COUNTING TOPOLOGICAL MANIFOLDS

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Introduction

The goal of this text is to prove a theorem by Cheeger and Kister stating that there are countably many compact topological manifolds. We will first look at precise classification theorems of compact manifolds in low dimensions, then at special cases of the statement for smooth and high-dimensional manifolds, and finally show the proof given by Cheeger and Kister in [CK70]. At the end, we will present an application to topological Morse theory discussed in that same paper.

1 Classifications of compact manifolds

It is sufficient to classify all compact connected manifolds of a given dimension to classify the compact manifolds. As a compact manifold can only have finitely many components, countability of the set of compact connected manifolds then automatically implies countability of the set of all compact manifolds, as those are exactly the finite disjoint unions of the connected compact manifolds.

- **0-manifolds.** There is only one compact connected 0-manifold, namely the point, and a 0-manifold cannot have nonempty boundary.
  So the set of all compact 0-manifolds up to homeomorphism is countable.
- **1-manifolds.** There is, up to homeomorphism, exactly one compact connected 1-manifold without boundary, namely the circle S^1, and one compact connected 1-manifold with nonempty boundary, namely the closed unit interval $I = [0, 1]$. A proof can be found in [FR84].
  Thus the set of all compact 1-manifolds up to homeomorphism is countable.
- **2-manifolds.** Orientable connected compact 2-manifolds with empty boundary can be obtained as finite connected sums of 2-tori, the number of tori being referred to as the *genus* of the manifold. The empty connected sum is defined as the sphere S^2.
  Non-orientable connected compact 2-manifolds with empty boundary are finite connected sums of $\mathbb{R}P^2$ and the number of $\mathbb{R}P^2$s in the connected sum is called the genus.
  As any compact 1-manifold without boundary is homeomorphic to a finite union of circles, compact 2-manifolds with nonempty boundary are obtained from compact 2-manifolds with empty boundary by removing finitely many disjoint disks. A detailed proof can be found in [Moi77, Chapter 22].
  So, up to homeomorphism, compact connected 2-manifolds can be classified by orientability, genus and number of boundary components, the last two being integers, and thus the set of all compact 2-manifolds up to homeomorphism is countable.

For higher dimension, classification results are far more difficult and not as complete. There has been a lot of progress in the classification of compact 3-manifolds, especially around the Thurston geometrization conjecture first stated in 1982 in [Thu82]. The conjecture was proved in 2003 using Ricci flow by Perelman in [Per03] (a brief discussion of these developments can be found in [Mil03]).

2 Smooth case

Theorem 2.1. *There are only countably many compact smooth manifolds, up to homeomorphism.*

Proof. As all compact smooth manifolds have a finite triangulation (as shown in [Cai61]), there can only be countably many compact smooth manifolds up to homeomorphism: every compact smooth manifold M can be constructed inductively in k steps by gluing a simplex to the existing structure in each step, where k is the number of simplices in the finite triangulation of M. In each step, there are only finitely many possible edges of simplices the new simplex can be glued to, so there are only countably many possibilities to construct a compact smooth manifold. □

We know that topological manifolds that admit smooth structures can have multiple smooth structures that are not diffeomorphic to each other, one well-known example being Milnor's exotic sphere (see [Mil56]). Thus, this proof can only classify compact smooth manifolds up to homeomorphism, as the simplices can have multiple non-diffeomorphic smooth structures, and the gluing maps need not preserve the smooth structure. So the question can be extended to ask whether there are countably many diffeomorphism types of smooth manifolds.

In dimensions up to three, topological and smooth manifolds coincide, i.e. every topological 3-manifold has exactly one smooth structure (up to diffeomorphism). A proof of this for dimension one can be found in the appendix of Milnor's book on differentiable topology ([Mil97, Appendix]), a proof for dimension two is given by Radó in [Rad25] and a proof for dimension three can be found in [Moi52].

Compact manifolds of dimension strictly greater than four only admit finitely many pairwise non-diffeomorphic structures on the same manifold. This follows from the facts that PL-able compact topological manifolds of dimension $n \geq 5$ only admit finitely many PL structures (this is a result by Kirby and Siebenmann that can be found in the lecture notes as Remark 17.12), and compact PL manifolds only admit a finite number of smoothings (see chapter 18 in the lecture notes).

This leaves only dimension four to be considered, which as usual behaves a bit differently than other dimensions. There are non-compact 4-manifolds that admit uncountably many pairwise non-diffeomorphic smooth structures (the most famous example being $\mathbb{R}^4$, as shown in [DF92]). However, compact 4-manifolds thankfully only admit countably infinitely many pairwise non-diffeomorphic smooth structures because the PL and DIFF categories coincide for dimension four, i.e. every PL 4-manifold has exactly one smooth structure up to diffeomorphism (see Theorem 18.3 in the lecture notes). By a combinatorial argument, there can only be countably many PL structures on a compact manifold. We know concrete examples of 4-manifolds that admit countably infinitely many non-isomorphic smooth structures, for example the $K3$-surface that was discussed in a previous talk. The proof that there are infinitely many exotic smooth structures on the $K3$-surface is due to work by Donaldson, Gompf and Mrowka in [Don90] and [GM93].

