THE DOUBLE SUSPENSION OF THE MAZUR HOMOLOGY SPHERE

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1 Introduction

The main objects of this text are *homology spheres*, which are defined below.

Definition 1.1. A manifold M of dimension n is called a homology n-sphere if it has the same homology groups as S^n; that is,

$$H_k(M) = \begin{cases} \mathbb{Z} & \text{if } k \in \{0, n\} \\ 0 & \text{otherwise} \end{cases}$$

A result of J.W. Cannon in [Can79] establishes the following theorem

Theorem 1.2 (Double Suspension Theorem)**.** *The double suspension of any homology n-sphere is homeomorphic to S^{n+2}.*

This, however, is beyond the scope of this text. We will, instead, construct a homology sphere, the *Mazur homology 3-sphere*, and show the double suspension theorem for this particular manifold. However, before beginning with the proper content, let us study the following famous example of a nontrivial homology sphere.

Example 1.3. . Let I be the group of (orientation preserving) symmetries of the icosahedron, which we recall is a regular polyhedron with twenty faces, twelve vertices, and thirty edges. This group, called the icosahedral group, is finite, with sixty elements, and is naturally a subgroup of $\mathrm{SO}(3)$. It is a well-known fact that we have a 2-fold covering $\xi : \mathrm{SU}(2) \to \mathrm{SO}(3)$, where $\mathrm{SU}(2) \cong S^3$, and $\mathrm{SO}(3) \cong \mathbb{R}P^3$. We then consider the following pullback diagram

$$\begin{array}{ccc} S^0 & & S^0 \\ \downarrow & & \downarrow \\ \widetilde{I} := \iota^*(\mathrm{SU}(2)) & \hookrightarrow & \mathrm{SU}(2) \\ \downarrow{\scriptstyle \iota^*\xi} & & \downarrow{\scriptstyle \xi} \\ I & \hookrightarrow & \mathrm{SO}(3) \end{array}$$

Then, $\widetilde{I}$ is also a group, where the multiplication is given by the lift of the map $\mu \circ (\iota^*\xi \times \iota^*\xi)$, where μ is the multiplication in I. Thus, $\widetilde{I}$ defines a subgroup of the compact Lie group $\mathrm{SU}(2)$, called the binary (or extended) icosahedral group. Furthermore, it is clear that this group consists of 120 elements; it can be further shown that $\widetilde{I} = \langle s, t \mid (st)^2 = s^3 = t^5 \rangle$. We now form the space $P^3 = {\mathrm{SU}(2)}/{\widetilde{I}}$, called the *Pioncaré homology sphere*, and note that it is itself a Lie group, as being the quotient of a Lie group by a finite subgroup. By covering space theory (more precisely, Proposition 1.40 from [Hat01]), one can show that $\pi_1 P^3 \cong \widetilde{I}$. Another way to see this is via a theorem of Gleason (cf. Corollary 1.4 in [Coh]), which states that $p : \mathrm{SU}(2) \to P^3$ is a principal $\widetilde{I}$-bundle. Since this principal bundle has discrete fibre, it is a covering space with the structure group being isomorphic to the fundamental group. That is, $\pi_1(P^3) \cong \widetilde{I}$.

It is a well-known fact that the binary icosahedral group is a perfect group, i.e. $\widetilde{I} = [\widetilde{I}, \widetilde{I}]$. From this, it follows that $H_1(P^3) \cong \pi_1^{ab}(P^3) = 0$. Furthermore, since P^3 is a Lie group, Poincaré duality holds, so that we get an isomorphism $H_2(P^3) \cong H^1(P^3)$; by the universal coefficient theorem, it also follows that $H^1(P^3) \cong \mathrm{Hom}(H_1(P^3), \mathbb{Z}) = 0$, so that $H_2(P^3) = 0$. Furthermore, P^3 is clearly connected, and hence $H_3(P^3) \cong \mathbb{Z}$. Thus, it is a homology sphere.

Remark: There are at least eight different constructions of that manifold, as found in [KS79]*. The above description is the most accessible one among them.*

The upshot of the above example is that homology equivalence does not classify spaces up to homotopy equivalence; indeed, P^3 and S^3 are homology equivalent, even though S^3 is simply connected, while $\pi_1(P^3) \cong \widetilde{I} \neq 0$.

2 Some technical lemmata

We now prove some technical lemmata, the first of which serves as a preliminary reduction of the double suspension theorem to proving that the double suspension is a manifold, while the second gives a criterion for a manifold to be a homology n-sphere. The third one will be used to show that the 4-manifold we construct in the next section is indeed contractible, so that the conditions of Lemma 2.3 are met.

Lemma 2.1. *If a compact manifold of dimension n can be written as $M = U_1 \cup U_2$ where $U_1 \cong U_2 \cong \mathbb{R}^n$, then $M \cong S^n$.*

Proof. By invariance of domain, we note that any U_1 and U_2 as in the above are open in M. Denote by φ the homeomorphism $\varphi : U_2 \to \mathbb{R}^n$. Observe that $M \setminus U_1$, being a closed set in the compact manifold M, is itself compact. Then, $\varphi(M \setminus U_1)$ is a compact subset of $\mathbb{R}^n$, so that, in particular, it is bounded in $\mathbb{R}^n$. Consider three closed, n-dimensional concentric balls in $\mathbb{R}^n$, containing $\varphi(M \setminus U_1)$ in their interiors, and let D be the middle ball. Then, $\varphi^{-1}(\partial D) \subseteq U_1$ is a bicollared $(n-1)$-dimensional sphere in U_1. Thus, the Schoenflies theorem tells us that $M \setminus \varphi^{-1}(\mathrm{int}(D))$ is itself homeomorphic to D^n. Consequently, we have shown that M can be written as a union of two homeomorphic copies of the standard disc D^n, attached along their boundaries. By the Alexander trick, we may extend the homeomorphism $\varphi^{-1}|_{\partial D}$ to homeomorphisms of the discs themselves, and as such, $M \cong S^n$. □

Corollary 2.2. *If the suspension ΣX of any topological space X is a compact n-manifold, then $\Sigma X \cong S^n$.*

