A NON PL-ABLE MANIFOLD

EKIN ERGEN

Contents

1 Introduction and outline

The main goal of this document is to construct a topological manifold that admits no PL structure. We present a construction due to Siebenmann [KS77]. An important step will be finding a PL automorphism $\alpha\colon D^2 \times T^n \to D^2 \times T^n$ that fixes boundary and satisfies certain properties. This automorphism will be used to create a TOP pseudoisotopy that Siebenmann referred to as a catastrophe, referencing French mathematician René Thom's catastrophe theory in a broader context [Tho74]. Having this pseudoisotopy, it will be easy to show that a certain manifold admits no PL structure.

Finding such an automorphism is not only nontrivial, but will yield constructions of some exotic manifolds as a byproduct. In addition, we will mention another counterexample in Section 4 that followed almost a decade later, by an additional discovery due to Freedman.

2 Constructing an automorphism α

In this section, we follow [KS77, Essay VI, Appendix B]. We will work in categories PL and DIFF, both of which we refer to as CAT as usual. The first goal is to come up with an explicit handle construction of an exotic manifold M that has analogous properties to α. This will allow us to create α, using the s-cobordism theorem on M that is regarded as an s-cobordism.

First, we want to recall some notions that define the 'exoticity' of a manifold. We shall start with the structure set.

Definition 2.1. The *structure set* $\mathcal{S}(M)$ of a manifold M^n is defined as the set of equivalence classes

$$\mathcal{S}(M) := \{(N^n, f\colon N \xrightarrow{\simeq} M)\} \big/ (h\text{-cobordism}) \tag{2.2}$$

As we shall consider maps that fix boundary throughout the section, we want to restrict ourselves to manifolds that are homotopy equivalent to M relative to boundary. That is, we want the boundary (and also a collar neighbourhood of it) to be fixed by the homotopy equivalence h. Therefore it is natural to consider the notion

(2.3)
$$\mathcal{S}(M^n \ \text{rel}\ \partial) := {\{(N^n, f\colon N \xrightarrow{\simeq} M) : f|_{\partial N\times[0,1]}\colon \partial N \times [0,1] \xrightarrow{\cong} \partial M \times [0,1]\}}\big/{(h\text{-cobordism})}$$

where $\partial N \times [0,1]$ resp. $\partial M \times [0,1]$ are to be understood as collar neighbourhoods of N resp. M.

The following is our main theorem, in which we construct a manifold M homotopy equivalent to $D^3 \times T^n$.

Theorem 2.4. *For $3 + n \geq 5$, there exists an element $[M^{3+n}, f]$ of $\mathcal{S}(D^3 \times T^n \ \text{rel}\ \partial)$ that is*

(1) nontrivial, i.e. $[M] \neq [D^3 \times T^n]$ [1];

(2) invariant under passage to standard finite coverings of $D^3 \times T^n$.

We first elaborate on the structure set itself as well as the exact meaning of the second claim.

Remark 2.5. Using previous knowledge from lectures [2] we can conclude that $\mathrm{Wh}(\pi(D^3 \times T^n)) = \mathrm{Wh}(\mathbb{Z}[\mathbb{Z}^n]) = 0$, so

$$\mathcal{S}(D^3 \times T^n \ \text{rel}\ \partial) = {\{(M,f)\}}\big/{(h\text{-cobordism})} \overset{\mathrm{Wh}=0}{=} {\{(M,f)\}}\big/{(s\text{-cobordism})} \underset{\text{thm.}}{\overset{\text{s-cob.}}{=}} {\{(M,f)\}}\big/{(\cong_{\mathrm{CAT}})}$$

meaning that the elements of the specific structure set $\mathcal{S}(D^3 \times T^n)$ are CAT-isomorphism classes of homotopy equivalent $3+n$-manifolds rel boundary.

If $[(M,f)] = [(M',f')] \in \mathcal{S}(D^3 \times T^n \ \text{rel}\ \partial)$, this means that there exists a CAT isomorphism $\varphi\colon M \to M'$ such that, restricting φ to ∂M, we obtain the commutative diagram

$$\begin{tikzcd}
\partial M \arrow[rr, "f|_{\partial M}"] \arrow[dr, "\varphi|_{\partial M}"] \arrow[dd, hook, "\mathrm{inc}"'] & & \partial(D^3 \times T^n) \arrow[dr, "\mathrm{Id}"] \arrow[dd, hook, "\mathrm{inc}'"] & \\
& \partial M' \arrow[rr, "f'|'_{\partial M}"] \arrow[dd, "\mathrm{inc}"] & & D^3 \times T^n \arrow[dd, "\mathrm{inc}'"] \\
M \arrow[rr, "f"] \arrow[dr, "\varphi"'] & & D^3 \times T^n \arrow[dr, "\mathrm{Id}"] & \\
& M' \arrow[rr, "f'"] & & D^3 \times T^n
\end{tikzcd}$$

Remark 2.6. In the second part of the theorem, we need to consider the pullback $\overline{p}\colon M' \to M$ of a covering map $p\colon D^3 \times T^n \to D^3 \times T^n$ along the homotopy equivalence $f\colon M \to D^3 \times T^n$ that comes from the tuple in the structure set, as shown in the next diagram.

$$\begin{tikzcd}
M' \arrow[r, "f'"] \arrow[d, "\overline{p}"'] & D^3 \times T^n \arrow[d, "p"] \\
M \arrow[r, "f"] & D^3 \times T^n
\end{tikzcd}$$

The second part of the theorem claims that $[(M,f)] = [(M',f')] \in \mathcal{S}(D^3 \times T^n \ \text{rel}\ \partial)$. The pullback of a covering map along any map is again a covering map, so the map $\overline{p}\colon M' \to M$ is

[1] From now on, we often suppress the homotopy equivalence in the notation, but we will indeed construct it as well.

