mathbb{N}^{PL}(X) \xrightarrow{\theta} L_4$ is exact, where $S(X)$ is the *P.L.* structure set. The actual argument intends to show that the composite

$$H_2(X;\ \mathbb{Z}_2) \xrightarrow{N} \pi_4(X) \xrightarrow{\tau} HE_{id}(X) \xrightarrow{n} \mathbb{N}^{TOP}(X)$$

maps onto the kernel of θ: $\mathbb{N}^{PL}(X) \to L_4$. However, Wall mis-applies Sullivan's Characteristic Variety Theorem in asserting that it suffices to compute the normal invariants (i.e, splitting invariants) along submanifolds representing a basis for $\ker(w_2(X))$. For example, if $X \simeq \mathbb{C}P(2) \# \mathbb{C}P(2)$, x generates $\ker(w_2(X))$ and $[Y] = x$, then $HE_{id}(X) \cong \mathbb{Z}_2$, generated by $f = \tau \circ N(x)$. But the splitting invariant of f along Y is $x \cdot x = 0$ (as we shall see below), so, if it were sufficient to calculate splitting invariants along Y, the above composite would be trivial, contradicting the assertions of Wall and Quinn. Sullivan does state in [18; p. 33] that it suffices to choose a basis for the "dual of $w_2^\perp$." The *Poincaré dual* of $w_2^\perp$ is $\ker(w_2(X))$ whereas a *Hom dual* (with respect to some basis) is a quite different subgroup. The latter is the correct interpretation of "dual" for this situation.

In addition, Wall's calculation of the splitting invariants of $\tau \circ N(z)$, along a fixed surface Y, is *only* correct if $z \in \ker(w_2(X))$. For example, if $X \simeq \mathbb{C}P(2) \# \mathbb{C}P(2)$ with $[Y] = x$ as above and $[V] = z$ representing a core $\mathbb{C}P(1)$, then Wall computes the splitting invariant of $\tau \circ N(z)$ along Y to be $x \cdot z = 1$. But, by 2.2, $\tau \circ N(z)$ is homotopic to the identity (or note $\pi_4(\mathbb{C}P(2)) \cong 0$ and the homotopy to the identity in $HE(\mathbb{C}P(2))$ extends to one in $HE(\mathbb{C}P(2) \# \mathbb{C}P(2))$). We shall provide details of a proof of the surjectivity of the above map, following the lines of Wall's original argument, in the case that X is *P.L.* This completes the proof of 2.1 of [15] for *P.L.* 4-manifolds.

Recall that the set of normal maps $\mathbb{N}^{TOP}(X)$ is in bijection with $[X, G/TOP]$ which is in bijection with $H^2(X; \mathbb{Z}_2) \times H^4(X, \mathbb{Z})$ (since, by [10; pp. 328–329], the 5-skeleton of G/TOP is the same as that of $K(\mathbb{Z}_2, 2) \times K(\mathbb{Z}, 4)$). Moreover, the image of the injection $\mathbb{N}^{PL}(X) \to \mathbb{N}^{TOP}(X)$ is $\{(a, b) \mid a^2 \equiv b \bmod 2\}$. The map $\mathbb{N}^{TOP}(X) \xrightarrow{\pi_2} H^4(X) \to L_4(\{e\}) \cong \mathbb{Z}$ is given by the surgery obstruction [19; p. 237]. Thus $K \equiv \ker(\mathbb{N}^{PL}(X) \to L_4)$ is in bijection with $\{a \in H^2(X; \mathbb{Z}_2) \mid a^2 \equiv 0 \bmod 2\}$ which is $w_2^{\perp}$. Since K and $\ker(w_2(X))$ are of the same cardinality, it suffices to show that $\ker(w_2(X)) \xrightarrow{\tau \circ N} HE(X) \xrightarrow{n} K$ is surjective.

Moreover, if X is a topological 4-manifold, then let K' be the intersection (in $\mathbb{N}^{TOP}$) of $\ker(\theta)$ and $\ker(\kappa)$ where κ is the difference of the Kirby–Siebenmann invariants. Thus, to show that "homotopy equivalences are determined by their normal invariants", it again suffices to demonstrate that $\ker(w_2(X)) \to HE(X) \to K'$ is surjective *or* injective. We proceed with an analysis in the PL case and discuss the topological case afterward.