Still, as there are only countably many compact smooth 4-manifolds up to homeomorphism, and each of these manifolds can only have countably many pairwise non-diffeomorphic smooth structures, there are also only countably many compact smooth 4-manifolds up to diffeomorphism.

3 High-dimensional case

Theorem 3.1. *There are only countably many closed topological manifolds of dimension $n \geq 6$ up to homeomorphism.*

We will only give an idea for a proof of this theorem. Using the handlebody theory for topological manifolds developed by Kirby and Siebenmann discussed in the lecture notes (Theorem 21.4), we know that closed topological manifolds of dimension $n \geq 6$ have a handle decomposition. This handle decomposition must be finite, as the manifold is compact: since the handles are compact, we can choose a covering of M by covering every handle with finitely many sets, and choose this covering in such a way that every open set in the covering covers at most one handle. As the manifold is compact, this covering has a finite subcovering, so there can only be finitely many handles. Similarly to Theorem 2.1 we can then construct every closed topological manifold of dimension $n \geq 6$ inductively in finitely many steps by gluing handles together. Up to homeomorphism, there are only countably many options of gluing a handle on, so there are only countably many possibilities to construct a closed manifold of dimension $n \geq 6$.

This proof can be extended to 5-manifolds using the work of Quinn, who showed in [Qui82] that 5-manifolds have topological handle decompositions.

4 Cheeger-Kister's proof

The main result of this part is the following theorem proved by Cheeger and Kister in [CK70].

Theorem 4.6. *Up to homeomorphism, there are only countably many compact topological manifolds (with possibly nonempty boundary).*

Although the statement of the theorem can be proven for many special cases (low dimensions, high dimensions, PL-able topological manifolds) as we discussed above, the statement of Theorem 4.6 is still an improvement, as we have seen that non-PL-able manifolds exist. A big advantage of the proof we will present is also that it works the same way for all dimensions.

To prepare for the proof, we state and prove the following useful facts about separable spaces.

Definition 4.1. A space X is called *separable* if there is a countable subset S of X that is dense in X.

Lemma 4.2. *Subspaces of separable metric spaces are separable.*

Proof. Let X be a separable metric space with a countable dense subset S and $Y \subseteq X$, and let d be the metric on X. We consider the distance $d(s, Y) := \inf\{d(s, y) \mid y \in Y\}$ for all $s \in S$. For every $s \in S$, find a sequence of points a_n^s in Y with $d(s, a_n^s) < d(s, Y) + \frac{1}{n}$.

We define the set $A := \{a_n^s \in Y \mid s \in S, n \in \mathbb{N}\}$. This set is a countable subset of Y because S is countable. It is also dense in Y because for any $y \in Y$ and $\varepsilon > 0$, we can find some $s \in S$ with $d(s, y) < \frac{\varepsilon}{3}$, because S is dense in X, and this implies $d(s, Y) \leq \frac{\varepsilon}{3}$. So there is some $a_n^s \in A$ with $d(s, a_n^s) < d(s, Y) + \frac{1}{n} < d(s, Y) + \frac{\varepsilon}{3} \leq \frac{2\varepsilon}{3}$, by choosing n big enough so that $\frac{1}{n} < \frac{\varepsilon}{3}$, and thus for this a_n^s we get $d(y, a_n^s) \leq d(s, y) + d(s, a_n^s) < \varepsilon$, so A is dense in Y and so Y is separable. □

Lemma 4.3. *If X is an uncountable separable metric space, there exists some $x \in X$ that is the limit point of a sequence $x_1, x_2, \dots \in X$ with $x_i \neq x$ for all $i \in \mathbb{N}$.*

Proof. Let X be an uncountable separable metric space. Assume there is no such point in X. Then we can find, for every $x \in X$, an open ball $B_{\varepsilon_x}(x) \subseteq X$ that contains only x. As X is uncountable, we can choose some $\varepsilon > 0$ such that there are uncountably many $x \in X$ with $\varepsilon_x > \varepsilon$ by observing the sets $X_n := \{x \in X \mid \varepsilon_x > \frac{1}{n}\}$. As X is uncountable and $\mathbb{N}$ is countable, there must be some $n \in \mathbb{N}$ with X_n uncountable, and we set $\varepsilon = \frac{1}{n}$.

We set $X' := X_n = \{x \in X \mid \varepsilon_x > \varepsilon\}$ and consider the smaller open balls $B_\varepsilon(x)$ for $x \in X'$. These are disjoint, as $B_\varepsilon(x)$ contains only x and there are uncountably many of them since X' is uncountable. But this is a contradiction to the separability of X, as all dense subsets of X must contain at least one element in each $B_\varepsilon(x)$. □

For the proof of Theorem 4.6 we first restrict to the case of manifolds with empty boundary.