Proof. Consider the coordinate neighborhoods U_1 and U_2 around the two suspension points of ΣX, which exist since we assumed that ΣX is a manifold. Then, by stretching these two neighborhoods along the $[-1,1]$ coordinate in $\Sigma X = {}^{(X \times [-1,1])}\!/\!_{\sim}$, we are reduced to the setting of Lemma 2.1. Thus, $\Sigma X \cong S^n$, as claimed. □

Lemma 2.3. *The boundary of any compact, contractible n-manifold is an $(n-1)$-homology sphere.*

Proof. We first recall that any simply connected manifold is orientable. It follows that M is a compact, orientable manifold with boundary, and thus Poincaré-Lefschetz duality holds, so that we have isomorphisms $H_k(M, \partial M) \cong H^{n-k}(M)$, for all $k \in \mathbb{N}$. Furthermore, since M is contractible, it follows that $H^k(M)$ is trivial in all dimensions except 0, where it is infinite cyclic. Inspecting the homology long exact sequence of the pair $(M, \partial M)$, we reach the following conclusions:

– For all $k > 1$, the fact that $H_k(M) = H_{k-1}(M) = 0$ implies that the connecting homomorphism $\partial_* : H_k(M, \partial M) \to H_{k-1}(\partial M)$ is an isomorphism; thus, $H_{n-1}(\partial M) \cong \mathbb{Z}$, while $H_k(\partial M) = 0$ for all $k \notin \{0, n-1\}$

– For $k = 1$, we have the following exact sequence:

$$0 = H_1(M, \partial M) \xrightarrow{\partial_*} H_0(\partial M) \xrightarrow{\iota_*} H_0(M) \xrightarrow{q_*} H_0(M, \partial M) = 0$$

Thus, ι_* is an isomorphism between $H_0(\partial M)$ and $H_0(M) \cong \mathbb{Z}$.

□

Lemma 2.4. *If a manifold M has a handle decomposition, then it is homotopy equivalent to a CW-complex whose cells are in bijection with the handles of M.*

Proof. The proof is rather straightforward: for every k-handle $D^k \times D^{n-k}$, attached via $\varphi_k : (\partial D^k) \times D^{n-k} \to M_{k-1}$, we contract D^{n-k} to its center 0; the resulting CW-complex will have, as attaching maps, the restriction of the handle attachments, namely $\varphi_k|_{D^k \times \{0\}}$. It then follows readily that the above two spaces are homotopy equivalent. □

3 Constructions

In light of Lemma 2.3, the Mazur homology 3-sphere will be defined as the boundary of a compact, contractible 4-manifold. This 4-manifold will be described via a handle decomposition using a 0-handle, a 1-handle, and a 2-handle. Let B_1 and B_2 be two disjoint 3-balls in ∂D^4, where D^4 is the 0-handle, and let h_1 and h_2 denote the homeomorphisms $h_i : D^3 \to B_i$. This then yields a map $f = h_1 \sqcup h_2 : S^0 \times D^3 \to B_1 \sqcup B_2$; we now form the manifold $D^4 \cup_f (D^1 \times D^3)$, which is readily seen to be the manifold $S^1 \times D^3$. This is represented in the following illustration:

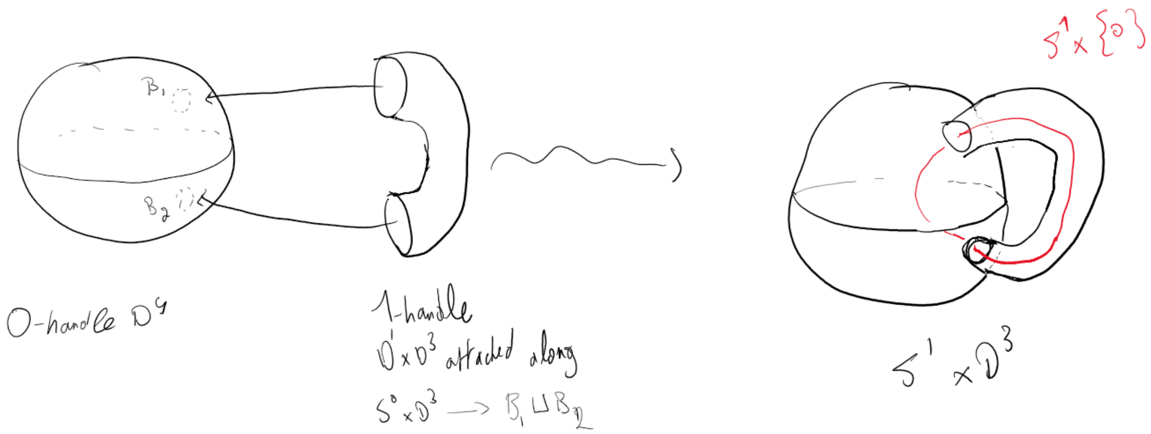

FIGURE 1. Attaching of the 1-handle

To attach the 2-handle onto the above manifold, we first consider the following inclusions:

$$S^1 \times D^2 \hookrightarrow S^1 \times S^2 \hookrightarrow S^1 \times D^3$$

The first inclusion is the result of viewing D^2 as one of the hemispheres of S^2, while the second follows from the fact that $\partial D^3 = S^2$; both maps are the identity on the S^1 factor. Thus, we may view $S^1 \times D^2$ as a subspace of $\partial(S^1 \times D^3)$. Let Γ_0 be the standard circle $S^1 \times \{0\}$ in $S^1 \times D^2$, and let Γ_1 be the knot embedded in $S^1 \times D^2 \subset \partial(S^1 \times D^3)$ shown in the following figure, taken from [Fer]

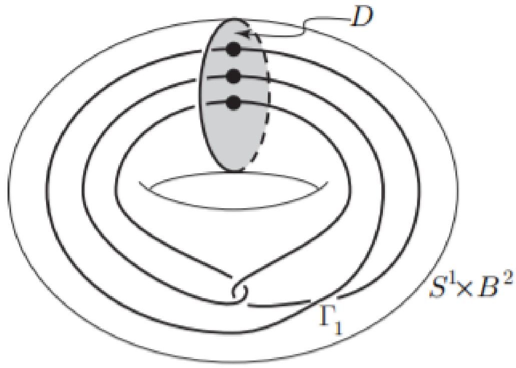

FIGURE 2. The Mazur link Γ_1

Let N be a thickened neighborhood of Γ_1, which is clearly homeomorphic to $S^1 \times D^2$. On the boundary of N, we have a pushoff β of Γ_1, which has linking number $lk(\Gamma_1, \beta) = 0$ with Γ_1. Consider the homeomorphism $\varphi : S^1 \times D^2 \to N$, mapping Γ_0 to Γ_1, and mapping a circle $S^1 \times *$ on the boundary of $S^1 \times D^2$ to a knot of the above type, i.e. having linking number 0 with Γ_1. Note that the attaching map of the 2-handle described above is an orientation preserving homeomorphism, as shown in Lemma 3.1. We now form the 4-manifold $W^4 := (S^1 \times D^3) \cup_\varphi (D^2 \times D^2)$.