Now, given a collection $\{Y_i\}$ of embedded, oriented surfaces representing a basis of $H_2(X; \mathbb{Z}_2)$, we shall recall the map of sets $S\colon \mathbb{N}^{PL}(X) \to H^2(X, \mathbb{Z}_2)$ (due to Sullivan [18]), where $S(f)$ assigns, to each Y_i, the "splitting invariant" of $f \in \mathbb{N}^{PL}(X)$ along Y_i. For X a PL 4-manifold, we shall compute the map $S \circ n \circ \tau \circ N\colon \ker(w_2(X)) \to H^2(X; \mathbb{Z}_2)$ and find it to be onto $w_2^{\perp}$. Hence the cardinality of $n \circ \tau \circ N(\ker(w_2(X)))$ must be that of K. This will show that the above maps define bijections between $\ker(w_2(X))$, $w_2^{\perp}$ and $\ker(\mathbb{N}^{PL}(X) \to L_4)$.

Henceforth assume X is a PL manifold and let $\mathbb{N}(X) \equiv \mathbb{N}^{PL}(X)$. Recall that the map $\mathbb{N}$ is not a homomorphism since $(x + y) \circ \eta^2 = (x + y) \circ \eta \circ \Sigma\eta = (x \circ \eta + y \circ \eta + [x, y]) \circ \eta = x \circ \eta^2 + y \circ \eta^2 + [x, y] \circ \Sigma\eta$ (see 0.4). For example, in case $X \simeq \mathbb{C}P(2) \# \mathbb{C}P(2)$, and x, y are the generators $\mathbb{C}P(1)$ of each factor, both $\tau(x \circ \eta^2)$ and $\tau(y \circ \eta^2)$ are zero in $HE_{id}(X)$ whereas $\tau((x + y) \circ \eta^2)$ is non-zero (2.2). Hence it is not enough to know $S \circ n \circ \tau \circ N$: $H_2(X; \mathbb{Z}_2) \to H_2(X;\mathbb{Z}_2)$ on generators. In fact,

THEOREM 5.1. *The map* $S \circ n \circ \tau \circ N\colon H_2(X, \mathbb{Z}_2) \to H^2(X, \mathbb{Z}_2)$ *is* $b \mapsto (1 + w_2(b))\hat{b}$ *where* $\hat{b}$ *is the Poincaré dual of* b.

Thus if $w_2(b) \neq 0$, the image of b is trivial (as necessitated by 2.2), whereas if $w_2(b) = 0$, b is sent to $\hat{b}$. Combining this with 2.2 we have that

THEOREM 5.2. *Suppose X is a piecewise-linear, 1-connected, closed, oriented 4-manifold. The maps* $\ker(w_2(X)) \xrightarrow{\tau \circ N} HE_{id}(X) \xrightarrow{S \circ n} w_2^{\perp}$ *are isomorphisms of groups, so* $HE_{id}(X) \xrightarrow{n} \mathbb{N}^{PL}(X)$ *is an injective map whose image is the kernel of* $\mathbb{N}^{PL}(X) \xrightarrow{\theta} L_4$.

Proof of 5.1. Suppose $[Y] = y \in H_2(X; \mathbb{Z}_2)$. We must show that $S \circ n \circ \tau \circ N(x)(y) = (1 + w_2(x))\hat{x}(y)$. This element of $\mathbb{Z}_2$ is calculated by "restricting" $\mathbb{N}(X) \xrightarrow{r} \mathbb{N}(Y) \xrightarrow{\theta} L_2 \approx \mathbb{Z}_2$ and computing the surgery obstruction. The restriction map is given as follows. Given a normal map,

$$\begin{array}{ccc} N_M & \to & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

one obtains a normal map for $Y \to X$ by

$$\begin{array}{ccc} N_M|_{f^{-1}(Y)} \oplus N(f^{-1}(Y), M) & \to & \xi|_Y \oplus N(Y, X) \\ \downarrow & & \downarrow \\ f^{-1}(Y) & \longrightarrow & Y \end{array}$$

where we assume that f is transverse to Y (so that $N(f^{-1}(Y), M) \to N(Y, X)$ is a pull-back). The surgery obstruction is then the Arf invariant of this normal map.