[2] Topological Manifolds, winter semester 20/21, University of Bonn

indeed a covering map. Moreover, the pullback of a homotopy equivalence along a fibration (e.g. a covering map) is again a homotopy equivalence, so $f'\colon M' \to D^3 \times T^n$ is also a homotopy equivalence. As a result, (M', f') defines an element in the structure set $\mathcal{S}(M^n \text{ rel } \partial)$.

The proof of Theorem 2.4 will be covered by the following three subsections. The construction is due to A. Casson.

2.1 Construction of M

(1) Recall the Poincaré homology 3-sphere P^3: it is a closed manifold given by $SO(3)/A_5$, where A_5 denotes the rotational symmetry group of an icosahedron. This group is isomorphic to the alternating group on 5 elements. Therefore P can be interpreted as the group of the positions of a unit icosahedron up to translation and symmetry. As the name suggests, $H_n(P) = H_n(S^3)$ for all $n \in \mathbb{Z}$. The fundamental group $\pi_1(P)$ of P is isomorphic to the binary icosahedron group of order 120 given by the presentation

$$\langle a, b \mid (ab)^2 = a^3 = b^5 \rangle.$$

We will denote $\pi_1(P)$ by π. Note that, as an orientable 3-manifold, P is parallelisable.

(2) Define $P_0^3 := P^3 \# D^3$. This is homeomorphic to the Poincaré homology 3-sphere minus an open ball, since the connected sum is created by removing a 3-ball from P^3 and D^3 each, the latter becoming an annulus $S^2 \times D^1$. This is glued onto $P^3 \setminus D^3$ along $S^2 \times \{0\}$, which only adds another collar to $P \setminus \mathring{D}^3$. The boundary of P_0 is $\partial P_0 = S^2$ as P is closed.

(3) Take $[0,1] \times P_0 \times D^n$. To $1 \times P_0 \times D^n$, we will attach handles that kill homotopy groups:[3]

(a) We have $\pi_1(1 \times P_0 \times D^n) \cong \pi_1(P_0) \cong \pi_1(P^3) =: \pi$. The latter isomorphism follows by the Seifert-van Kampen Theorem for $D^3 \cup_{S^2 \times (-\varepsilon,\varepsilon)} P_0$. We can derive from the fibration $A_5 \to SO(3) \to P$ (where A_5 is a discrete Lie group) and its associated long exact sequence

$$\ldots \to \underbrace{\pi_1(A_5)}_{0} \to \pi_1(SO(3)) \to \pi_1(P) \to \pi_0(A_5) \to \underbrace{\pi_0(SO(3))}_{0} \to \ldots$$

we obtain the short exact sequence

$$0 \to \pi_1(SO(3)) \xrightarrow{i} \pi \xrightarrow{p} A_5 \to 0. \tag{2.7}$$

We know that $\pi_1(SO(3)) \cong \mathbb{Z}/2$. Take an element γ of π that is not in the subgroup $i(\mathbb{Z}/2)$ and consider the smallest normal subgroup $\langle\langle\gamma\rangle\rangle$ that contains γ. We want to show that $\langle\langle\gamma\rangle\rangle = \pi$. Note that $p(\langle\langle\gamma\rangle\rangle)$ is a normal subgroup of A_5. Because A_5 is simple, this image is either 0 or A_5. Because $\gamma \notin i(\mathbb{Z}/2)$, $p(\gamma) \neq 0$ by exactness and therefore $i(\langle\langle\gamma\rangle\rangle) = A_5$.This implies that $[\pi : \langle\langle\gamma\rangle\rangle] \in \{1, 2\}$. If $[\pi : \langle\langle\gamma\rangle\rangle]$ were 2, then the sequence 2.7 would be split exact. This yields a map $s\colon \pi \to \mathbb{Z}/2$ such that $s \circ i = \mathrm{Id}_{\mathbb{Z}/2}$. This induces maps between abelianisations $\mathbb{Z}/2 \xrightarrow{\bar{i}} \pi/[\pi,\pi] \xrightarrow{\bar{s}} \mathbb{Z}/2$ such that the composition is the identity. However, this is not possible: Using Hurewicz's Theorem, we see that $\pi/[\pi,\pi] \cong H_1(P) \cong H_1(S^3) = 0$. Therefore $[\pi : \langle\langle\gamma\rangle\rangle] = 1$ and $\pi = \langle\langle\gamma\rangle\rangle$.

We want to attach a 2-handle h_2 ($\cong D^2 \times D^{n+2}$) along this loop γ to make the resulting space simply connected. Up to homotopy, we have $|[S^1, SO(n+2)]| = 2$ choices for the attaching map $S^1 \times D^{n+2} \to S^1 \times D^{n+2}$, where the S^1-component of the target is the image of a loop representing γ. One of these choices indeed kills the fundamental group and gives rise to another parallelisable manifold: P_0 is parallelisable, as is $1 \times P_0 \times D^n$, so it has a trivial tangent bundle. Identifying this

[3] In Diff, we need to do smoothing of edges. We will not go into any detail of this, as the primary goal of the talk is the category PL anyway.

tangent bundle with the trivial tangent bundle of $\partial(D^2) \times D^{n+2}$ in a compatible way, we obtain another parallelisable manifold.
The resulting space is a simply connected CAT cobordism rel ∂ from $0 \times P_0 \times D^n$ to a simply connected CAT manifold Q. The homology of Q is the same as the homology of $D^{3+n}\#(S^2 \times S^{n+l})$, just as if P_0 were D^3.

(b) Now that one end of the cobordism is 1-connected, we want to achieve 2-connectivity. By 1-connectivity and Hurewicz's theorem,

$$\pi_2(Q) \cong H_2(Q) \cong H_2(D^{3+n}\#(S^2 \times S^{n+1})) \cong H_2(S^2 \times S^{n+1}) \cong \mathbb{Z}$$

so we can glue a 3-handle h_3 along a 2-sphere in $\operatorname{Int} Q$ that represents a generator δ of $\pi_2(Q)$ to make Q as well as the entire cobordism 2-connected. Here we use that $n+3 \geq 5$ to ensure that the attaching map can be embedded. Note that the handles are added to the interior, so the boundary has not changed.

