4-manifold topology

Lecture I: the Disc Embedding Theorem

Mark Powell

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Theorem (The disc embedding theorem, Freedman '82)

Let M be a topological 4-manifold with $\pi_1(M)$ good. Let

$$f \colon (D^2, \partial D^2) \to (M, \partial M)$$

be a generic immersion and let $g \colon S^2 \looparrowright M$ be such that $\lambda(f, g) = 1 \in \mathbb{Z}\pi_1(M)$ and $\lambda(g, g) = \mu(g) = 0$.

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$$\overline{f}\colon(D^2,\partial D^2)\hookrightarrow(M,\partial M)$$

with a geometric dual $\overline{g}\colon S^2\looparrowright M$ such that $f|_{\partial D^2}=\overline{f}|_{\partial D^2}$, $|\overline{f}\pitchfork\overline{g}|=1$, and $g\sim\overline{g}$.

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The aim of this talk is to outline a proof of the disc embedding theorem.

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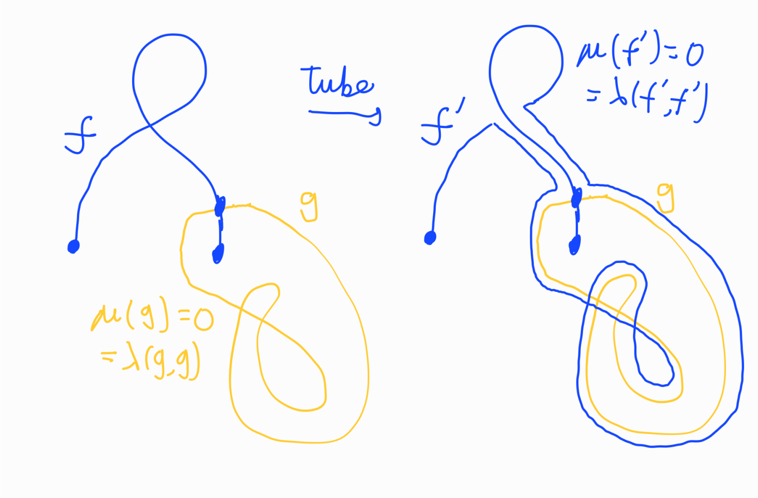
- Exotic $\mathbb{R}^4$s (F and Donaldson).
- There are non-triangulable manifolds (F and Casson).

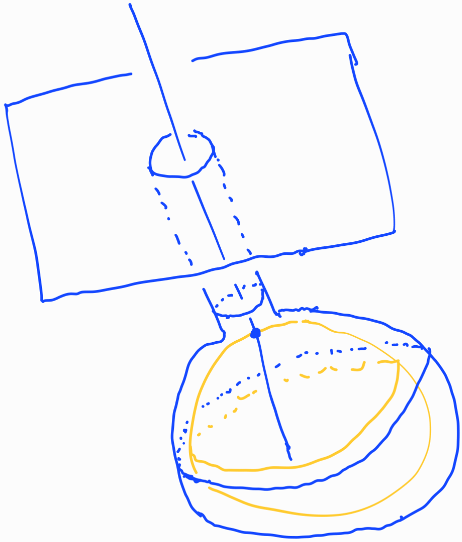
Step 1: Tubing.

We want to replace f by a *capped surface* f'' in M.

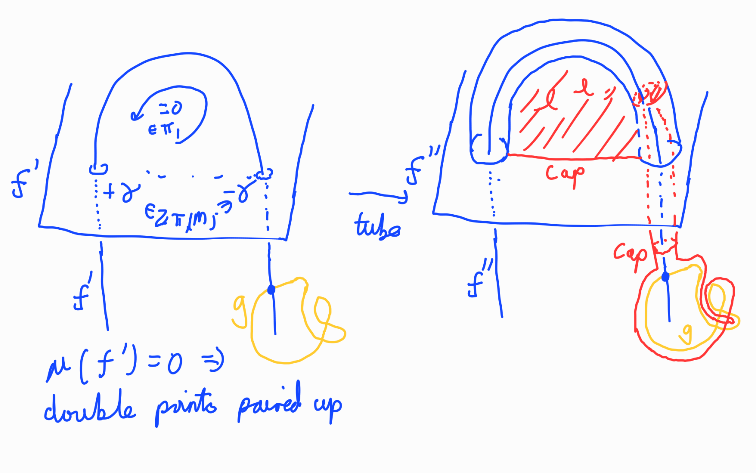
Turn $f \rightsquigarrow f'$ with algebraically trivial self-intersections.