To compute this in case f: $X \to X$ is the composite $\tau \circ N(x)$

$$X \to X \vee S^4 \xrightarrow{1 \vee \eta^2} X \vee S^2 \xrightarrow{1 \vee x} X$$

we may first assume that $S^2 \to X$ is an immersion representing $x \in \pi_2(X)$ which is transverse to Y in the target X. Then $f^{-1}(Y)$ consists of Y together with the pre-images (torii) in S^4 (top-cell) of $|x \cdot y|$ oriented points in S^2. The associated quadratic form, defined on $\ker(H_1(f^{-1}(Y)) \xrightarrow{f} H_1(Y))$, is the direct sum of its restrictions to $H_1(T)$ for each component torus, so that it suffices to compute the appropriate "Arf invariant" for a single T (this will be independent of T). Since T maps to a point in Y, both $\xi|_Y$ and $N(Y, X)$ pull back to trivialized bundles over T. Thus the quadratic form on $H_1(T)$ is the sum of a quadratic form associated to

$$\begin{array}{ccccc} N(T, X) & \to & N(*, S^2) & \to & N(Y, X) \\ \downarrow & & \downarrow & & \downarrow \\ T & \to & * & \to & Y \end{array}$$

and a quadratic form associated to

$$\begin{array}{ccccc} N_X|_T & \to & \xi|_* & \to & \xi|_Y \\ \downarrow & & \downarrow & & \downarrow \\ T & \to & * & \to & Y \end{array}$$

The first Arf invariant is clearly 1 since the first diagram "represents" η^2 in $\tilde{\pi}_4^S(S^2)$. The second Arf invariant will be shown to be $w_2(x)$. This will yield the desired total of $(1 + w_2(x))|x \cdot y| = (1 + w_2(x))\hat{x}(y)$, completing the proof.

We claim that, by careful choice of $S^4 \xrightarrow{g} S^2$ representing η^2, we may assume not only that $g^{-1}(*)$ is a torus T, but that there are embedded circles α, β generating $H_1(T)$ which bound embedded 2-disks in the 4-cell D^4 of X. Furthermore, for either disk, the map $(D^2, S^1) \to (S^4, S^1) \to (S^2, *)$ is of degree ± 1. To see this, consider the restriction of the bundle projection $H \to S^2$ (H a Hopf 2-disk bundle over S^2) to its boundary. This representative η: $S^3 \to S^2$ has a decomposition

$$D^2 \times S^1 \cup_\psi D^2 \times S^1 \to D^2 \cup_{S^1} D^2 \quad \text{where } \psi(x, \theta) = (x, x\theta).$$

If C denotes the equatorial circle of $S^2 \equiv D^2 \cup_{S^1} D^2$, then $\eta^{-1}(C)$ is $S^1 \times S^1$ where $(S^1 \times \text{pt.})$ and $\psi^{-1}(S^1 \times \text{pt.})$ are the desired generating circles bounding disks. Thus $(\Sigma\eta)^{-1}(C)$ is this torus. Hence it suffices to choose a representative f of η so that $f^{-1}(*) = C$ lying in the equatorial S^2; this is possible since, for our representative η described above, $\eta^{-1}(*)$ is unknotted. The claims about degree may be easily checked.

We then have the diagram

$$\begin{array}{ccccc} N_X|_{S^1} & \to & N_X|_T & \to & \xi|_* \\ \downarrow & & \downarrow & & \downarrow \\ S^1 & \xrightarrow{\alpha} & T & \to & * \end{array}$$

which induces a trivialization of $N_X|_{S^1}$. The quadratic form on α takes the value 0 or 1 according as this trivialization "agrees" with the trivialization induced by the "unique" trivialization of $N_X|_{D^2}$. But note that the composition of $(D^2, S^1) \to X$ with our homotopy equivalence is essentially the original immersion $S^2 \to X$ representing x. Then the diagram

$$\begin{array}{ccc} N_X|_{D^2} & \to & \xi|_{S^2} \\ \uparrow & & \uparrow \\ N_X|_{S^1} & \to & \xi|_* \end{array}$$

shows that the value of the quadratic form on α (and β) is precisely $w_2(\xi|_{S^2})$. Here we may assume without loss of generality that ξ reduces to the stable normal bundle of X (since X is a manifold), so the desired value is $x \cdot x \equiv w_2(x) \bmod 2$. Thus the Arf invariant on T is $w_2(x)$. □

This concludes the piecewise linear case. The details of the arguments for the topological case seem not to have been published. However, using Quinn's proof of the Annulus Conjecture, one knows that $X - \{pt\}$ has a smooth structure. Using this it seems that a topological normal bordism to the identity gives a smooth normal bordism of proper self-homotopy equivalences $X - \{pt\} \to X - \{pt\}$, from which one shows that the splitting invariants are obstructions. This would show that $\ker(w_2(X)) \to K'$ is injective.