(c) After adding h_2 and h_3, the resulting space is a CAT cobordism rel ∂ from $0 \times P_0 \times D^n$ to an $(n+3)$-manifold that we denote $(P_0 \times D^n)^\#$. A sketch of this construction is given in Figure 1.

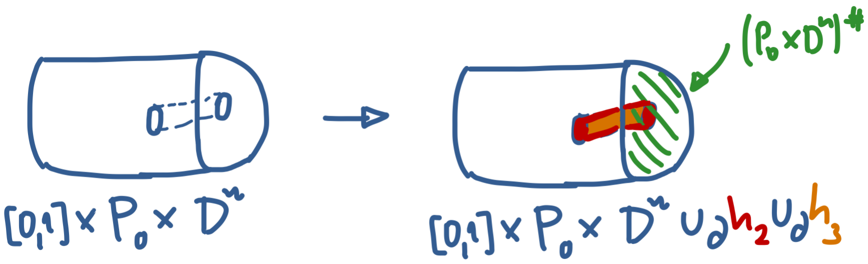


FIGURE 1. A visually inaccurate sketch of the construction of $(P_0 \times D^n)^\#$.

Claim. $(P_0 \times D^n)^\#$ *is contractible.*

Proof. We already know that $(P_0 \times D^n)^\#$ is 2-connected. Using

$$H_*(Q) = H_*(D^{3+n}\#(S^2 \times S^{n+1}))$$

we will apply Hurewicz's Theorem to see that all homotopy groups vanish, which implies that $(P_0 \times D^n)^\#$ is contractible by Whitehead's Theorem.
For most of the homology groups, the vanishing is obvious. Only possible nontrivial degrees could be 3 and $n+1$.

- To Q, we glue a 3-handle, i.e. a $D^3 \times D^{n+1}$ along the S^2-factor of the 2-handle (this is because the 2-handle generates the nontrivial homotopy group, as observed above). Doing so, no nontrivial third homology can be created.
- For degree $n+1$, we can use Poincare duality and universal coefficient theorem to see that $H_{n+1}((P_0 \times D^n)^\#) \cong H^3((P_0 \times D^n)^\#) \cong \operatorname{Hom}(H_3((P_0 \times D^n)^\#), \mathbb{Z}) = 0$.

As a result, $\pi_i(P_0 \times D^n) = 0$ for all $i \leq 1$ and therefore $(P_0 \times D^n)^\# \simeq \{\text{pt}\}$. □

By an analogous argument, we can see that the CAT cobordism $([0,1] \times P_0 \times D^n) \cup h_2 \cup h_3$ is also contractible.

(4) Identifying $D^n = [1/4, 3/4]^n$ and considering it as a subset of T^n by $[1/4, 3/4]^n \subset [0,1]^n/\sim = T^n$, we can include

$$([0,1] \times P_0 \times D^n) \cup h_2 \cup h_3 \hookrightarrow ([0,1] \times P_0 \times T^n) \cup h_2 \cup h_3 =: X^{n+4} \tag{2.8}$$

The attaching maps of the handles shall be the same as before. Again, X^{n+4} is a cobordism relative boundary from $0 \times P_0 \times T^n$ to

$$(P_0 \times T^n)^\# := (P_0 \times D^n)^\# \cup_\partial (P_0 \times (T^n \setminus \operatorname{Int} D^n))$$

which gives us the $(n+3)$-manifold M that we claim fulfills the desired properties in Theorem 2.4. In other words, we define $M^{n+3} := (P_0 \times T^n)^\#$. Figure 2 illustrates the inclusion (2.8).

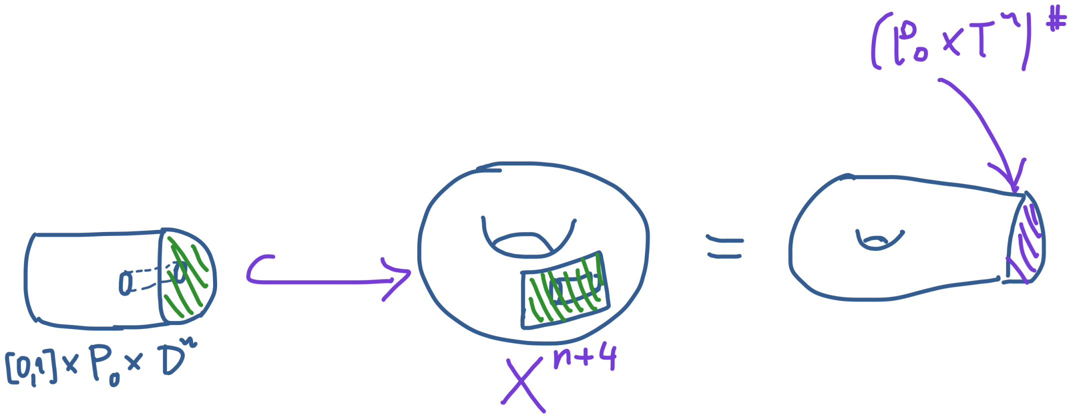


FIGURE 2. A visually even more inaccurate sketch of the construction of $M = (P_0 \times T^n)^\#$.

Claim. *M is homotopy equivalent to $D^3 \times T^n$ rel ∂.*

Proof. The boundaries of M and $D^3 \times T^n$ are homeomorphic because adding the handles has not changed the boundary. It is therefore immediate that the diagram in Remark 2.5 commutes.