Lemma 3.1. *$\varphi : S^1 \times D^2 \xrightarrow{\cong} N$ is an orientation preserving homeomorphism.*

Proof. By the relative Künneth formula, we get isomorphisms

$$\begin{aligned} H^3(S^1 \times D^2, \partial(S^1 \times D^2)) &= H^3(S^1 \times D^2, (\emptyset \times D^2) \cup (S^1 \times S^1)) \\ &\cong \bigoplus_{i+j=3} (H^i(S^1) \otimes_{\mathbb{Z}} H^j(D^2, S^1)) \\ &\cong H^1(S^1) \otimes_{\mathbb{Z}} H^2(S^2) \end{aligned}$$

Furthermore, we have that $H^1(S^1) \otimes_{\mathbb{Z}} H^2(S^2) \cong H^1(S^1)$, via the map $x \mapsto x \otimes 1$. Thus, the degree of the map φ is determined by its restriction onto Γ_0; that is, we have the following diagram.

$$\begin{array}{ccc} H^3(S^1 \times D^2, \partial(S^1 \times D^2)) & \xrightarrow{\varphi_*} & H^3(S^1 \times D^2, \partial(S^1 \times D^2)) \\ \cong\downarrow & & \downarrow\cong \\ H^1(S^1) & \xrightarrow{(\varphi|_{\Gamma_0})_*} & H^1(S^1) \end{array}$$

A standard computation shows that the degree of the lower horizontal map is the identity; thus, the upper map also has degree 1, and hence is orientation preserving. $\square$

Lemma 3.2. *W^4 is contractible*

Proof. By Lemma 2.4, W^4 is homotopy equivalent to a 2 dimensional CW-complex X; this CW-complex is constructed using one 0-cell, one 1-cell, and one 2-cell. The 2-cell is attached to the circle S^1 via a degree one map $\tilde{\varphi}$ wrapping $\partial D^2 = S^1$ around the 1-skeleton twice in one direction, and once in the opposite direction, courtesy of the fact that the knot Γ_1 winds twice in one direction, and once in the opposite direction. This CW-complex can be easily shown to be contractible, as follows. All its reduced homology groups are trivial, since its cellular cochain complex ends takes the following form

$$0 \to \mathbb{Z} \xrightarrow{id} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \to 0$$

In the above, the second map is the identity as a result of the local degree formula, and the third map is trivial since the space is connected. Furthermore, a presentation of the fundamental group is given by $\pi_1(X) = \langle a \mid a^2 a^{-1} = 1 \rangle$, which is clearly the trivial group. Consequently, by successive iterations of the Hurewicz theorem, it follows that $\pi_n(X) = 0$, for all $n \geq 1$. Thus, the map $X \to *$ is a weak homotopy equivalence, so that contractibility follows from the Whitehead theorem. $\square$

As a consequence of Lemma 2.3, it follows that $H^3 := \partial W^4$ is a homology 3-sphere, called the Mazur homolgy 3-sphere. We note that in the above, the precise choice of the attaching map φ is irrelevant, as long as $\varphi|\Gamma_0$ winds, homtopically, once around $S^1 \times D^2$.

It is noteworthy to mention that a presentation of $\pi_1 H^3$ is given by

$$\pi_1 H^3 = \langle a, b \mid a^7 = b^5,\, b^4 = a^2 b a^2 \rangle$$

Mazur showed this group is nontrivial, as stated in [Dav87]. Thus, H^3 is not homotopy equivalent to S^3.

4 The Giffen disc

In this section, we prove that W^4 contains a 2-cell B inside its interior, called the *Giffen disc*, which will play a vital role in the proof of the double suspension theorem, in this setting. Furthermore, we show that this 2-cell is a *pseudo-spine*, as defined below.

Definition 4.1. A compact subset X of a manifold with boundary M is called a pseudo-spine if $M \setminus X \cong \partial M \times [0, 1)$

We now construct the Giffen disc. Begin by cutting $S^1 \times D^2$ along the disc D indicated in Figure 2, and let $\{B_i^2\}_{i \in \mathbb{N}}$ be a countable collection where each of the B_i's is a cylinder resulting from the above cutting. Form $D^2 \times [0, \infty)$ by attaching $B_i^2 \times [i-1, i]$ to $B_{i+1}^2[i, i+1]$ in such a way that the curves inside the cylinders align; finally, let C^3 be the one point compactification of the above space, as represented in the Figure 3, taken from [Dav87]

Let $L = (\bigcup_{i \in \mathbb{N}} L_i) \cup \{\infty\}$, where L_i are as in the above figure. Clearly, C^3 is a 3-cell (hence the notation), and L has two connected components; furthermore, the component of ∞ is a Fox-Artin arc, so that, as a consequence, the embedding $L \hookrightarrow \mathbb{R}^3$, resulting from the standard embedding $C^3 \hookrightarrow \mathbb{R}^3$, is wild. Let $\sigma : C^3 \to C^3$ be the shift homeomorphism defined as $\sigma(b, t) = (b, t+1)$, and $\sigma(\infty) = \infty$. Form the mapping torus $T(\sigma) = {C^3 \times \mathbb{I}}/{\sim}$, where $\sim$ identifies points $(x, 0)$ with $(\sigma(x), 1)$; additionally, let Ω be the mapping torus of $\sigma|L : L \to L$. We first note that $T(\sigma)$ is homeomorphic to $D^3 \times S^1$; to see this, observe that ${C^3 \times \mathbb{I}}/{\sim}$ only skips $B_1^2 \times [0, 1]$, which can be homeomorphically "flattened", whereby we get a homeomorphism $T(\sigma) \cong {C^3 \times \mathbb{I}}/{(C^3 \times \{0\} \sim C^3 \times \{1\})} \cong C^3 \times S^1 \cong D^3 \times S^1$. We further claim that Ω is an