Theorem 4.4. *Up to homeomorphism, there are only countably many compact topological manifolds with empty boundary.*

Proof. The proof of the theorem will be by contradiction, so we first assume that there are uncountably many compact manifolds with empty boundary that are pairwise non-homeomorphic. Then, there must be some dimension n such that there are uncountably many n-manifolds (as there are only countably many options for n). We choose such an n and fix it for the rest of the proof. Throughout the rest of the text, $B_r(x)$ will denote the closed ball of radius r centered at x in $\mathbb{R}^n$. For this n, we denote the collection of all homeomorphism types of compact n-manifolds by $\mathcal{M} = \{M_\alpha\}_{\alpha \in \mathcal{A}}$, where $\mathcal{A}$ is uncountable.

For each manifold M_α in $\mathcal{M}$, choose finitely many embeddings of $B_2(0)$ into M such that M_α is covered by the images of the restrictions to $B_1(0)$, i.e. we choose embeddings $h_{\alpha_j}\colon B_2(0) \to M_\alpha$ so that $\{h_{\alpha_j}|_{B_1(0)}\}_{j=1}^{k_\alpha}$ covers M_α. Such a covering can be constructed by covering M_α with open balls $h_{\alpha_x}(\mathring{B}_2(0))$ around each point x in M_α, as the manifold is locally euclidean, and restricting to the images of $\mathring{B}_1(0)$. Since M is a compact manifold, there exists a finite subcovering of this collection that still covers M_α, and the embeddings of this subcovering, extended to the closed balls of radius 2, meet the condition.

We then choose a k such that there are uncountably many n-manifolds in $\mathcal{M}$ with $k_\alpha = k$. This choice is possible by the same argument as for the choice of n. By an abuse of notation, we also denote this new uncountable collection of manifolds with $k_\alpha = k$ by $\mathcal{M}$. We then modify the maps h_{α_j} by fixing them on $B_1(0)$ and reparametrizing such that $h_{\alpha_j}|_{B_1(0)}$ is extended to an embedding of $B_{k+1}(0)$ with the same image as h_{α_j}, and continue referring to this modified embedding as h_{α_j}, i.e. we now have embeddings $h_{\alpha_j}\colon B_{k+1}(0) \to M_\alpha$ for $1 \le j \le k$ and $M_\alpha \in \mathcal{M}$.

Every n-manifold M_α can be embedded in $\mathbb{R}^{2n+1}$, as we have seen in a previous talk (the result is due to Hanner [Han51]). We set $l \coloneqq 2n+1$ and fix an embedding of M_α into $\mathbb{R}^l$ for all $\alpha \in \mathcal{A}$. We assume henceforth that $M_\alpha \subseteq \mathbb{R}^l$ for all $M_\alpha \in \mathcal{M}$.

Let d be the standard metric in $\mathbb{R}^l$ and define

$$\varepsilon_{\alpha,j,m} \coloneqq d(h_{\alpha_j}(B_m(0)), M_\alpha \setminus h_{\alpha_j}(\mathring{B}_{m+1}(0)))\ \forall \alpha \in \mathcal{A}, j, m \in \{1, \dots, k\}.$$

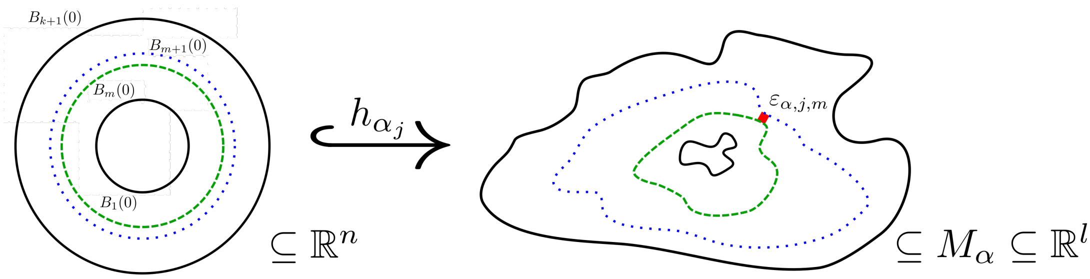


FIGURE 1. This figure shows the intuition behind the definition of $\varepsilon_{\alpha,j,m}$.

We then define $\varepsilon_\alpha \coloneqq \min_{j,m}\{\varepsilon_{\alpha,j,m}\}$. This minimum is well-defined because $j \in \{1, \dots, k\}$ and $m \le k$.