For the homotopy equivalence, consider the pushouts

$$\begin{tikzcd}
P_0 \times D^n \arrow[r, hook] \arrow[d, hook, "i_1"] & (P_0 \times D^n)^\# \arrow[r, "\simeq"] \arrow[d, hook, "i_2"] & D^3 \times D^n \arrow[d, hook, "i_3"] \\
P_0 \times T^n \arrow[r, hook] & (P_0 \times T^n)^\# \arrow[r, "f"] & D^3 \times T^n
\end{tikzcd}$$

The top right homotopy equivalence can be defined as the composition $(P_0 \times D^n)^\# \to \mathrm{pt} \to D^3 \times D^n$. The inclusions i_2 and i_3 are cellular, and hence a cofibration. Therefore f is a homotopy equivalence. □

2.2 Invariance under coverings

Now we want to verify that $[M'] = [M]$ in the pullback in Remark 2.6. In other words, we want to show that M and M' are s-cobordant (which implies that they are isomorphic by the s-cobordism theorem).

Let $c\colon T^n \to T^n$ be a CAT covering map of degree d. We can consider the corresponding covering map of $X^{n+4} = [0,1] \times P_0 \times T^n \cup h_2 \cup h_3$, where the handles are glued onto $1 \times P_0 \times T^n$. That means, the total space $\widetilde{X}^{4+n}$ is a copy of $[0,1] \times P_0 \times T^n$ with d 2-handles and d 3-handles attached, all to $1 \times P_0 \times T^n$. Figure 3 provides a sketch of the construction of this total space.

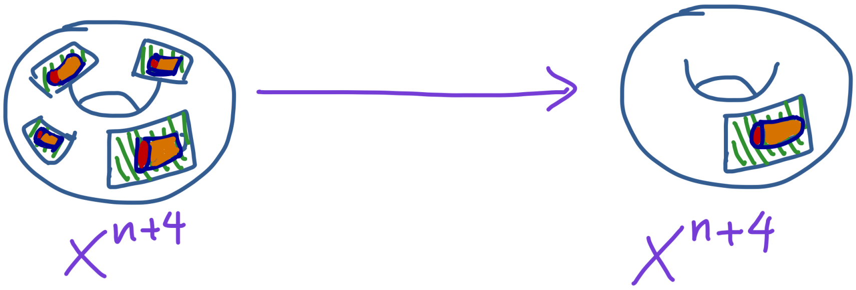


FIGURE 3. The torus-component of the covering space.

Let us glue X to $\widetilde{X}$ along their 0-ends (i.e. $0 \times P_0 \times T^n$) with the identity map to obtain Y^{n+4}. This is a CAT cobordism rel ∂ from M to M'. Moreover, Y is an h-cobordism as the union of two h-cobordisms. By applying Seifert-van Kampen's theorem on X and $\widetilde{X}$ at neighbourhoods of h_2 and copies of h_2, respectively, we see that $\pi_1(Y)$ is free abelian. Therefore $\tau(Y) = 0$.As a result, Y is an s-cobordism and $M \cong M'$ as desired.

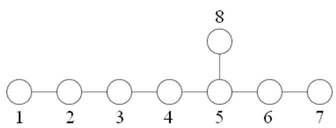
2.3 Interlude: Milnor's E_8 plumbing

We mention some concepts that will be key to obtaining contradictions in the next subsection.

Definition 2.9. [Bro69, Chapter V] Let ζ_i^n be a rank n vector bundle over an n-dimensional smooth manifold M_i for $i = 1, 2$. Let E_i be the total space of the associated disk bundle and suppose ζ_i, M_i and E_i are oriented in a compatible way. If we pick $x_1 \in M_1$ and $X_2 \in M_2$, and consider a ball neighbourhood of x_i in M_i, the preimage of these will be $D_i^n \times D_i^n$, neighbourhoods of the fiber over x_i. Let $h\colon D_1^n \to D_2^n$ and $k\colon D_2^n \to D_1^n$ be two diffeomorphisms, either both orientation preserving or both orientation reversing. Then we can define the *plumbimg* of the spaces E_1 and E_2 to be the quotient space $P = E_1 \cup_{(k,h)} E_2$.

Remark 2.10. One can inductively plumb more than two total spaces, as well as two different points in one space. In the first case, one can use graphs (in particular, trees) to determine the pairs of spaces that will be plumbed.

Definition 2.11. The Dynkin diagram E_8 looks like this:



Consider the disc bundle over S^2 with Euler number 2. We can plumb 8 copies of this bundle according to the Dynkin diagram given above to obtain *Milnor's E8 plumbing*, which we denote by P_{E_8}. As a smooth 4-manifold, this can also be considered as a PL manifold.

Remark 2.12. The intersection form on P_{E_8} is given by the matrix

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}$$

Rows and columns are identified with the enumeration in the figure.

This matrix is positive definite, therefore the signature $\sigma(P_{E_8})$ of P_{E_8} is 8. Moreover, it is unimodular. It is well-known that the map $H_2(P_{E_8}) \to H_2(P_{E_8}, \partial P_{E_8})$ from the long exact homology sequence can be identified with this intersection form because $H_1(P_{E_8})$ is torsion-free. As a result, $H_2(\partial P_{E_8}) = 0 = H_2(\partial P_{E_8})$. The boundary ∂P_{E_8} is connected and oriented, therefore it is a homology sphere. In fact, it is CAT isomorphic to the Poincaré homology sphere P^3.

2.4 Proof of nontriviality

Finally we show that $[M] \neq [D^3 \times T^n] \in \mathcal{S}(D^3 \times T^n \operatorname{rel} \partial)$, where the right hand side is represented by the canonical CAT structure on $D^3 \times T^n$. In other words, we want to prove that $M \not\cong D^3 \times T^n \operatorname{rel} \partial$.

For the sake of contradiction, we assume that $M \cong D^3 \times T^n \operatorname{rel} \partial$. Then we could glue M to $D^3 \times T^n$ along their common boundary to obtain $M_2 := M \cup_{S^2 \times T^n} D^3 \times T^n \cong S^3 \times T^n$.