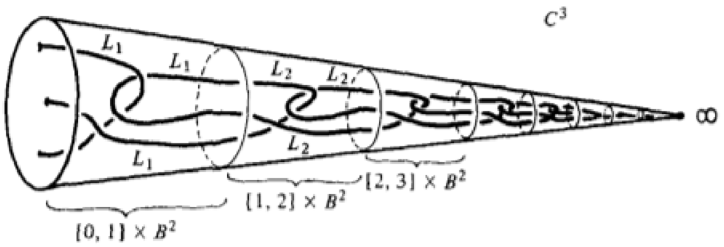

FIGURE 3. Cone containing a Fox-Artin arc

annulus $S^1 \times \mathbb{I}$. This follows after straightening both components of L, since $L \cong I_1 \sqcup I_2$, where each I_i is an interval (however, it is of course not ambiently isotopic to it, since the Fox-Artin arc is wild). Then, Ω is the quotient space represented in the following figure.

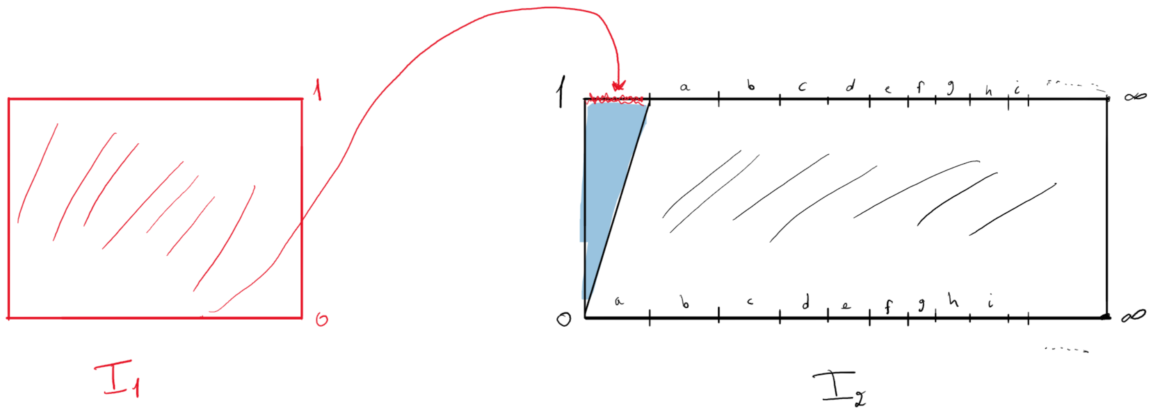

FIGURE 4. Quotient space homeomorphic to the mapping torus Ω

In the figure, I_2 is partitioned into countably infinitely many subintervals, where the partition points lie on the intersetion of the middle part of the Fox-Artin arc with the discs $B_i^2 \times \{i\}$ in the above cone. Then, $I_1 \times \{0\}$ is identified with a (strict) subinterval of the first interval in this partition; furthermore, due to the shift map, the subintervals $[i-1, i] \times \{0\} \subset I_2 \times [0, 1]$ are identified with the shifted ones, namely $[i, i+1] \times \{1\} \subset I_2 \times [0, 1]$; these are represented on the above figure by shifted Latin letters. It then follows easoly that Ω is an annulus. Consider the natural embedding $\Omega \hookrightarrow T(\sigma)$, resulting from the fact that $L \subset C^3$; this embedding is such that $\Omega \cap \partial T(\sigma) = \partial\Omega$. It is imperative to note that $\partial\Omega$ consists of a standard circle on one of its connected components, and of a Mazur link Γ_1 on the other; this can be seen geometrically from the figure of the cone C^3 above, where the standard circle results from the ∞ point. This embedding (or more precisely a part thereof) is represented in Figure 5

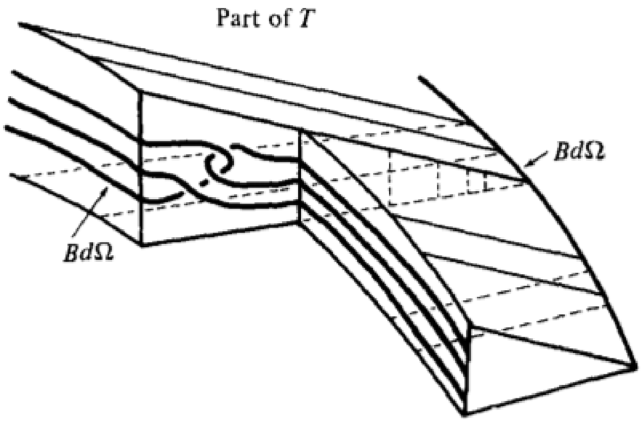

FIGURE 5. Transversal cut of $T(\sigma)$

In $W^4 = (S^1 \times D^3) \cup h$, where h is a 2-handle such that $h \cap (S^1 \times D^3)$ is a neighborhood as occurring in the above construction, and where $\Gamma_1 = \partial D^2 \times \{0\} \subset D^2 \times D^2 = h$, let $B := \Omega \cup (D^2 \times \{0\})$. By the Alexander trick, B is easily seen to be a 2-cell in $int(W^4)$.

Theorem 4.2. *B is a pseudo-spine of W^4.*

Before attempting to prove Theorem 4.2, we will have a detour to see some results from regular neighborhoods and piecewise-linear topology.

5 Regular Neighborhoods

In this section, we will list some definitions and results from PL-topology and regular neighborhoods, without proofs.

Definition 5.1. A Δ-complex structure on a space X is a collection of maps $\{\sigma_\alpha : \Delta^{n_\alpha} \to X\}$ from standard simplices to X, where n_α depends on α, that satisfy the following

(1) The restriction $\sigma_\alpha|_{int(\Delta^{n_\alpha})}$ is injective, and each point of X lies in the image of exactly one such restriction $\sigma_\alpha|_{int(\Delta^{n_\alpha})}$
(2) Each restriction of σ_α onto the faces of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$, where we identify Δ^{n-1} with a face of Δ^n by a linear homeomorphism
(3) A set $A \subset X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for all α.