We then choose an uncountable subcollection of $\mathcal{M}$ so that there exists some $\varepsilon > 0$ such that $\varepsilon_\alpha > \varepsilon$ for all manifolds in this subcollection. This new subcollection, which we will continue denoting by $\mathcal{M}$, can be chosen by defining $\mathcal{M}_n \coloneqq \{M \in \mathcal{M} \mid \varepsilon_\alpha > \frac{1}{n}\}$ and noting that if $\mathcal{M}_n$ were countable for every $n \in \mathbb{N}$, then $\mathcal{M}$ would be countable. As it is not, we can find an n so that the collection of manifolds $\mathcal{M}_n$ is uncountable and set $\varepsilon \coloneqq \frac{1}{n}$ and $\mathcal{M}_n$ to be our new $\mathcal{M}$. This ε will be used later in the proof.

It will be useful to think about j as counting the embedding and m as describing the size of the ball, and to remember that M is already covered with $j = k$ and $m = 1$.

Each manifold M_α determines an embedding $g_\alpha\colon B_{k+1}(0) \to \mathbb{R}^{kl}$ by

$$g_\alpha(x) = (h_{\alpha_1}(x), \dots, h_{\alpha_k}(x)).$$

We set $\mathcal{G} \coloneqq \{g_\alpha \mid \alpha \in \mathcal{A}\}$ to be the uncountable set of all such embeddings.

Claim 1. *The set $\mathcal{G}$ is separable and metrizable.*

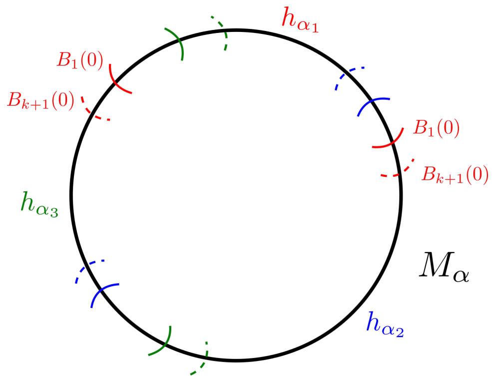


FIGURE 2. This is an example of a covering of a 1-manifold $M_\alpha \cong S^1$ with $k = 3$.

Proof. We define the uniform metric on $\mathcal{G}$ by

$$d(g_\alpha, g_\beta) := \max_{x \in B_{k+1}(0)} d(g_\alpha(x), g_\beta(x)),$$

which is well-defined since $B_{k+1}(0)$ is compact, and equip $\mathcal{G}$ with the induced topology.

As $\mathcal{G} \subseteq \mathcal{C}(B_{k+1}(0), \mathbb{R}^{kl})$ and we previously showed that subsets of separable metric spaces are separable, we now only need to prove that $\mathcal{C}(B_{k+1}(0), \mathbb{R}^{kl})$ with the uniform metric is separable, as $\mathcal{G}$ then has the subspace topology and is thus also separable. This is a consequence of the theorem of Stone-Weierstrass (a general version and proof can be found in [Rud91, Chapter 5]), which states that every real-valued function from a compact Hausdorff space can be approximated by polynomials. The polynomials can then be approximated by polynomials with rational coefficients, of which there are countably many. To obtain the statement about functions into $\mathbb{R}^{kl}$ instead of real-valued functions, we apply Stone-Weierstrass in each variable separately.

Thus, the set of functions from a compact subset of $\mathbb{R}^n$ into $\mathbb{R}^{kl}$ is separable, and the polynomials with rational coefficients form a countable dense subset. So $\mathcal{G}$ is separable. □

Claim 2. *There exists some $g_{\alpha_0} \in \mathcal{G}$ that is the limit point of a sequence of embeddings $g_{\alpha_1}, g_{\alpha_2}, \ldots$ in $\mathcal{G}$ with $g_{\alpha_i} \neq g_{\alpha_0}$ for all $i \in \mathbb{N} \setminus \{0\}$.*

Proof. We know that $\mathcal{G}$ is an uncountable separable metric space and can thus apply Lemma 4.3. □

We will now produce a contradiction by constructing a homeomorphism from M_{α_0} to M_{α_i} for i sufficiently large. This homeomorphism will be arbitrarily close to the identity as measured by the metric d. Thus, as we had assumed that all elements in $\mathcal{M}$ are pairwise non-homeomorphic and that $g_{\alpha_i} \neq g_{\alpha_0}$ for all $i \in \mathbb{N}$, we will obtain a contradiction.

To simplify the notation, we will denote M_{α_i} for some fixed but arbitrarily large i by M'. Then define the sets

$$V_j(m) := h_{\alpha_0,j}(B_m(0)) \subseteq M \text{ and } V'_j(m) := h_{\alpha_i,j}(B_m(0)) \subseteq M'$$

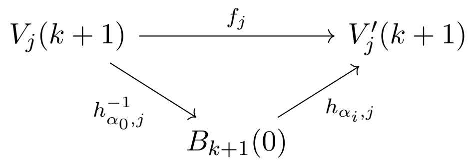
with $j = 1, \ldots, k$ and $m = 1, \ldots, k+1$, and let

$$U_j(m) := \cup_{p=1}^{j} V_p(m) \subseteq M \text{ and } U'_j(m) := \cup_{p=1}^{j} V'_p(m) \subseteq M'.$$

We observe a few properties of these sets:

- $U_k(1) = M$ and $U'_k(1) = M'$ hold.
- Also, $U_j(m) \subseteq U_{j+1}(m)$ and $U_j(m) \subseteq U_j(m+1)$ hold, and so do the analogous statements for $U'_j(m)$.