An idealised tubing (with g embedded).

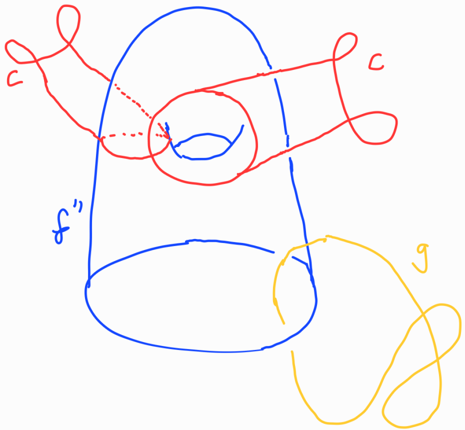


More tubing to convert f' to a capped surface f''.



The arcs persist in the past and future, so also represent sheets of surfaces.

A schematic of a capped surface f'' with a dual sphere g. The caps still have intersections, but they are 'higher order' intersections. I claim this constitutes progress.



Note that $\partial f'' = \partial f$.

Step 2: Towers.

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Casson's idea: push intersection problems 'to infinity'.

Definition

A (finite) *tower* is a compact 4-manifold $G \cong \natural^k S^1 \times D^3$ with a preferred attaching region

$$\partial_- \colon S^1 \times D^2 \hookrightarrow \partial G$$

and a decomposition as a union of standard pieces:

(s) thickened surfaces $\Sigma_{g,1} \times D^2$;

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Each piece has an attaching region (s) $\partial\Sigma_{g,1} \times D^2$; (c) $\partial D^2 \times D^2$, and a collection of *tip regions*, shown in yellow;
(s) symplectic basis of curves; (c) double point loops.

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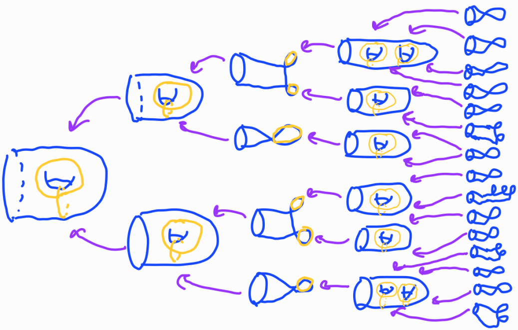

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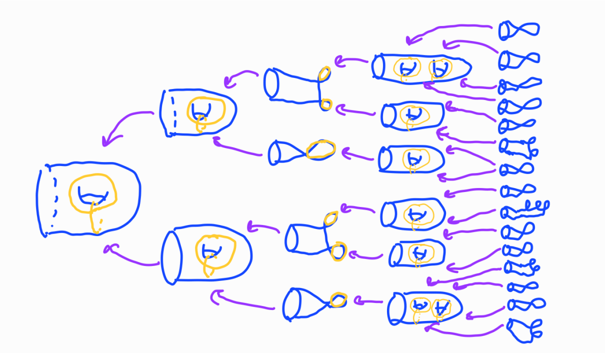

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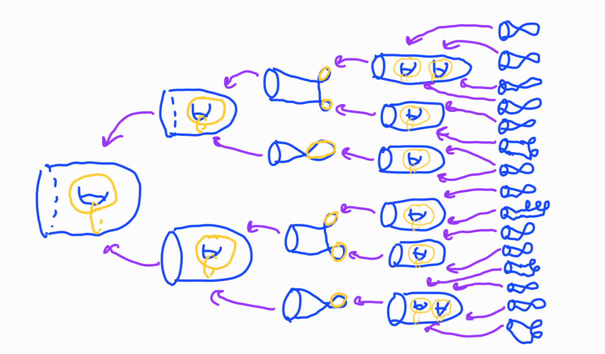