If X is a manifold, then a splitting ϕ of 3.1 may be easily derived as follows. Given $A \in \mathrm{Aut}(H_2(X), \pm \cdot)$, choose $g \in HE(X)$ so that $g_* = A$. By 5.2, $-(S \circ n(g))$ is in the image of some $\alpha \in \pi_4(X)$. Let $\phi(A)$ be the result of acting on g by α. It is easily verified that ϕ is a homomorphism.

Endnote 1

In [9, p. 31, lines 4–5], Kahn claims that $\pi_4\left(\bigvee^r S^2\right)$ and hence $\pi_4(X)$ are finite. But the Hilton–Milnor theorem gives the rank of the former as $(r^3 - r)/3$, and we compute the rank of the latter to be $(r^3 - 4r)/3$, $r > 0$. Hence his Lemma 2, page 30, is incorrect as stated (image ψ has elements of infinite order). Theorem 1 remains correct (since in Lemma 2, "image Σb^* consists entirely of elements of order 2" is correct since $\pi_4\left(\bigvee^r S^3\right)$ is 2-torsion). Corollary 1, the corollary to Theorem 1 and the corollary to Theorem 3 are correct as stated, but are falsely proven for $n = 2$. The proofs given were based on 4.4 where the Hilton–Milnor decomposition was used incorrectly for $n = 2$. However, a correct proof, based on our 2.2, can be substituted.

Endnote 2

This concerns errors in the proof of 2.1 of [15]. The argument for injectivity is invalidated because the *Claim* on page 348 is incorrect. The major difficulty seems to be in the next-to-last line of the proof. If the claim were true, then $\tau([[x_i, x_j], x_k])$ and

$\tau([x_i, x_j]\cdot\Sigma\eta)$ would always be trivial. But for $S^2 \tilde{\times} S^2$ (the boundary of the non-trivial 3-disk bundle over S^2) with x, y representing the base and fiber, the relation $A\cdot\Sigma\eta$ in π_4 gives that $x\cdot\eta^2 = [x, y]\cdot\Sigma\eta$. The Claim would thus imply that the image of $\ker(w_2) \xrightarrow{\tau\cdot N} HE_{id}$ were trivial (since $x\cdot x = 0$). This would be inconsistent with Quinn's thesis (and our work). Additionally, page 348, line 22, is missing a minus sign in the formula (see our 1.6). Also page 348, line -14, says that $w_2(x_i) \equiv \alpha_{ii} \bmod 2$ where $\alpha_{ii} = \langle x_i^* \cup x_i^*, [X]\rangle$ for x_i^* the Hom-duals of the basis $\{x_i\}$ for $H_2(x)$. This is false for $S^2 \tilde{\times} S^2$ in the basis given above.

In general, the argument seems to assume that certain maps are homomorphisms, when in fact they are not.

Finally, as discussed in §0 and §5, the referenced argument of Wall [19, p. 237] is incorrect.

APPENDIX I

We organize some homotopy theory related to the attaching map $A\in\pi_3\left(\bigvee^r S^2\right)$ of a 1-connected 4-manifold $X \simeq \vee S^2 \cup_A e^4$. We make no claims to originality, although we cannot find a calculation of A in the literature.

First recall the Hilton–Milnor analysis of $\pi_n\left(\bigvee_{i=1}^r S^2\right)$ [21]. Suppose $\{x_i\}$ is the obvious basis of π_2, and suppose that $w(x_1, \ldots, x_r)$ is an (iterated) commutator of weight k in the symbols $\{x_i\}$, so that w represents an iterated Whitehead product in $\pi_{k+1}\left(\bigvee^r S^2\right)$. Note that for any $\alpha\in\pi_n(S^{k+1})$, $w\circ\alpha\in\pi_n\left(\bigvee^r S^2\right)$. The Hilton–Milnor theorem says that the latter group has a basis of such composites where α varies over a basis of $\pi_n(S^{k+1})$, w varies over all *basic* commutators of weight k, and $1\leqslant k\leqslant n-1$. Thus $\pi_3\left(\bigvee^r S^2\right) \cong r(\pi_3(S^2)) \oplus r(r-1)/2\ (\pi_3(S^3))$ generated by $\{x_i\circ\eta\}$ and $\{[x_i, x_j],\ i>j\}$. Similarly $\pi_4\left(\bigvee^r S^2\right) \cong r\pi_4(S^2)\oplus(r(r-1)/2)\ \pi_4(S^3)\oplus(r^3-r)\ \pi_4(S^4)$ generated by $\{x_i\circ\eta^2\}$, $\{[x_i, x_j]\circ\Sigma\eta,\ i>j\}$ and $\{[[x_i, x_j], X_k],\ i>j\leqslant k\}$ where η is the Hopf map and $\eta^2 \simeq \eta\circ\Sigma\eta$.