Next, we consider $P_{E_8} \times T^n$. Its boundary is $\partial P_{E_8} \times T^n \cong P \times T^n$. To $P \times \{\mathrm{pt}\} \subset \partial(P_{E_8} \times T^n)$ we can attach handles as in the construction in Section 2.1. This way, we obtain a new space $P_{E_8} \times T^n \cup_\partial h_2 \cup_\partial h_3 =: V$.

Claim. $\partial V \cong_{\mathrm{CAT}} M_2$.

Proof. First, recall that $P_0 = P \# D^3$ which is CAT homeomorphic to P minus a 3-ball. This implies $P = P_0 \cup_{S^2} D^3$ and hence $\partial(P_{E_8} \times T^n) \cong P \times T^n \cong (P_0 \cup_{S^2} D^3) \times T^n \cong P_0 \times T^n \cup_{S^2 \times T^n} D^3 \times T^n$. The gluing of handles is identical as in the construction of M. Moreover, we may assume that the handles are attached to $P_0 \times T^n$, so if we consider ∂V as a cobordism relative boundary, this is equal $M \cup_{S^2 \times T^n} D^3 \times T^n$ by definition of M. □

Assuming $M_2 \cong S^3 \times T^n$ and using the CAT homeomorphism in the above claim as the attaching map, we can glue a $D^4 \times T^n$, which yields a closed CAT manifold W. To express the homotopy type of W, we introduce $E := {P_{E_8}}/{\partial P_{E_8}}$.

Claim. $W \simeq E \times T^n$.

Proof. $P_{E_8} \times T^n$ can include in both $W = V \cup_{M_2} D^4 \times T^n$ and $E \times T^n$. The remainder (i.e. the space that is glued to $Q \times T^n$ to yield the respective space) in W is $h_2 \cup h_3 \cup D^4 \times T^n$. Consider the map $f\colon W \to E \times T^n$ constructed as follows: We idenfify $\operatorname{Int} P_{E_8} \times T^n$ with $(E \setminus \{\mathrm{pt}\}) \times T^n$ by the inclusions of $\operatorname{Int} P_{E_8} \times T^n$ into both spaces as described above. The handles h_2 and h_3 are contracted to a point in $S^3 \times T^n \subseteq D^4 \times T^n$,which is then collapsed to $\{\mathrm{pt}\} \times T^n \subset E \times T^n$ by the contraction of the D^4 component. One can show that f induces isomorphisms under H_k for all k using the Mayer-Vietoris exact sequence.

If W and $E \times T^n$ were simply connected, Hurewicz's Theorem would directly imply that f is a homotopy equivalence; however, this is evidently not the case. Therefore we consider the lift $\tilde{f}\colon \widetilde{W} \to \widetilde{E \times T^n}$ of f along universal coverings of both spaces. The identifications of $P_{E_8} \times T^n$ are again identifications when lifted, and the collapses mentioned above also lift to nullhomologous maps. Using Mayer-Vietoris exact sequence, we can see that $H_k(f)$ is an isomorphism for all k. This implies $\pi_k(\widetilde{W}) \cong \pi_k(\widetilde{E \times T^n}) \cong \pi_k(W) \cong H_k(E \times T^n)$ for $k \geq 2$. Therefore, f is a homotopy equivalence. □

Theorem 2.13 (Farrell)**.** [Far67] *Let $f\colon M^m \to S^1$ be a map of compact CAT manifolds such that $f|_{\partial M}$ is a CAT locally trivial fibration. Then f is homotopic rel $f|_{\partial M}$ to a CAT locally trivial fibration if the following hold:*

(1) $\dim M \geq 6$.

(2) The covering $\widetilde{M}$ from the pullback

$$\begin{tikzcd} \widetilde{M} \arrow[r, "\bar{f}"] \arrow[d, "\bar{p}"'] & \mathbb{R} \arrow[d, "p"] \\ M \arrow[r, "f"] & S^1 \end{tikzcd}$$

where $p\colon \mathbb{R} \to S^1$ denotes the standard universal covering, has finite homotopy type.

(3) $\pi_1(M)$ is free abelian.

Note that condition (2) is assured if $M \simeq X \times S^1$ for X homotopy equivalent to a finite CW-complex. Using the proof of Claim 2.4 as well as the observation above, we observe that conditions (2) and (3) are satisfied for $W^{4+n} \xrightarrow{\simeq} E_8 \times T^n \xrightarrow{\text{proj}} S^1$ so that we can use Farrell's Theorem to promote this map to a CAT locally trivial fibration. Then we take a fiber W^{n+3}, which should factor through $E \times T^{n-1}$, again homotopy equivalent to W^{n+3}. Iterating this process, we can create a chain of closed subsets

$$W^{4+n} \supset W^{3+n} \supset \ldots \supset W^5 \simeq E^4 \times S^1. \tag{2.14}$$

Finally we need a 4-manifold X^4 to finish our claim, but cannot use Farrell's Theorem anymore, so we just make the map $W^5 \xrightarrow{\simeq} E \times S^1 \xrightarrow{\text{proj}} S^1$ transverse to 0 to obtain an orientable manifold $W^4 \subset W^5$ which we assert contradicts the following theorem due to Rochlin.

Theorem 2.15 (Rochlin)**.** *Every closed, oriented, smooth or PL 4-manifold W^4 with second Stiefel-Whitney class $w_2(W)$ zero has signature $\sigma(M) \in \mathbb{Z}$ divisible by 16.*

Next week's talk will be about the proof of this theorem. We are rather interested in deriving the following, which will lead to a contradiction:

Claim.
(1) $w_2(W^4) = 0$.
(2) $\sigma(W^4) = 8$.

Proof. (1) Recall that parallelisable manifolds have trivial Stiefel-Whitney classes. P_{E_8} is parallelisable by construction and hence so is $P_{E_8} \times T^n$. In particular, $w_2(P_{E_8} \times T^n) = 0$.