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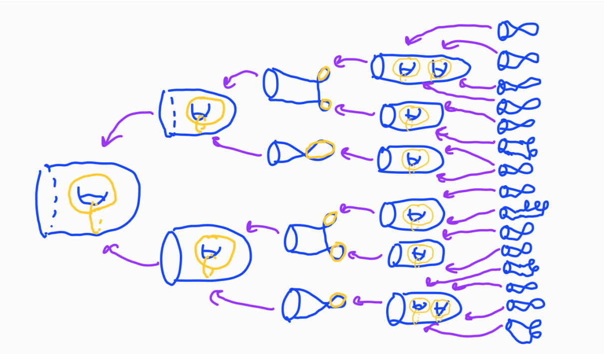

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(iii) A tower of type $s \dots sc \dots s \dots sc$ with k cap stages is a *k-storey tower* $\mathcal{T}_k$. Each storey is a capped grope.

Theorem (Grope height raising)

Given a height k capped grope G_k^c, $k \geq 2$, for all $r \in \mathbb{N}$ there exists an embedding $G_r^c \hookrightarrow G_k^c$ with the same attaching region ∂_-.

Definition

The fundamental group $\pi_1(M)$ is *good* if for all

$$(G^c_{k+2}, \partial_-) \hookrightarrow (M, \partial),$$

there exists

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Unknown whether all groups are good.

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If $\pi_1(M)$ is good, there exists an infinite tower

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- Iterate, applying this argument to top storey. □

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There exists an embedding

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as before, such that

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A priori $\widehat{\mathcal{T}}_\infty$ need not be a manifold.

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Then

$$D^2 \times \{0\} \hookrightarrow \widehat{\mathcal{T}}_\infty \hookrightarrow M$$

gives the desired locally flat embedding

$$\overline{f}\colon (D^2, \partial) \to (M, \partial M).$$

Write $\mathcal{T}_{[k,\ell]}$ for storeys k through ℓ of $\widehat{\mathcal{T}}_\infty$ and let $\{\mathcal{T}_{[k,k+1]}(m)\}$ be the path components of $\mathcal{T}_{[k,k+1]}$.

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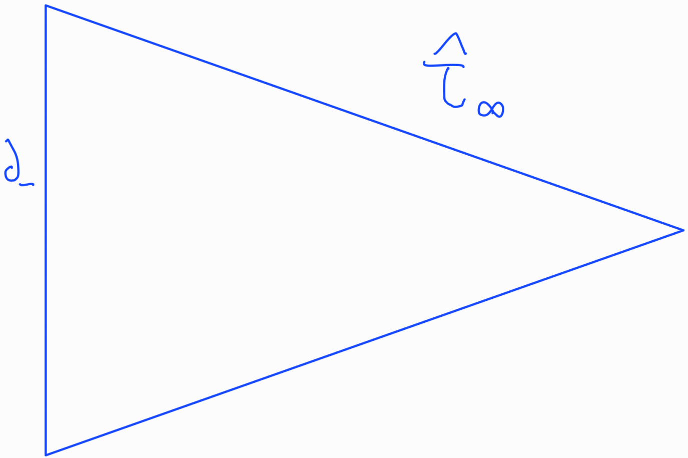
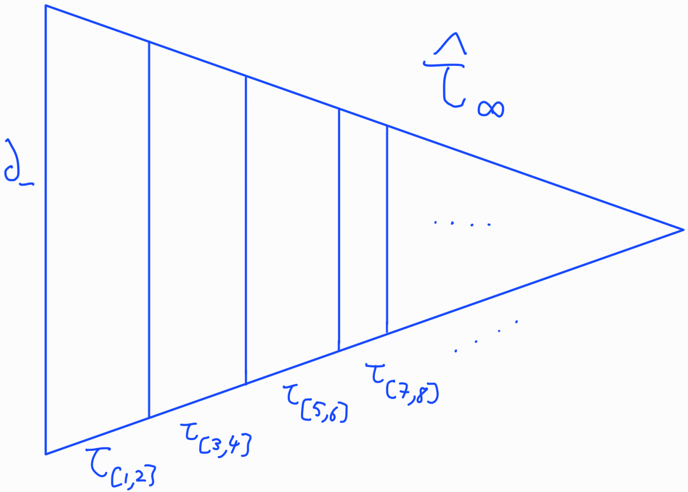
We use this to fill our initial $\widehat{\mathcal{T}}_\infty$ as much as possible, with a Cantor set's worth of embedded $\widehat{\mathcal{T}}_\infty$s.