In [13] it is shown that the oriented homotopy type of X is that of $\bigvee^r S^2 \cup_A e^4$ for a unique A, that A determines this type, and that A is determined by the intersection form $(H_2(X), \cdot)$. (This refines the theorem of J. H. C. Whitehead [24].) The map A is *not* computed. Fixing the obvious basis $\{x_i\}$ of $\pi_2\left(\bigvee^r S^2\right)$ induces the aforementioned basis of $\pi_3\left(\bigvee^r S^2\right)$. To describe A in this basis it suffices to give a symmetric integral matrix $\alpha = \{\alpha_{ij} \mid 1\leqslant i\leqslant r, 1\leqslant j\leqslant r\}$ of coefficients, i.e. $A = \sum_{i=1}^r \alpha_{ii}(x_i\circ\eta) + \sum_{i>j}\alpha_{ij}[x_i, x_j]$. Since the intersection form $\{x_i\cdot x_j\}$ is such a matrix, it is tempting to conclude that $\alpha_{ij} = x_i\cdot x_j$. However, α is actually the *inverse* of this intersection matrix! In fact, $\alpha_{ij} = \langle x_i^*\cup x_j^*, [X]\rangle$ where x_i^* is the *Hom-dual* basis of $\{x_i\}\in H_2(X)$. Of course, $x_i\cdot x_j = \langle \hat{x}_i\cup\hat{x}_j, [X]\rangle$ where $\hat{x}_i$ is the *Poincaré* dual of x_i. This form for A is stated without proof in [20; p. 182]. It may be deduced by carefully tracing through Milnor's (unexplicit) correspondence. Perhaps one reason that people sometimes confuse the intersection matrix with the matrix α is that for many of the simplest and best known examples of 4-manifolds, it *is* possible to choose a basis for $H_2(X)$ so that the two matrices *are* equal!

We offer a simple geometric argument (for handlebodies) below. But first, we make some further remarks about α. Suppose $\{y_j\}$ is a basis of $H_2(X)$ so that $y_j \cdot x_i = \delta_{ji}$. Then $\hat{y}_j(x_i) = y_j \cdot x_i = \delta_{ij}$ so $\hat{y}_j = x_j^*$. Hence $\alpha_{ij} = \langle x_i^* \cup x_j^*, [X] \rangle = \langle \hat{y}_i \cup \hat{y}_j, [X] \rangle = y_i \cdot y_j$ so that α *is* the intersection matrix of $H_2(X)$ with respect to this "dual" basis. Furthermore, it follows that $y_j = \sum_i \alpha_{ij} x_i$, so $x_j = \sum_i \beta_{ij} y_i$, where (β_{ij}) is the inverse matrix to α. Therefore $x_i \cdot x_j = \left(\sum_k \beta_{ki} y_k \right) \cdot \left(\sum_l \beta_{lj} y_l \right) = \sum_{k,l} \beta_{ki} \beta_{lj} \alpha_{kl} = \sum_k \beta_{ki} \delta_{kj} = \beta_{ji} = \beta_{ij}$. Hence α is the inverse of the intersection matrix in the basis $\{x_i\}$ as claimed.

Now assume that X is a handlebody consisting of one 0-handle, one 4-handle, and r 2-handles. Consider the obvious wedge of 2-spheres which is a retract of X-(4-handle)^0. Then A is simply the restriction of this retraction to the boundary of the 4-handle. Suppose x_i represents the ith 2-sphere. It is fairly well known that, if $f: S^3 \to \bigvee^r S^2$ is transverse to the north poles n_i, then the coordinates of $[f]$ with respect to the Hilton–Milnor basis (above) are given by the linking numbers in S^3, $\alpha_{ij} = lk(L_i, L_j)$ and self-linking numbers $\alpha_{ii} = lk(L_i, L_i^+)$ where L is the framed link corresponding to $f^{-1}(n_i)$. In our case, L_i is the boundary of the co-core of the ith 2-handle. Let y_j be a homology class of the union of the jth co-core and the cone (into the 4-handle) of its boundary. Then $y_j \cdot x_i = \delta_{ij}$ and $y_i \cdot y_j$ is well known to be $lk(L_i, L_j)$ for $i > j$ and $lk(L_i, L_i^+)$ if $i = j$ [16]. Thus $\alpha_{ij} = y_i \cdot y_j$ as desired. Note that we have suppressed orientation considerations (signs) throughout.