We will, however, not need this generality; our study restricts to $X \subset \mathbb{R}^n$. A finite collection K of simplices in $\mathbb{R}^n$ is called a simplicial complex if for $\sigma \in K$, and $\tau < \sigma$, where $<$ means "is a subface of", then $\tau \in K$, and if $\sigma, \tau \in K$, then $\sigma \cap \tau < \sigma$ and $\sigma \cap \tau < \tau$. In the above case, the geometric realisation of K is by definition $|K| = \bigcup_{\sigma \in K} \sigma$, and is called a *polyhedron*, while K is a *triangulation* of $|K|$.

Definition 5.2. A locally finite simplicial complex is a (possibly infinite) collection K of simplices $\sigma \subset \mathbb{R}^n$ such that

(1) If $\sigma \in K$ and $\tau < \sigma$, then $\tau \in K$
(2) If $\sigma, \tau \in K$, then $\sigma \cap \tau < \sigma$ and $\sigma \cap \tau < \tau$
(3) Every point of K has an open cover that intersects only finitely many of the simplices of K non trivially.

Two disjoint simplexes $\sigma, \tau \subset \mathbb{R}^n$ are said to be *joinable* if there exists a simplex γ that is spanned by their vertices. In this setting, σ and τ are said to be opposite faces of γ, and γ

is called the join of σ and τ, denoted $\gamma = \sigma * \tau$. We remark that we will later discuss a more general operation on topological spaces, called join, not to be confused with the one here. Two finite simplicial complexes K, L are said to be joinable if all $\sigma \in K$ and $\tau \in L$ are joinable, and if for $\sigma, \sigma' \in K$ and $\tau, \tau' \in L$, we have that $\sigma * \tau \cap \sigma' * \tau'$ is a common face of $\sigma * \tau$ and $\sigma' * \tau'$. We now have reached the definition which was behind this entire excursion into PL-topology.

Definition 5.3.
- Let K be a simplicial complex, and let L be a subcomplex. We say that there is an elementary collapse of K onto L if $K \setminus L$ consists of two simplexes A and B such that $A = a * B$, where a is a vertex of A. Thus, $|K| = |L| \cup A$, and $|L| \cap A = a * \partial B$
- The complex K is said to collapse to the subcompex L if there is a finite sequence of elementary collapses that eventually land in L.
- If P is a polyhedron in a PL manifold M, then N is a regular neighborhood of P if
  1. N is a closed neighborhood of P
  2. N is a PL manifold
  3. N collapses to L

We now quote the regular neighborhood theorem, as used in the proof of theorem 6.

Theorem 5.4 (Regular Neighborhood Theorem)**.** *Let P be a polyhedron in the PL manifold M. Then, there exists a regular neighborhood N of P in M, that is unique up to PL homeomorphism, rel. P.*

6 Joins

The current section is devoted to a discussion on joins of topological spaces. This operation is quite interesting in of its own, as it is used in the Milnor construction of universal G-bundles; furthermore, it is quite relevant to our discussion here.

Definition 6.1. Let X and Y be two topological spaces. We define $X * Y$ as the space ${(X \times T \times \mathbb{I})}/{\sim}$ where $\sim$ identifies the following points:
- $(x, y_1, 0) \sim (x, y_2, 0)$, for all $x \in X$, and $y_1, y_2 \in Y$
- $(x_1, y, 1) \sim (x_2, y, 1)$, for all $x_1, x_2 \in X$ and $y \in Y$

The first result that we will prove is the fact that joins behave nicely for spheres, in the following sense

Lemma 6.2. $S^n * S^m \cong S^{n+m+1}$

Proof. Define the map $\varphi : S^n \times S^m \times \mathbb{I} \to S^{n+m+1}$, mapping $(x, y, t) \mapsto x \cos \frac{\pi t}{2} + y \sin \frac{\pi t}{2} \in \mathbb{R}^{n+1} \times \mathbb{R}^{m+1}$ which has norm 1, i.e. $\varphi(x, y, t) \in S^{n+m+1}$. First, note that for $y_1, y_2 \in S^m$, we have $\varphi(x, y_1, 0) = \varphi(x, y_2, 0) = x$, while for $x_1, x_2 \in S^n$, we also have $\varphi(x_1, y, 1) = \varphi(x_2, y, 1)$. Thus, φ respects the equivalence relation on $S^n * S^m$, so that φ descends to the quotient to a map $\widetilde{\varphi} : S^n * S^m \to S^{n+m+1}$, such that $\varphi = \widetilde{\varphi} \circ q$, where q is the quotient map. We note that the map φ is surjective. Indeed, let $z \in S^{n+m+1}$. We first distinguish two cases; if z has all coordinates of one of the factors $\mathbb{R}^{n+1}$ or $\mathbb{R}^{m+1}$ equal to zero when z is seen as an element of $\mathbb{R}^{n+m+2}$. Let $x_z = proj_{n+1}(z)$, and $y_z = proj_{m+1}(z)$ be the projections in the product $\mathbb{R}^{n+1} \times \mathbb{R}^{m+1}$. Then, we have that $z = \varphi(x_z, y, 0)$ (for arbitrary y) or $z = \varphi(x, y_z, 1)$ (for arbitrary x).
In the second case, both x_z and y_z are nonzero. Then, we have that $|x|^2 + |y|^2 = 1$, so that there exists a unique $t \in (0, 1)$ such that $|x^2| = \cos \frac{\pi t}{2}$ and $|y|^2 = \sin \frac{\pi t}{2}$, with both non zero. In that setting, it is easy to see that $\varphi(\frac{x}{\cos \frac{\pi t}{2}}, \frac{y}{\sin \frac{\pi t}{2}}, t) = z$, and hence surjectivity of φ. Since $\varphi = \widetilde{\varphi} \circ q$, it follows that $\widetilde{\varphi}$ is also surjective.
For injectivity, we note that the only points of $z \in S^{n+m+1}$ that have $|\varphi^{-1}(z)| > 1$ are those

points such that $x_z = 0$ or $y_z = 0$, each having preimages $\{x_z\} \times S^m \times \{0\}$ and $S^n \times \{y_z\} \times \{1\}$. These are precisely those sets that $\sim$ identifies, and hence $\widetilde{\varphi}$ is also injective, and thus a bijective continuous map. Finally, note that $S^n * S^m$ is a compact space, as being image of a compact space, while S^{n+m+1} is compact. By the compact-Hausdorff lemma, this continuous bijective map is actually a homeomorphism, and thus $S^n * S^m \cong S^{n+m+1}$. □