$\hat{\mathcal{T}}_\infty$
∂_-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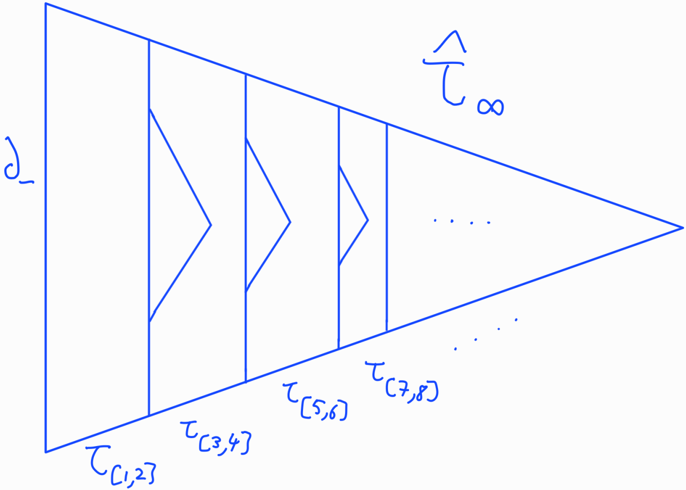
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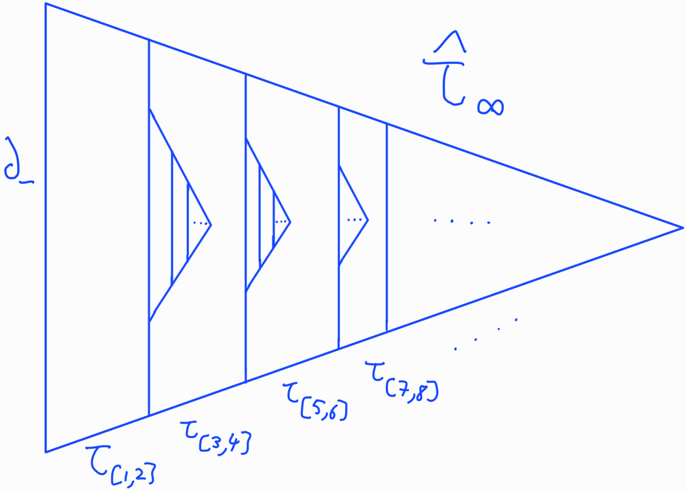