The referee points out that it would be more to the point to make use of the "functional cup products" as explained in §3 of Chapter XI of [21]. On the one hand, the functional cup product suffices to give an effective homotopy classification of maps $f: S^3 \to S^2 \vee S^2 \ldots \vee S^2$; on the other hand, the functional cup products are intimately related to cup products in the mapping core for f. The reader might also peruse §5 of the same chapter to read about Hopf's derivation of the connection to linking numbers. □

Acknowledgements—The authors wish to acknowledge conversations and correspondence with Mike Freedman, Matthias Kreck, Bill Massey, Ed Miller, Frank Quinn, Larry Taylor and Pierre Vogel.

REFERENCES

1. W. D. Barcus and M. G. Barratt: On the homotopy classification of the extensions of a fixed map, *Trans. Am. math. Soc.* **88** (1958), 57–74.
2. J. Baues: On homotopy classification problems of J. H. C. Whitehead, in *Algebraic Topology: Göttingen 1984*, Lecture Notes in Math. #1172, Springer, Berlin, (1985).
3. A. Dold: Chern classes in general cohomology, Symposia Mathematica, Vol. 5, Academic Press, (1971).
4. B. Eckmann and P. Hilton: Operators and co-operators in homotopy theory, *Math. Annln* **141** (1960), 1–21.
5. S. Eilenberg and S. MacLane: On the groups $H(\pi, n)$, II, *Ann. Math.* **60** (1954), 49–138.
6. M. H. Freedman: The topology of 4-dimensional manifolds, *J. Diff. Geom.* **17** (1982), 357–453.
7. A. Haefliger: Enlacements de spheres en codimensionl superieure à deux, *Commentarii Math. Helv.*
8. S. Hu: *Homotopy Theory*, Academic Press, (1959).
9. P. Kahn: Self-equivalences of $(n-1)$-connected $2n$-manifolds, *Maths Annln* 180 (1969), 26–47.
10. R. Kirby and L. Siebenmann: *Foundational Essays on Topological Manifolds, Smoothings, and Triangulations*, Annals of Math. Studies #88 (Princeton University Press, Princeton, 1977).
11. R. Mandlebaum: Four-dimensional topology: an introduction, *Bull. Am. Math. Soc.* **2** (1) (1980), 1–160.
12. J. Milnor: On simply-connected 4-manifolds, *Symposia International Topologia Alg.* (1958).
13. J. Milnor and D. Husemoller: *Symmetric Bilinear Forms* (Springer, (1973)).
14. I. Namioka: Maps of pairs in homotopy theory, *Proc. Lond. math. Soc.* **12** (1962), 725–738.
15. F. Quinn: Isotopy of 4-manifolds, *J. Diff. Geom.* **25**(4) (1986).
16. D. Rolfsen: *Knots and Links*, Publish or Perish, Berkeley (1976).

17. V. A. ROCHLIN: New results in the theory of 4-dimensional manifolds, *Doklady Akad. Nauk SSSR* (N.S.) **84** (1952), 221–224.
18. D. SULLIVAN: *Geometric Topology*, part I, mimeographed lecture notes, Massachusetts Inst. of Technology (1970).
19. C. T. C. WALL: *Surgery on Compact Manifolds*, Academic Press, (1971).
20. C. T. C. WALL: Classifications of $(n-1)$-connected $2n$-manifolds, *Ann. Math.* (1962), 163–189.
21. G. W. WHITEHEAD: *Homotopy Theory*, Springer-Verlag Graduate Texts in Math. **61**, springer, (1978).
22. G. W. WHITEHEAD: A generalization of the Hopf invariant, *Ann. Math.* **51** (1) (1950), 192–237.
23. J. H. C. WHITEHEAD: On simply-connected 4-dimensional polyhedra, *Comment math. Helv.* **22** (1949), 48–92.
24. J. H. C. WHITEHEAD: A certain exact sequence, *Ann. Math.* **52** (1950), 51–110.

Department of Mathematics
Rice University, Houston
Texas
U.S.A.

Department of Mathematics
University of Georgia
Athens, Georgia 30602
U.S.A.