The following lemma is the main reason why we included this section in this text:

Lemma 6.3. *For any topological space X, $\Sigma X \cong S^0 * X$. By associativity of the join operator, it then follows that $\Sigma^2 X \cong (S^0 * S^0) * X \cong S^1 * X$*

Proof. Write $S^0 = \{\alpha, \beta\}$. The proof follows easily after unraveling what $\sim$ does in this particular setting. First, $S^0 \times X \times \mathbb{I}$ is homeomorphic to a disjoint union of two cylinders $X \times \mathbb{I}$, which we write as $(X \times \mathbb{I})_\alpha \sqcup (X \times \mathbb{I})_\beta$. Then, $\sim$ first identifies all points $(\alpha, x, 0) \sim (\alpha, x', 0)$ and $(\beta, x, 0) \sim (\beta, x', 0)$, so collapses the 0-th part of both cylinders to a point. Thus, we get a disjoint union of two cones over X. Then, we identify the points $(\alpha, x, 1) \sim (\beta, x, 1)$, i.e. we glue the full part of the cone together via the identity. The resulting space is clearly the suspension of X, as being homeomorphic to ${}^{(X \times [0,2])}\!\diagup_{(X \times \{0\}), (X \times \{2\})}$. Below is an illustration of the proof □

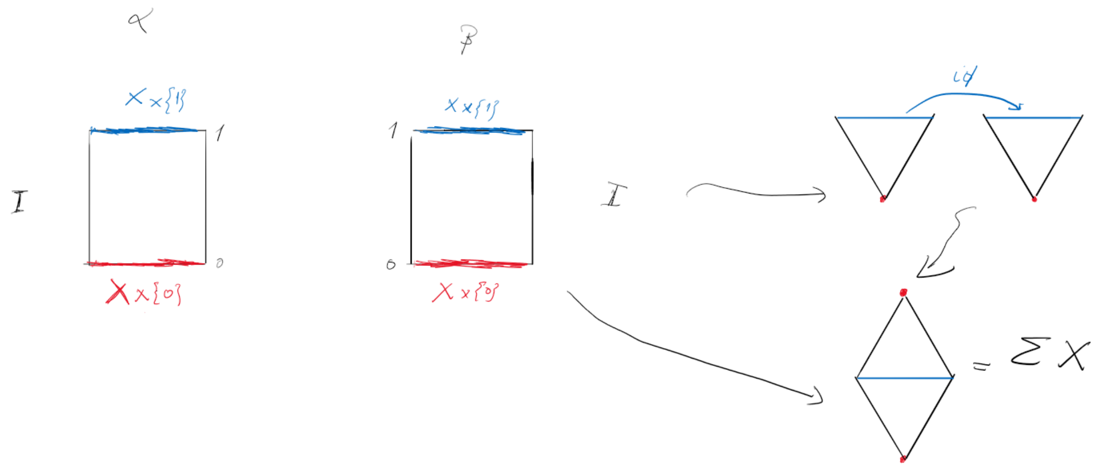

FIGURE 6. Visual representation of the above proof

In our situation, we note that $\Sigma^2 H^3 = {}^{(H^3 \times \mathbb{I} \times \mathbb{I})}\!\diagup_{\sim}$, where $\sim$ identifies the sets $\{H^3 \times \{0\} \times \{t\}\}_{t \in [0,1]}$, $\{H^3 \times \{1\} \times \{t\}\}_{t \in [0,1]}$, $H^3 \times \mathbb{I} \times \{0\}$ and $H^3 \times \mathbb{I} \times \{1\}$ each to a point. Define $\Gamma \subset \Sigma^2 H^3$ as being the following set, where q denotes the quotient map

$$\Gamma := q\Big(\bigcup_{t \in [0,1]} H^3 \times \{0\} \times \{t\}) \cup (\bigcup_{t \in [0,1]} H^3 \times \{1\} \times \{t\}) \cup (H^3 \times \mathbb{I} \times \{0\}) \cup (H^3 \times \mathbb{I} \times \{1\})$$

This set is called the *suspension circle*, for obvious reasons. Note that away from this set, the topology of the set $H^3 \times (0,1) \times (0,1)$ is unaltered, as the quotient does not affect it, so that on these points, we do have a manifold of dimension 5. By lemma 1, the proof of the double suspension theorem for the Mazur homology 3-sphere would follow from showing that the points in Γ are also manifold points, i.e. have neighborhoods homeomorphic to $\mathbb{R}^5$. This, however, is not as simple as it may sound. It will use the following Proposition 6.4, Theorem 4.2 (which is yet to proved), and a theorem by Bryant which we will be stated without proof.

Proposition 6.4. Γ *is locally homeomorphic to cone*$(H^3) \times \mathbb{R}$*, in the sense that for all* $x \in \Gamma$*, there exists a neighborhood* $\mathcal{U}$ *of* x *in* $\Sigma^2 H^3$ *such that* $\mathcal{U} \cong$ *cone*$(H^3) \times \mathbb{R}$

Proof. Let $\alpha \in \Gamma \subset \Sigma^2 H^3$, which is an element of the form $[x, t, s]$, where [] denotes the equivalence class under the relation $\sim$. In this setting, we need to distinguish two cases: $s \notin \{0, 1\}$, and $s \in \{0, 1\}$.
In the first case, we may assume $t = 0$, as the case $t = 1$ follows with the same proof. We view $X \times \mathbb{I} \times \{s\}$ as a copy of ΣX, along the transversal cut at s. Let $cone(x)$ be the cone $p(X \times [0, \varepsilon])$, for $0 < \varepsilon < 1$, where p is the quotient map $p : X \times \mathbb{I} \to \Sigma X$, centered at that x, which is clearly an open set containing $(x, 0)$. Since we assumed that $s \notin \{0, 1\}$, then the topology at that point coincides with the topology of $\Sigma X \times \mathbb{I}$. Let $\delta < \min(s, 1 - s)$, chosen so that the following open neighborhood avoids the suspension points, namely the open neighborhood of α given by $cone(x) \times (s - \delta, s + \delta)$. This is clearly the required one, as $cone(x) \cong cone(H^3)$ (by construction), and $(s - \delta, s + \delta) \cong \mathbb{R}$.