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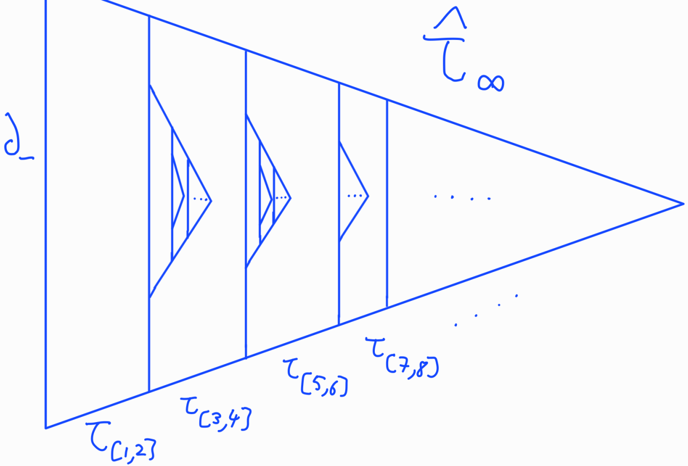
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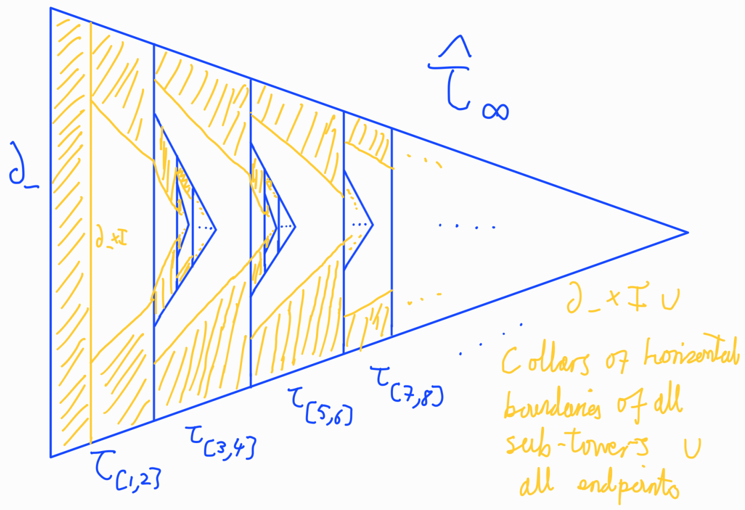
$\hat{\tau}_\infty$
∂_-
$\partial_- \times I$
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$\partial_- \times I \cup$
Collars of horizontal boundaries of all sub-towers $\cup$ all endpoints

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This is a “known” subset.

Step 5: shrinking.

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We want identify the connected components of $\widehat{\mathcal{T}}_\infty \setminus \mathfrak{D}_\infty$ to points, show this does not change the homeomorphism type, and recognise the quotient space as $D^2 \times D^2$.

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We will usually equate a decomposition with its non-singleton sets.

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In particular X and $X/\mathcal{D}$ are homeomorphic.

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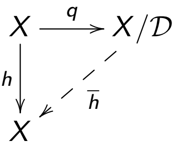

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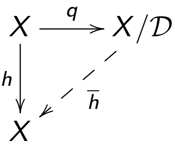

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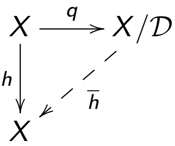

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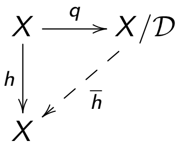

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$$\overline{h}^{-1} \circ h_i \to \overline{h}^{-1} \circ h = \overline{h}^{-1} \circ \overline{h} \circ q = q,$$

so q is ABH.

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We see that

$$[0, 1] \cong [0, 1]/\mathcal{D}.$$

Proposition (Bing, Ancel-Starbird)

The frontier of each $\widehat{\mathcal{T}}_\infty$ *is homeomorphic to* $S^3 = \partial_- \cup D^2 \times S^1$ *provided that* $\sum_i \frac{a_i}{2^i}$ *diverges, where* a_i *is the number of surface stages in the* i*th storey of* $\widehat{\mathcal{T}}_\infty$.

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So we already have many embedded discs, just none are yet known to be locally flat.

Sketch proof that $\operatorname{Fr}\widehat{\mathcal{T}}_\infty \cong S^3$.

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Stage of tower $\leftrightarrow$ Stage of nested solid tori T_i in S^3.