The map $j^*\colon H^2(W^{n+4}, \mathbb{Z}/2) \to H^2(P_{E_8} \times T^n, \mathbb{Z}/2)$ is injective, since the map

$$j_*\colon H_2(P_{E_8} \times T^n, \mathbb{Z}/2) \to H_2(W^{n+4}, \mathbb{Z}/2)$$

induced by $j\colon P_{E_8} \hookrightarrow W^{n+4}$ is surjective. Therefore the preimage of $w_2(Q \times T^n)$ is also 0. This is precisely $w_2(W^{n+4})$ by the naturality of Stiefel-Whitney classes.

Inductively, we can argue that each W^k has $w_2(W^k) = 0$ as follows: Consider the map $i^*\colon H^2(W^{n+1}, \mathbb{Z}/2\mathbb{Z}) \to H^2(W^n, \mathbb{Z}/2)$ induced by inclusion $i\colon W^n \hookrightarrow W^{n+1}$. Since W^n is bicollared in W^{n+1},

$$\begin{aligned} & \mathrm{T}W^n \oplus \varepsilon \cong \mathrm{T}W^{n+1}|_{W^n} \\ \Rightarrow \quad & w_2(\mathrm{T}W^n \oplus \varepsilon) \cong i^*(w_2(\mathrm{T}W^{n+1})) \\ \Rightarrow \quad & w_2(\mathrm{T}W^n) \cong i^*(w_2(\mathrm{T}W^{n+1})) \end{aligned}$$

As a result, $w_2(\mathrm{T}W^{n+1}) = 0$ implies $i^*(w_2(\mathrm{T}W^{n+1})) = w_2(\mathrm{T}W^n) = 0$, which yields the induction step. After finitely many steps, we reach $w_2(W^4) = 0$.

(2) Recall some properties of σ:
  (a) Signature is cobordism invariant,
  (b) $\sigma(\mathbb{C}P^2) = 1$ and therefore $\sigma(X \times \mathbb{C}P^2) = \sigma(X)$.

  So first, $\sigma(W^4) = \sigma(W^4 \times \mathbb{C}P^2)$. The latter space is cobordant to a space $V^8 \simeq \mathbb{C}P^2 \times P_*$. We obtain V^8 by applying Farrell's Theorem to the map $\mathbb{C}P^2 \times W^5 \xrightarrow{\simeq} \mathbb{C}P^2 \times P_* \times S^1 \to S^1$. The cobordism lies e.g. in the infinite cyclic covering of $\mathbb{C}P^2 \times W^5$. Therefore

$$\sigma(W^4) = \sigma(W^4 \times \mathbb{C}P^2) = \sigma(V^8) = \sigma(\mathbb{C}P^2 \times E) = \sigma(E) = 8$$

  The last equality follows because the intersection pairing of P_* is the matrix E_8. In fact P_* is an important example of why we use manifolds and not homology manifolds in Rochlin's Theorem.

$\square$

The above claims imply that our assumption $M \cong D^3 \times T^n \operatorname{rel} \partial$ cannot hold. We have found a representative for a nontrivial element in $\mathcal{S}(D^3 \times T^n \operatorname{rel} \partial)$!

2.5 Applications with the nontrivial element

After having found an exotic homotopy $D^3 \times T^n$ that we have called M^{3+n}, we want to proceed to produce further exotic spaces.

Theorem 2.16 (Exotic homotopy $S^3 \times T^n$)**.** *If we glue $D^3 \times T^n$ to M along the common boundary, we obtain*[4] *a CAT manifold homotopy equivalent to $S^3 \times T^n$ but not CAT isomorphic to $S^3 \times T^n$.*

Proof. The first assertion is clear with the canonical homotopy equivalence between the assembled homotopy equivalent parts, i.e. $M \cup_\partial D^3 \times T^n \simeq D^3 \times \cup_\partial D^3 \times T^n$. The fact that $M \cup_\partial D^3 \times T^n \ncong S^3 \times T^n$ follows by the above proof, as $M \cup_\partial D^3 \times T^n \cong S^3 \times T^n$ was assumed from the second step onwards, which has lead to a contradiction. $\square$

The following application can be found in [KS77].

Theorem 2.17 (Exotic homotopy torus)**.** *Identifying opposite ends of the three interval factors $D^3 \cong [0,1]^3$, we derive from M a CAT exotic homotopy T^{3+n}.*

Another interesting construction regarding $D^k \times T^n$, $k \leq 3$ is given in [Sie70, Section 5].

2.6 Finding α at last

Recall that we are looking for $\alpha\colon D^2 \times T^n \to D^2 \times T^n$ fixing boundary such that

(1) identifying the opposite endpoints of D^2 to obtain a torus T^2 induces a map $\bar{\alpha}\colon T^{n+2} \to T^{n+2}$ that has a mapping torus $T(\bar{\alpha}) := [0,1] \times T^{n+2}/((0\ x) = (1\ \beta(x)))$ not CAT isomorphic to T^{n+3}, and
(2) for any 2^n-fold standard covering map $p\colon D^2 \times T^n \to D^2 \times T^n$, the covering automorphism α' that comes from the lifting

$$\begin{array}{ccc} D^2 \times T^n & \xrightarrow{\alpha'} & D^2 \times T^n \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ D^2 \times T^n & \xrightarrow{\alpha} & D^2 \times T^n \end{array}$$

[4] We may need to smooth corners in DIFF.

is PL pseudoisotopic to α rel boundary. That is, there exists a PL automorphism H of $([0,1],\{0,1\}) \times D^2 \times T^n$ fixing $[0,1] \times \partial D^2 \times T^n$ such that $H_{0\times D^2\times T^n} = 0 \times \alpha$ and $H_{1\times D^2\times T^n} = 1 \times \alpha'$.