Let now $s \in \{0, 1\}$, i.e. α be one of the suspension points. Again, assume that $s = 0$, as the proof works, *mutatis mutandis*, for $s = 1$. Again, in this setting, $q(X \times \{\frac{1}{2}\} \times \mathbb{I})$ is a copy of ΣX; let $cone(x)$ be the cone in that copy of ΣX of the pole x. Then, $cone(x)$ is an open neighborhood covering the X and s factors in the product; taking $\mathcal{U} := cone(x) \times (\frac{1}{4}, \frac{3}{4})$, the result follows. We again include an illustration:

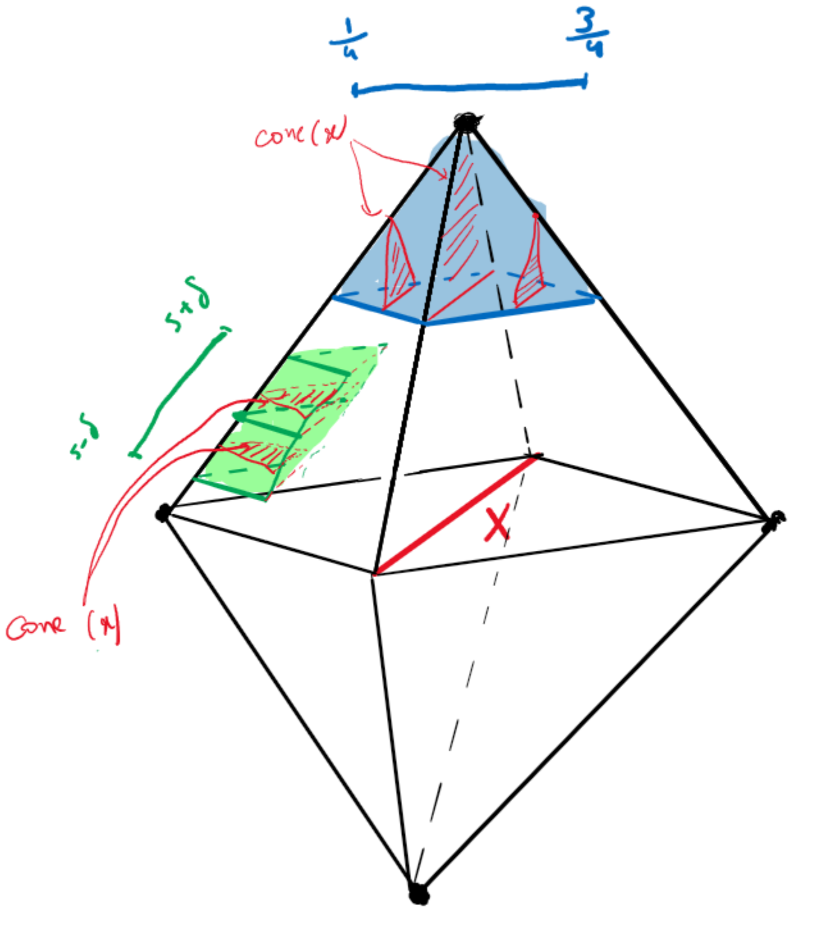

FIGURE 7. Local neighborhoods homeomorphic to $cone(H^3) \times \mathbb{R}$

The green neighborhood corresponds to the construction of the first case, while the blue one is that of second case. □

7 Proof of the Double Suspension Theorem for H^3

We again delay the proof of Theorem 4.2 above, which will be crucial in what follows. We quote the following result by Bryant, established in [Bry68].

Theorem 7.1. *If M^n is an n-manifold, and $D \subset int(M^n)$ is homeomorphic to D^k, for $k \leq n$, then* ${M^n}/{D} \times \mathbb{R} \cong M^n \times \mathbb{R}$

The proof of the double suspension theorem for H^3 follows easily from Proposition 6.4, Theorems 4.2 and 7.1; indeed, we have a homeomorphism $\varphi : W^4 \setminus B \xrightarrow{\cong} H^3 \times [0, 1)$, since the Giffen disc B is a pseudo-spine of W^4 by Theorem 4.2. Then, it follows that ${W^4}/{B} \cong cone(H^3)$, where the homeomorphism $\Phi : {W^4}/{D} \to cone(H^3)$ is defined as being the map φ on $W^4 \setminus B$, while mapping $[B]$ (the point onto which B collapses) to the coning point $[H^3 \times \{1\}]$ in $cone(H^3)$. Another way to see the above homeomorphism is to note that they are both the one point compactification of $H^3 \times [0, 1)$, and that one point compactification of Hausdorff locally compact spaces is unique up to homeomorphism.

By Lemma 2.1, it is sufficient to show that $\Sigma^2 H^3$ is a manifold, since it is clearly compact. As mentioned above, all points in $H^3 \setminus \Gamma$ are clearly manifold points, i.e. have neighborhoods homeomorphic to 5 dimensional Euclidean space. For points $x \in \Gamma$, Proposition 6.4 yields a neighborhood $x \in \mathcal{U} \subset H^3$ such that $\mathcal{U} \cong cone(H^3) \times \mathbb{R}$. From the above discussion, we note that $\mathcal{U} \cong {W^4}/{B} \times \mathbb{R}$. By Bryant's Theorem, i.e. Theorem 7.1 above, we get that $\mathcal{U} \cong W^4 \times \mathbb{R}$, which is itself a manifold of dimension 5. Thus, $\Sigma^2 H^3$ is locally 5-Euclidean. It is easy to check that $\Sigma^2 H^3$ is second countable and Hausorff, and consequently, $\Sigma^2 H^3$ is a 5 dimensional compact topological manifold, and thus by Lemma 2.1, $\Sigma^2 H^3 \cong S^5$.