We define the map $f_j := h_{\alpha_i,j} \circ h^{-1}_{\alpha_0,j} : V_j(k+1) \to V'_j(k+1)$.



This map can be arbitrarily close to the identity as measured by the metric d, because the embeddings $h_{\alpha_{i,j}}$ and $h_{\alpha_{0,j}}$ can be arbitrarily close by choosing i big enough, and is a homeomorphism.

To construct the homeomorphism from M to M', we will proceed inductively. The induction starts with the embedding $g_1 = f_1|_{V_1(k)} \colon U_1(k) = V_1(k) \hookrightarrow V_1'(k+1) \subseteq M'$. We already know that this embedding can be arbitrarily close to the identity. Given an embedding $g_j \colon U_j(m) \hookrightarrow M'$ that is close to the identity, we will use theorem [EK71, Theorem 5.1] to construct an embedding $g_{j+1} \colon U_{j+1}(m-1) \hookrightarrow M'$ that is also close to the identity. Thus, by setting $k = m$ in the first step, we obtain an embedding $g_k \colon M = U_k(1) \hookrightarrow M'$ in $k-1$ steps, which we will then show is surjective.

Claim 3. *If g_j is close to the identity relative to the ε defined above and i is big enough,*

$$g_j(U_j(m) \cap V_{j+1}(m)) \subseteq V'_{j+1}(m+1)$$

holds.

Proof. Let the embedding $g_j \colon U_j(m) \hookrightarrow M'$ be close to the identity relative to ϵ, which was chosen so that

$$\varepsilon < \varepsilon_\alpha \coloneqq \min_{j,m}\{d(h_{\alpha_j}(B_m(0)), M_\alpha \setminus h_{\alpha_j}(\mathring{B}_{m+1}(0)))\}$$

for all $\alpha \in \mathcal{A}$. So

$$g_j(U_j(m) \cap V_{j+1}(m)) = g_j((\cup_{p=1}^{j} V_p(m)) \cap h_{\alpha_0,j}(B_m(0))) \subseteq h_{\alpha_i,j+1}(B_{m+1}(0)) = V'_{j+1}(m+1)$$

must hold, because g_j was close enough to the identity relative to ε, which measured the distance of $h_{\alpha_j}(B_m(0)$ and $M_\alpha \setminus h_{\alpha_j}(\mathring{B}_{m+1}(0))$ over all $j \in \{1, \ldots, k\}$. □

The composition

$$F = f_{j+1}^{-1} \circ g_j \colon U_j(m) \cap V_{j+1}(m) \to V_{j+1}(m)$$

is then well-defined and close to the identity, as both f_{j+1} and g_j were close to the identity.

We apply the following theorem, [EK71, Theorem 5.1], a proof of which is given in the lecture notes, Theorem 14.8, to extend F to $V_{j+1}(m)$ while it stays fixed on an open set $N \subseteq M$ with

$$U_j(m-1) \cap V_{j+1}(m-1) \subseteq N \subseteq U_j(m) \cap V_{j+1}(m).$$

This is similar to the application of the theorem in the proof of the isotopy extension theorem ([EK71, Corollary 1.2, Corollary 1.4], Theorem 14.9 in the lecture notes).

Theorem 4.5. *Let M be a manifold and $C \subseteq U \subseteq M$ where U is an open neighbourhood of the compact set C. Then there exists a neighbourhood P of the inclusion $\eta \colon U \to M$ and a deformation*

$$\phi \colon P \times [0,1] \to \mathrm{Emb}(U, M)$$

into $\mathrm{Emb}_C(U, M)$ modulo the complement of a compact neighbourhood of C in U, and fixing η.

The manifold M can have nonempty boundary, the proof in this case is similar to the case with empty boundary sketched in the lecture notes, but uses a boundary collar. Details can be found in [EK71].

We want to obtain a homeomorphism $\widetilde{F} \colon V_{j+1}(m) \to V_{j+1}(m)$ that is equal to F on N. So, for the application of the theorem, we set $M = V_{j+1}(m)$, $U = (U_j(m) \mathring{\cap} V_{j+1}(m))$ and C some compact set with

$$U_j(m-1) \cap V_{j+1}(m-1) \subseteq N \coloneqq \mathring{C} \subseteq C \subseteq U \subseteq U_j(m) \cap V_{j+1}(m).$$

This C exists because we can find open disjoint neighbourhoods of $U_j(m-1) \cap V_{j+1}(m-1)$ and of $M \setminus U_j(m) \cap V_{j+1}(m)$ and choose C as the closure of the open neighbourhood of $U_j(m-1) \cap V_{j+1}(m-1)$.