By Claim 2.1, $M = (P_0^3 \times T^n)^\#$ is homotopy equivalent to $D^3 \times T^n \operatorname{rel} \partial$, in particular, $\partial M \cong S^2 \times T^n$, so M can be seen as an h-cobordism relative boundary from $0 \times D^2 \times T^n$ to $1 \times D^2 \times T^n$. Moreover, the Whitehead torsion $\tau(M)$ vanishes, so that M is an s-cobordism. As $2 + n \geq 5$, the s-cobordism theorem gives rise to a PL homeomorphism $h\colon [0,1] \times D^2 \times T^n \xrightarrow{\cong} M$.

Note that by choosing $h|_{0\times D^2\times T^n}$ to be the identity by precomposition with $(\mathrm{Id}_{[0,1]} \times h|_{0\times D^2\times T^n})$, we induce another automorphism at the other end $1 \times D^2 \times T^n$, which we name $\alpha\colon D^2 \times T^n \to D^2 \times T^n$. Indeed, this map cannot have a mapping torus homeomorphic to T^{3+n}, which can be seen by Theorem 2.17.

Finally we need to show the second property. But we have almost established this in Section 2.2 : We have constructed an s-cobordism Y between M and a covering space M' induced by an arbitrary covering map of T^n. By s-cobordism theorem, this gives us a CAT isomorphism $k\colon M' \to M$. Following this construcion, the map α' can be created just like α, i.e. if we consider $M' \cong [0,1] \times D^2 \times T^n$ as an s-cobordism, this yields an isotopy h' from α' to $\mathrm{Id}_{D^2\times T^n}$. Concatenating h with h', we obtain a pseudoisotopy from α to α'.

3 The catastrophe

The next goal is to construct a PL pseudoisotopy $\operatorname{rel} \partial$ from α to $\mathrm{Id}_{D^2\times T^n}$. In Thom's terminology, this would be referred to as a catastrophe.

Let $p\colon D^2 \times T^n \to D^2 \times T^n$ be derived from scalar multiplication by 2. Define $\alpha_0 := \alpha$ and $H_0 := H$ as in the second property of α. Iteratively, pick α_i to be the lift of α_{i-1} and H_i to be the pseudoisotopy $\operatorname{rel} \partial$ from α_i to α_{i+1}. Here it is worth noting that, as $[M] = [M']$, the covers are always exotic and the maps α_i with mapping tori not PL homeomorphic to the standard T^{3+n}, by induction.

Next, define a PL automorphism H' of $[0,1) \times D^2 \times T^n$ as follows: for $a_k := 1 - \frac{1}{2^k}$, consider the oriented linear bijection $l_k\colon [a_k, a_{k+1}] \to [0,1]$. Let $H'(x,d,t) = H_k(l_k(x),d,t)$ for $x \in [a_k, a_{k+1}]$. As the H_k fix boundaries, this map is well defined for $x = a_k$ for some k.

Extend H' to $[0,1) \times \mathbb{R}^2 \times T^n$ by the identity (which is again possible because the boundaries are fixed).

Define $\phi\colon [0,1) \times D^2 \times T^n \to [0,1) \times D^2 \times T^n$ by $\phi(t,x,y) = (t,(1-t)x,y)$ Define another continuous bijection of $[0,1) \times D^2 \times T^n$ by $H'' := \phi H' \phi^{-1}$.

Claim. *H'' is a well-defined continuous bijection.*

Proof. Contunuity as well as bijectivity are obvious. We should see that $\phi^{-1}(t,x,y) = (t, \frac{1}{1-t}x, y)$. If $|x| \geq 1-t$, H' maps $\phi^{-1}(t,x,y)$ to itself by construction, so such points are fixed by H''. If $x \leq 1-t$, H' maps the second component to again something in D^2, and so does ϕ afterwards. This shows that the map is well-defined. □

The proof also shows that H'' fixes the boundary, setting $x = 1$ and $t = 0$.

The following figure from [Sie77] sketches the map H'' on $[0,1] \times D^2 \times T^n$, each factor shown by one dimension. Note that in the figure, the number of segments that are mapped via α, i.e. the "squares" that are marked with α are doubled for each k. If the figure were dimensionally accurate, they would be multiplied by 2^n instead.

Finally, we extend H'' to a bijection $H''\colon [0,1] \times D^2 \times T^n \to [0,1] \times D^2 \times T^n$ by $H''|_{1\times D^2\times T^n} = \mathrm{Id}_{1\times D^2\times T^n}$. It is immediate that bijectivity and well-definedness are preserved. Moreover, the domain is compact and the codomain Hausdorff, so that by the compact-Hausdorff argument[5],

[5] A continuous bijective map $f\colon A \to B$ where A is compact and B is Hausdorff is a homeomorphism.

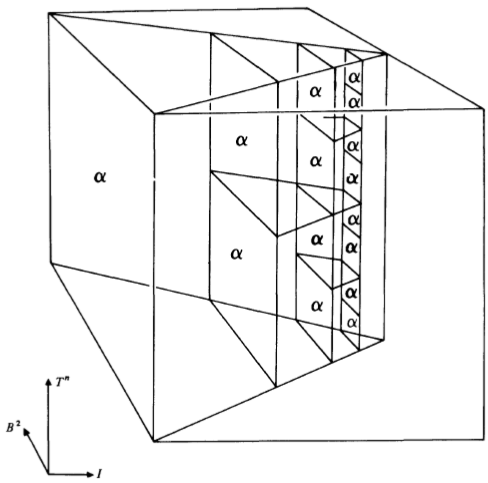


FIGURE 4. Schematic description of H''.

we only need to show that this map is continuous at $1 \times D^2 \times T^n$ in order to prove that it is a TOP homeomorphism.

Claim. *H'' is continuous at $1 \times D^2 \times T^n$.*

Proof. By construction, the part of the D^2-component of $[0,1] \times D^2 \times T^n$ that is not fixed by H'', which is $[a_k, a_{k+1}]$ shrinks strictly for $t \to 1$. Therefore, at $t = 1$, (t, x, y) is fixed everywhere with $x \neq 0$.