8 Proof of Theorem 4.2

In this section, we give a sketch of the proof of Theorem 4.2. We begin by showing that $T(\sigma)$ collapses to Ω under an infinite sequence of collapses. To see this, consider $[0, 1] \times B_1^2 \times [0, 1]$, where $[0, 1] \times B_1^2$ is as in Figure 3. Then, we have the following collapse in the above set:

$$[0, 1] \times B_1^2 \times [0, 1] \searrow (L_1 \times [0, 1]) \cup ([0, 1] \times B_1^2 \times \{1\}) \cup (\{1\} \times B_1^2 \times [0, 1])$$

A very rough illustration is given in Figure 8.
Then, taking the image of the above collapse in the mapping torus, this would be the first step of the collapse $T(\sigma) \searrow \Omega$. That is, for $n \in \mathbb{N}$, where all the B_i^2, for $i \leq n$, have been collapsed to their underlying subcylinder, we consider the collapse

$$[n, n+1] \times B_{n+1}^2 \times [0, 1] \searrow (L_{n+1} \times [0, 1]) \cup ([n, n+1] \times B_{n+1}^2 \times \{1\}) \cup (\{n+1\} \times B_{n+1}^2 \times [0, 1])$$

Since $W^4 = (S^1 \times D^3) \cup_\varphi (D^2 \times D^2)$, we get, by the definition of collapses, that

$$W^4 \searrow (S^1 \times D^3) \cup (D^2 \times \{0\})$$

We thus get the following sequence of collapses:

$$W^4 \searrow (S^1 \times D^3) \cup_\varphi D^2 \times \{0\} \cong T(\sigma) \cup (D^2 \times \{0\}) \searrow \Omega \cup (D^2 \times \{0\}) = B$$

However, the last collapse here differs from our definition, as it is the composition of infinitely many collapses, namely the ones defined inductively in the above. However, this can be resolved

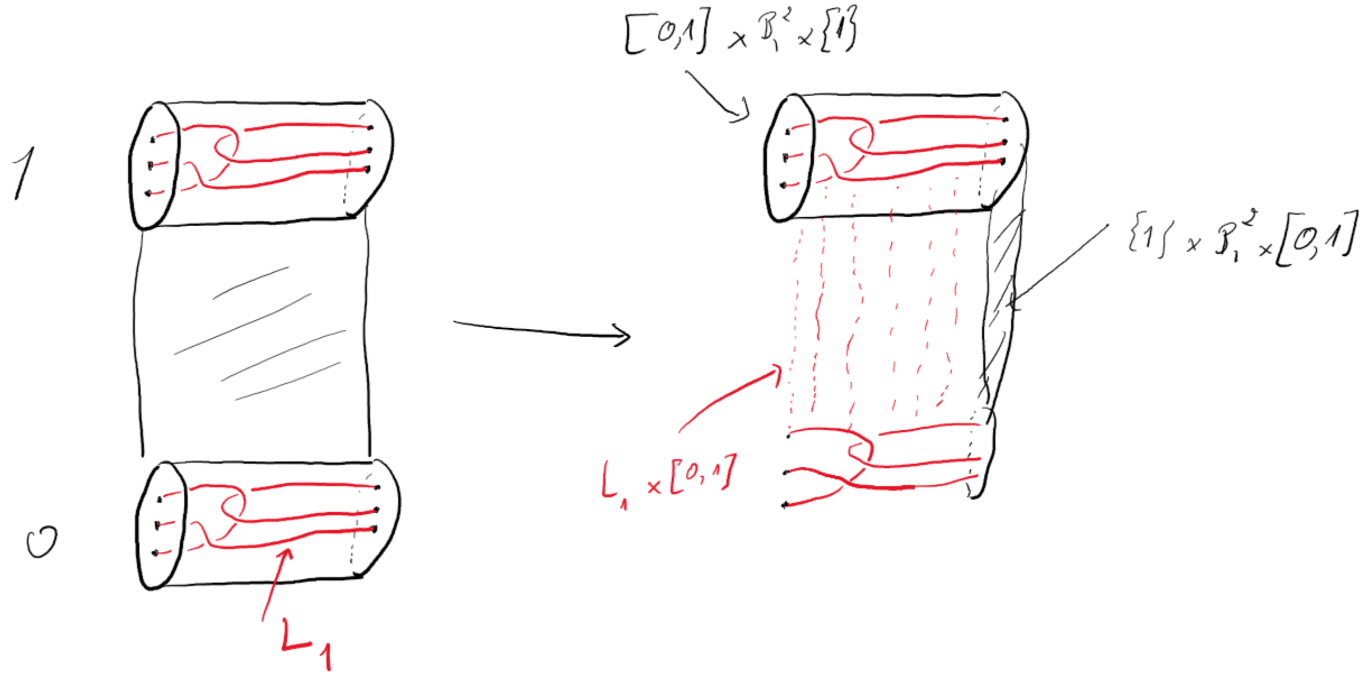

FIGURE 8. Sketch of the above collapse

via the regular neighborhood theorem. Indeed, we have written B as $B = \bigcap_{i \in \mathbb{N}} K_i$, where K_i are the sets above. Then, W^4 collapses to each K_i, as there are finitely many collapses connecting them. For any small enough regular neighborhood N_i of K_i, which exists by the regular neighborhood theorem, we have $N_i \setminus int(N_{i+1}) \cong \partial N_i \times [0,1]$. Then, we have the following equalities

$$W^4 \setminus B = W^4 \setminus (\bigcap_{i \in \mathbb{N}} K_i) = W^4 \setminus (\bigcap_{i \in \mathbb{N}} N_i) = \bigcup_{i \in \mathbb{N}} (W^4 \setminus N_i)$$

Since W^4 collapses to N_i, we have that $W^4 \setminus N_i \cong \partial W^4 \times [0,1)$; but since $N_i \setminus int(N_{i+1}) \cong \partial N_i \times [0,1]$, it follows that the above union is exactly $\partial W^4 \times [0,\infty) = H^3 \times [0,\infty)$; thus, B is indeed a pseud-spine, which concludes the proof of the double suspension theorem in the setting of the Mazur homology 3-sphere.

References

[Bre97] Glen E. Bredon. *Topology and Geometry*, volume 139 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997.
[Bry68] J. L. Bryant. Euclidean space modulo a cell. *Fundamenta Mathematicae*, 63:43–51, 1968.
[Can79] J. W. Cannon. Shrinking cell-like decompositions of manifolds. codimension three. *Annals of Mathematics*, 110(1):83–112, 1979.
[Coh] Ralph L. Cohen. The topology of fibre bundles.
[Dav87] Robert J. Daverman. *Decompositions of Manifolds*. Pure and Applied Mathematics. Academic Press INC. (London) LTD., London, 1987.
[Fer] Steve Ferry. Geometric topology notes.
[Hat01] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge, United Kingdom, 2001.
[KS79] R.C. Kirby and M.G. Scharlemann. *Eight faces of the Poincare homology 3-sphere*. Academic Press, 1979.

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