The theorem provides a neighbourhood P of the inclusion and a continuous deformation

$$\phi\colon P \times [0,1] \to \mathrm{Emb}(U, M)$$

into $\mathrm{Emb}_C(U, M)$ modulo the complement of some compact neighbourhood $C \subseteq W \subseteq U$, i.e. $\phi(P \times \{1\}) \subseteq \mathrm{Emb}_C(U, M)$ and $\phi(h, t)|_{U\setminus W} = h|_{U\setminus W}$ for all $h \in P$ and $t \in [0, 1]$.

As $F\colon U \to V_{j+1}(m)$ can be obtained to be as close to the inclusion as wished, we can set i to be large enough so that $F \in P$. Applied to F, the theorem gives an isotopy from $\phi(F, 0) = F$ to $G := \phi(F, 1) \in \mathrm{Emb}_C(U, M)$ with $\phi(F, t)|_{U\setminus W} = F|_{U\setminus W}$ for all $t \in [0, 1]$.

We define a map $\widetilde{F}\colon V_{j+1}(m) \to V_{j+1}(m)$ by

$$\widetilde{F} = \begin{cases} FG^{-1}(x) & x \in G(U) \\ x & x \in M \setminus G(W). \end{cases}$$

Claim 4. *The map $\widetilde{F}$ is well-defined, continuous, a homeomorphism onto $V_{j+1}(m)$ and coincides with F on C, and thus extends F as we wished.*

Proof.

- $\widetilde{F}$ is well-defined:
  We know that $G(U) \cap (M \setminus G(W)) = G(U \setminus W)$. For $x \in G(U \setminus W)$, choose $z \in G^{-1}(x)$. As the deformation was modulo $U \setminus W$, the relation $x = G(z) = \phi(F, 1)(z) = F(z)$ holds. We then have
  $$FG^{-1}(x) = \phi(F, 0)\phi(F, 1)^{-1}(x) = x.$$
- $\widetilde{F}$ is continuous:
  The map is continuous as the maps F and G^{-1} are continuous.
- $\widetilde{F}$ is a homeomorphism:
  Both F and G are embeddings, so FG^{-1} maps $G(U)$ homeomorphically onto $F(U)$. On $M \setminus G(U)$ we have $\widetilde{F}(x) = x$, so $\widetilde{F}$ is a homeomorphism.
- $\widetilde{F}(x) = F(x)$ on C
  On the set C, the maps F and $\widetilde{F}$ coincide because $C \subseteq G(U)$ as G can be made close enough to the identity and U is a neighbourhood of C, and thus
  $$\widetilde{F}(x) = FG^{-1}(x) = F(x)$$
  because $G \in \mathrm{Emb}_C(U, M)$.

□

So we have extended F to $\widetilde{F}$.

We now define the map $g_{j+1}\colon U_{j+1}(m-1) \to M'$ as

$$g_{j+1}(x) = \begin{cases} g_j(x) & x \in U_j(m-1) \\ f_{j+1}\widetilde{F}(x) & x \in V_{j+1}(m-1). \end{cases}$$

Claim 5. *The map g_{j+1} is well-defined, continuous and an embedding, more precisely a homeomorphism onto $U'_{j+1}(m-j)$.*

Proof.

- g_{j+1} is well-defined:
  On $U_j(m-1) \cap V_{j+1}(m-1)$, the relation
  $$f_{j+1}\widetilde{F}(x) = f_{j+1}FG^{-1}(x) = f_{j+1}F(x) = f_{j+1}f_{j+1}^{-1}g_j(x) = g_j(x)$$
  holds, because $U_j(m-1) \cap V_{j+1}(m-1) \subseteq C$, so g_{j+1} is well-defined.

- g_{j+1} is continuous:
  The map is continuous as the maps $\widetilde{F}$, f_{j+1} and g_j are continuous.
- g_{j+1} is an embedding and a homeomorphism onto $U'_{j+1}(m+1-j)$:
  All the maps F, f_{j+1} and g_j are embeddings and can be arbitrarily close to the identity. By induction, we know that g_j is a homeomorphism onto $U'_j(m+2-j)$. The map g_{j+1} is injective on $U_j(m-1) \setminus N$ (which is a compact set with an open set removed) and $V_{j+1}(m-1) \setminus N$ by definition. The distance between those sets is strictly greater than zero, and on N, the map is injective too. Thus, g_{j+1} is injective if i is big enough, as that means that it is close enough to the identity.
  The fact that g_{j+1} is a homeomorphism follows from the fact that $\widetilde{F}$ is.