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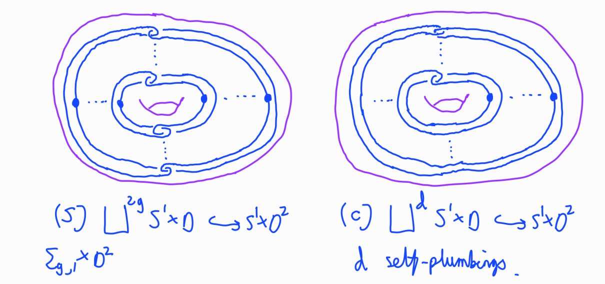

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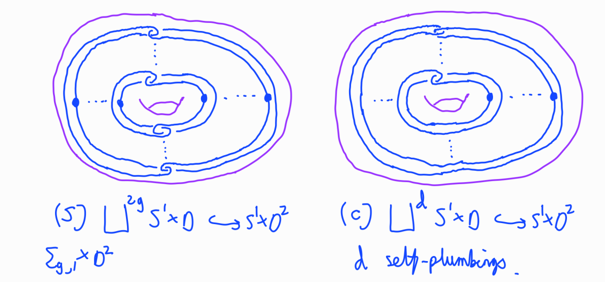

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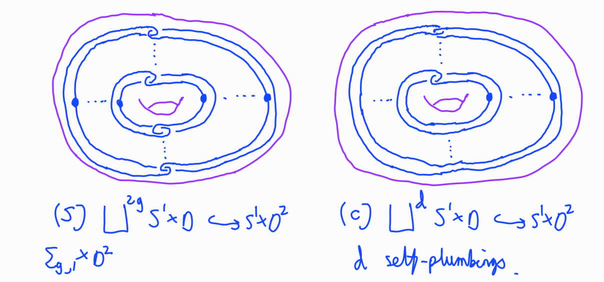

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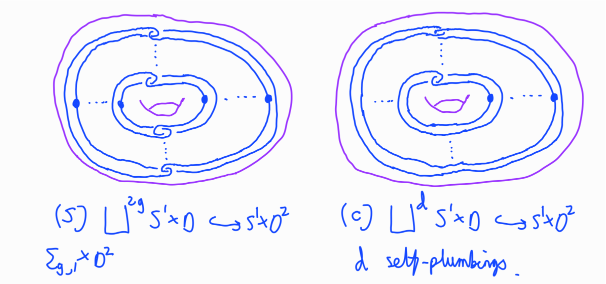

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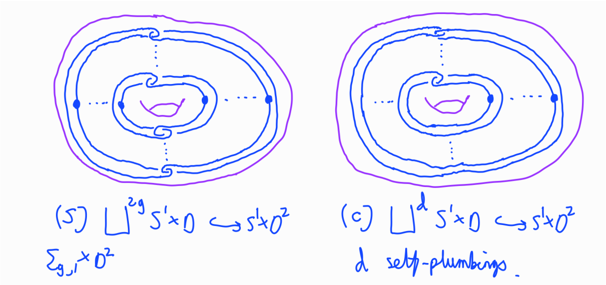

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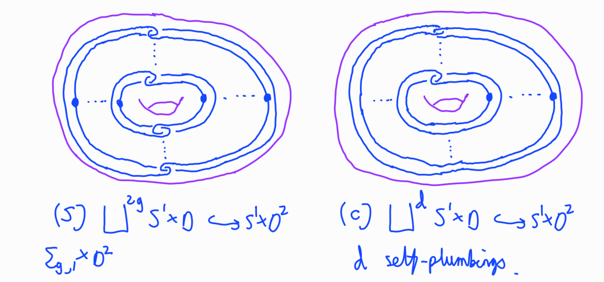

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End of Disc Embedding Theorem proof.

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$$(\widehat{\mathcal{T}}_\infty, \partial_-) \cong (H, \partial H) := (D^2 \times D^2, S^1 \times D^2).$$

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(Should really talk about slight modifications $\mathcal{G}^+$ and $\mathcal{H}^+$ here, omitted for time reasons.)

There is a common quotient:

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Thus $(\widehat{\mathcal{T}}_\infty, \partial_-) \cong (H, \partial H)$ as desired.

This was a stand-alone lecture. The next two lectures will be somewhat distinct from this one. It won't be necessary to know anything about the proof of DET to follow them.