Let $(q_i)_{i \in \mathbb{N}}$ be a sequence of points converging to $q = (1, 0, y)$. Let p_i denote the projection onto the i-th component ($i = 1, 2, 3$). Then $p_i(H''(q_i)) \to p_i(H''(1, 0, y_*))$, showing the convergence in the first two factors.

To see the convergence in the third factor, let $\widetilde{H_k}$ be the lift

$$\begin{tikzcd}
{[0,1] \times D^2 \times \mathbb{R}^n} \arrow[r, "\widetilde{H_k}"] \arrow[d, "p"] & {[0,1] \times D^3 \times \mathbb{R}^n} \arrow[d, "p"] \\
{[0,1] \times D^2 \times T^n} \arrow[r, "H_k"] & {[0,1] \times D^2 \times T^n}
\end{tikzcd}$$

of H_k that fixes $[0,1] \times \partial D^2 \times \mathbb{R}^n$. For $z \in [0,1] \times D^2 \times \mathbb{R}^n$,

$$\sup |p_3(z) - p_3(\widetilde{H_k}(z))| =: d_k \tag{3.1}$$

is finite. Moreover, $\widetilde{H_k}$ can be expressed as $\theta_k^{-1} \circ \tilde{H}_0 \circ \theta_k$ with $\theta_k(t, x, y) = (t, x, 2^k y)$ by construction of the H_k. Therefore

$$|p_3(z) - p_3(\tilde{H}_k(z))| = |p_3(z) - p_3(\theta_k^{-1} \circ \tilde{H}_0 \circ \theta_k)(z)| \leq \frac{1}{2^k} d_0 \tag{3.2}$$

which can be seen by induction: $k = 0$ is obvious and conjugation by $\theta_k = \underbrace{\theta_1 \circ \cdots \circ \theta_1}_{k \text{ times}}$ means that the image will be shrunk by a factor of 2. As a result, we see that $d_k \to 0$ as $k \to \infty$. Passing to H_k, we see that $\lim p_3(H''(q_i)) = p_3(H_k(q_i)) \xrightarrow{k \to \infty} p_3(\lim q_i) = p_3(q) = p_3(H''(q))$, as desired. $\square$

We see that $H''|_{0 \times D^2 \times T^n} = 0 \times \alpha$ and $H''|_{1 \times D^2 \times T^n} = \mathrm{Id}_{D^2 \times T^n}$. In particular, this gives the pseudo-isotopy that we wanted at the beginning of the section. As a result of the s-cobordism theorem, we conclude $M_2 \cong_{\mathrm{TOP}} D^2 \times T^n$. However, this means that the construction of W^{n+4}

as in 2.4 carries through in the category TOP. In other words, the topological manifold W^{n+4} exists, but its PL-ability is a contradiction to Rochlin's theorem, so it is not PL-able.

4 Freedman's work 10 years later

So far, we have been able to construct a manifold that does not admit a PL structure. The key point to non-PL-ability was the contradiction to Rochlin's Theorem, where we used the homotopy type of the E_8 plumbing modulo boundary, which we called E. Instead of this, one might have been tempted to show analogous assertions for E. However, at the time Siebenmann described the above counterexample, it was not known whether P_* is indeed homotopy equivalent to a manifold.

Theorem 4.1. [Fre82] *Every homology 3-sphere bounds a fake 4-ball, i.e.a 4-dimensional, compact, contractible manifold.*

Using the theorem above, we can start with Milnor's plumbing $P^4_{E_8}$, which has boundary the Poincare homology sphere P^3. To P_{E_8}, we shall glue a fake 4-ball that also has P^3 as its boundary, along the boundary with the identity map. This way, we obtain a manifold E' homotopy equivalent to E. In particular, it is homotopy equivalent to a manifold. If this was known in 1970, one could have avoided the construction above by directly showing analogous claims to Claim 2.4 with E', instead of W^4. In other words, we immediately see that X has no CAT structure by Rochlin's Theorem. On the other hand, the construction we have presented is still considered the easiest, as proving Theorem 4.1 requires more technical work and knowledge.

An immediate consequence of the non-PL-ability of the manifold E' is that Rochlin's Theorem does not hold for the category TOP, as then E' is a spin topological manifold of dimension 4 that has signature 8. Again, it is worth noting that this was not known at the time [KS77] was published; indeed, it is noted in the aforementioned chapter that Rochlin's Theorem is undecided in the category TOP.

It is also worth noting that E' is also non-triangulable even without requiring the triangulation to be a PL triangulation, as shown in [Fre82].

References

[Bro69] W. Browder. *Surgery on Simply Connected Manifolds.* Department of Mathematics, Princeton University, 1969.

[Far67] F. Farrell. The obstruction to fibering a manifold over a circle. *Bulletin of the American Mathematical Society*, 73:737–740, 1967.

[Fre82] Michael Hartley Freedman. The topology of four-dimensional manifolds. *Journal of Differential Geometry*, 17(3):357 – 453, 1982.

[KS77] Robion C. Kirby and Laurence C. Siebenmann. *Essay VI, Foundational essays on topological manifolds, smoothings, and triangulations.* Princeton University Press, 1977.

[Sie70] Laurence Siebenmann. Disruption of low-dimensional handlebody theory by Rochlin's Theorem. *Topology of Manifolds*, 01 1970.

[Sie77] L. C. Siebenmann. Annex C. Topological Manifolds. *Foundational Essays on Topological Manifolds, Smoothings, and Triangulations. (AM-88)*, page 307–337, 1977.

[Tho74] René Thom. Stabilité structurelle et morphogenèse. *Poetics*, 3(2):7–19, 1974.

Universität Bonn
Email address: ekinergen@uni-bonn.de