□

Thus, the embedding $g_k(1)\colon U_k(1) = M \to M' = U'_k(1)$ is a homeomorphism, and it is arbitrarily close to the identity. This means that the manifolds M and M' are homeomorphic, which is a contradiction to the statement that the elements of the sequence were distinct. So our assumption must be false, thus we have proven that there are only countably many manifolds without boundary up to homeomorphism. □

Theorem 4.6. *Up to homeomorphism, there are only countably many compact topological manifolds (with possibly nonempty boundary).*

Proof. The argument for manifolds with non-empty boundary is similar to the case with empty boundary: we first assume that there are uncountably many compact topological manifolds with boundary and fix some n so that there are uncountably many n-manifolds with boundary. As before, $B_r(x)$ denotes the closed ball of radius r centered at x in $\mathbb{R}^n$. We then choose an $(n-1)$-manifold B without boundary such that the set of all n-manifolds whose boundary is homeomorphic to this manifold is uncountable, and restrict to this case. As we have proven above that there are only countably many closed $(n-1)$-manifolds, such a manifold B must exist, and this restriction is no loss of generality. So we get an uncountable set $\mathcal{M} = \{M_\alpha\}_{\alpha\in\mathcal{A}}$ of n-manifolds, each with boundary B.

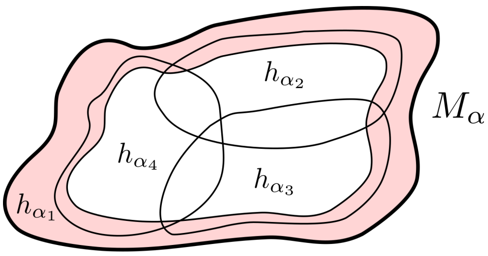


FIGURE 3. This is an example of a covering of a 2-manifold with boundary $M_\alpha \cong D^2$ with $k = 4$.

We choose finite covers for the manifolds in $\mathcal{M}$ analogously to the case with empty boundary and an integer k such that there are uncountably many compact manifolds covered by k embedded closed balls in the same way. By the collaring theorem (first shown in [Bro62], Theorem 6.5 in the lecture notes), each manifold in this set has a collared boundary, so for M_α there exists an embedding $h_{\alpha_1}\colon B \times [0, k+1] \to M_\alpha$ with $h_{\alpha_1}(b, 0) = b$ for all $b \in B$. The finite cover of M_α gives us embeddings $h_{\alpha_j}\colon B_{k+1}(0) \to M_\alpha$ such that M_α is covered by $\{h_{\alpha_1}(B \times [0, 1]), h_{\alpha_2}(B(1)), \ldots, h_{\alpha_k}(B(1))\}$. We embed M_α in $\mathbb{R}^{2n+1} = \mathbb{R}^l$ as in the first part of the proof of Theorem 4.4 and define an embedding $g_\alpha\colon B \times [0, k+1] \times B_{k+1}(0) \to \mathbb{R}^{kl}$ by

$$g_\alpha(x, t, y) = (h_{\alpha_1}(x, t), h_{\alpha_2}(y), \ldots, h_{\alpha_k}(y))$$

for every manifold M_α in $\mathcal{M}$. We denote the set of all such embeddings for $\alpha \in \mathcal{A}$ by $\mathcal{G}$.

By the same arguments as before, there is an embedding $g_{\alpha_0} \in \mathcal{G}$ that is the limit point of a sequence $g_{\alpha_1}, g_{\alpha_2}, \dots$ in $\mathcal{G}$ with $g_{\alpha_0} \neq g_{\alpha_i}$ for all $i \in \mathbb{N}$. As in the proof for manifolds with empty boundary, we set $M' = M_{\alpha_i}$ for some i that is fixed, but arbitrarily large. We define $V_1(m) := h_{\alpha_0,1}(B \times [0,m])$ and $V_1'(m) := h_{\alpha_i,1}(B \times [0,m])$ with $m = 1, \dots, k+1$. The sets $V_j(m)$ and $V_j'(m)$ for $j = 2, \dots, k$ as well as $U_j(m)$ and $U_j'(m)$ are defined as before, and the rest of the proof constructs a homeomorphism from M to M' in the same way as the proof for manifolds with empty boundary, as all statements used can also be applied to manifolds with boundary. □

5 Application to Morse theory

In their paper [CK70], Cheeger and Kister also present a topological submersion theorem that is an application of their results and useful in topological Morse theory. We will now state these results and sketch a proof.

Definition 5.1. A map $f\colon X \to Y$ is called *proper* if the preimages of compact sets are compact, i.e. for all compact $C \subseteq Y$, the set $f^{-1}(C)$ is compact. We call a continuous map $f\colon X \to Y$ *monotone* if the inverse image of any point in $f(X)$ is a connected subset of X.

Let Y be an m-manifold and X be an n-manifold, and thus metrisable. Let d be a metric on X. Let $f\colon X \to Y$ be a proper monotone map satisfying the following condition:

($\star$) for every $x \in X$ there are closed neighbourhoods $f(x) \in U \subseteq Y$ and $x \in V \subseteq X$ and a homeomorphism $h\colon B_2(0) \times U \to V$ such that $f \circ h$ is the projection map onto U.

We define $M_y := f^{-1}(y)$ for $y \in Y$.

Proposition 5.2. *For every $y \in Y$, M_y is a compact connected topological $(m-n)$-manifold.*

Proof. We know that M_y is compact and connected because the map f is proper and monotone. It is also Hausdorff as a subset of a Hausdorff space.

To show it is locally $(m-n)$-euclidean, take any $x \in M_y \subseteq X$. By the condition mentioned above, there is some neighbourhood V of x that is homeomorphic to $B_2(0) \times U$. As Y is an n-manifold, there is some neighbourhood of $f(x)$ in U that is isomorphic to $\mathbb{R}^n$. We can now restrict h to $B_2(0)$ cross this neighbourhood and obtain that M_y is locally $(m-n)$-euclidean and thus a compact topological $(m-n)$-manifold.

As M_y is compact and locally $(m-n)$-euclidean, it can be covered by finitely many open balls. We know that $\mathbb{R}^{m-n}$ is second-countable, so we can find a countable basis of the topology of M_y by considering the images of the bases of the balls that we embedded. □

We fix $y_0 \in Y$ and can find a collection of embeddings

$$\{h_{y_j}\colon B_2(0) \to M_y \,|\, y \in U, j = 1, 2, \dots, k\}$$

where U is a closed neighbourhood of y_0 in Y, and $\{h_{y_j}(B_1(0))\}_{j=1}^k$ covers M_y. For fixed j, the embeddings h_{y_j} vary continuously in y as embeddings of $B_2(0)$ into X. We apply the method from the proof of Theorem 4.4 to construct a homeomorphism $g\colon M_{y_0} \to M_y$ for y in a small enough neighbourhood U' of y_0. This is canonical and continuous in y because the result from [EK71] was too.

Then the map $g\colon M_{y_0} \times U' \to X$ that we define as $g(x, y) = g_y(x)$ is a local trivialization of f.

This result can be applied to topological Morse theory. For this purpose, we first want to define some very basic concepts from topological Morse theory, as they can be found in [Sch99].

Definition 5.3. Let X be a connected topological n-manifold and $f\colon X \to \mathbb{R}_+$ a continuous function. Then $x \in X$ is an *ordinary point* of f if there exists an open neighbourhood V of x

in X and a homeomorphic parametrization of V by n parameters such that one of them is f. Otherwise, x is a *critical point* of f.

A critical point x of f is called *non-degenerate* if there exists an open neighbourhood V of x in X, a homeomorphic parametrization of V by parameters $y_1, \dots, y_n$ and an integer $0 \leq j \leq n$ such that, for all $u \in U$,

$$f(u) - f(x) = \sum_{i=1}^{j} y_i^2 - \sum_{i=j+1}^{n} y_i^2$$

holds.

Such a function f is called a *topological Morse function* if all critical points of f are non-degenerate.

Applying the previous consideration to this case results in the following statement:

Proposition 5.4. *Let X be a compact connected topological $(n+1)$-manifold, $Y = [0,1]$ and $f\colon X \to Y$ be a topological Morse function without critical points, i.e. all point $x \in X$ are ordinary points. Then f is a trivial bundle map with fiber a compact manifold.*

Proof. The map f, which is a topological Morse function without critical points, is proper and monotone. To prove that f is proper, let $C \subseteq [0,1]$ be a compact set, and thus in particular closed, as $[0,1]$ is a Hausdorff space. Then $f^{-1}(C) \subseteq X$ is also compact because it is a closed subset of a compact manifold.

Let $t \in [0,1]$ be any point. The preimage of t under f cannot be empty. We can see this as the image is nonempty, connected, closed, because X is compact, and open, because we can find an open neighbourhood around any point using the condition that any point of X is an ordinary point of f. Thus, the image must be all of Y. Similarly to the proof of Proposition 5.2, we can see that $f^{-1}(t)$ is a compact manifold. If we assume that the preimage of t is not connected, we can find two points $x, y \in X$ with $f(x) = f(y) = t$ that are in different path components of $f^{-1}(t)$, but that can be connected by a path $\gamma \subseteq X$ in X, as X is a connected, and thus path-connected manifold. Then there must be a point in $f(\gamma)$ that is not an ordinary point of f, which is a contradiction to our first assumption that f has no critical points.

The map f meets the condition $\star$ by definition by setting V to be the closure of the open neighbourhood in the definition of a Morse function and $U = f(V)$. The homeomorphic parametrization yields exactly the necessary homeomorphism such that $f \circ h$ is the projection.

We can then analogously define M_t as $f^{-1}(t)$ and construct a homeomorphism $g\colon M_{t_0} \to M_t$ for t in a small enough neighbourhood of t_0, and the map $g\colon M_{t_0} \times U' \to X$, defined as $g(x,y) = g_y(x)$ is a local trivialization of f. As these homeomorphisms change continuously and all points of f are ordinary points, the map f is a trivial bundle map with fiber a compact n-manifold. ☐

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