1. CW Complex

1.1. **Cellular Complex.** Given a space X, we call *n-cell* a subspace that is homeomorphic to the interior of an n-disk.

Definition 1.1. Given a Hausdorff space X, we call *cellular structure* on X a collection of disjoint cells of X, say $\{\{e^n_\alpha\}_{\alpha\in\mathcal{A}_n}\}_{n\in\mathbb{N}}$ (where the $\mathcal{A}_n$ are indexing sets), together with a collection of continuous maps called *characteristic maps*, say $\{\{\Phi^n_\alpha\colon D^n_\alpha \to X\}_{\alpha\in\mathcal{A}_n}\}_{n\in\mathbb{N}}$, such that, if $X^n := \{e^k_\alpha | 0 \leq k \leq n\}$ (for all n), then :

(1) $X = \bigcup_{n\in\mathbb{N}} (\bigcup_{\alpha\in\mathcal{A}_n} e^n_\alpha)$,
(2) $\Phi^n_\alpha \restriction_{Int(D^n)}$ is a homeomorphism of $Int(D^n)$ onto e^n_α,
(3) and $\Phi^n_\alpha(\partial D^n) \subset \bigsqcup_{e^k_\alpha \in X^{n-1}} e^k_\alpha$.

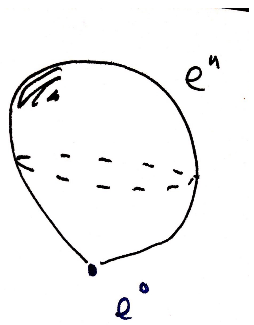
We shall call X together with a cellular structure a *cellular complex.*

Remark 1.2. We call $X^n \subset \{e^n_\alpha\}$ the n-skeleton of X. Also, we shall identify X^n with $\bigcup_{e^k_\alpha \subset X^n} e^k_\alpha$ and vice versa for simplicity of notations. So (3) becomes $\Phi^n_\alpha(\partial D^n_\alpha) \subset \bigcup X^{n-1}$.

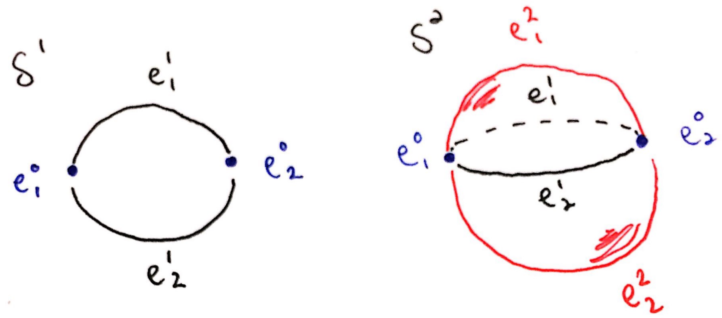
Since X is a Hausdorff space (in the definition above), we have that $\overline{e^k_\alpha} = \Phi^k_\alpha(D^n)$ (where the closure is taken in X). Indeed, we easily have that $\Phi^k_\alpha(D^n) \subset \overline{e^k_\alpha}$ by continuity of Φ^n_α. Moreover, since D^n is compact, we have that its image under Φ^k_α also is, and since X is Hausdorff, $\Phi^k_\alpha(D^n)$ is a closed subset containing e^k_α. Hence we have the other inclusion (Remember that the closure of a subset is the intersection of all closed subset containing it). In particular, this implies that $\overline{e^k_\alpha}$ is compact (and hence closed) for every cell e^k_α in its cellular structure.

Example 1.3. Any Hausdorff space X has a *trivial cellular structure.* Namely, we define $X^0 := X$ and $\Phi^0_x\colon \{x\} \to X$, for all $x \in X^0$.

Example 1.4. We can put on S^n the cellular structure consisting of a single 0-cell, say e^0, together with a single n-cell, say e^n. The associated characteristic maps are $\Phi^0\colon \{*\} \to S^n$ and $\Phi^n\colon D^n \to S^n$, the quotient map which sends ∂D^n to e^0.



Example 1.5. We can also put on S^1, S^2 and so on, the cellular structures represented on the following picture.



Definition 1.6. A subspace A of a cellular complex $(X, \{e_\alpha^n\}, \{\Phi_\alpha^n\}))$ is a *subcomplex* if it is a disjoint reunion of cells of X and is such that $\overline{e_\alpha^n} \subset A$, for all cell $e_\alpha^n \subset A$.

Example 1.7. Given a cellular complex X, each of its n-skeleton X^n is a subcomplex of X.

Example 1.8. In the Example 1.5 we can see S^1 as a subcomplex of S^2.

Obviously, if we define $A^n := \{e_\alpha^k \in X^n | e_\alpha^k \subset A\}$, for all n, the set $\bigcup_{n\in\mathbb{N}} A^n$ together with the corresponding characteristic maps corresponds to a cellular structure on A. Therefore, A with this cellular structure is a cellular complex in its own right. Consequently, by a subcomplex $A \subset X$, we shall always mean the cellular complex induced in this way together with the subspace topology. Note that if $A \subset X$ is a disjoint reunion of cells of X and a closed subspace of X, A is a subcomplex of X, by definition (since the closure of its cells will be included in its closure, which is itself). However the converse is not necessarily true (think of $\mathbb{R}$ with its canonical cellular structure: any open interval is a subcomplex).

Moreover, by definition of subcomplex, if we allow $\emptyset$ to be a subcomplex, we have that the intersection of any number of subcomplexes of X is again a subcomplex of X. Therefore, given a subset $E \subset X$, we can define $X(E)$ to be the subcomplex that corresponds to the intersection of all subcomplexes containing E. This leads us to defining the following property :

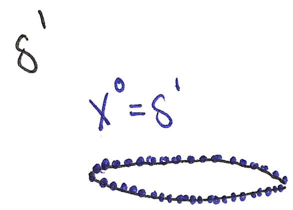
Definition 1.9. We say a cellular complex X is *closure finite* if $X(e)$ is a finite subcomplex for all cell e in the cellular structure of X.

Remark 1.10. If p is a point of X contained in the cell e of X, then $X(p) = X(e) = X(\overline{e})$.

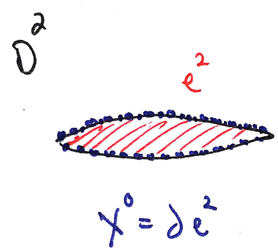
In another order of idea, the cellular structure on a space doesn't impose its topology to the whole space, just to the closure of the cells. Indeed, a subset $A \subset \overline{e_\alpha^n}$ is closed in $\overline{e_\alpha^n}$ if and only if $(\Phi_\alpha^n)^{-1}(A)$ is closed in D_α^n. Hence, the space doesn't necessarily have a nice topology relatively to its cells. However, there is a topology that naturally fits with the cellular structure.

Definition 1.11. We say a cellular complex $(X, (\{e_\alpha^n\}, \{\Phi_\alpha^n\}))$ has the *weak topology* relative to its cells, if $A \subset X$ is closed (open) if and only if $\overline{e_\alpha^n} \cap A$ is closed (relatively open) for each cell $\overline{e_\alpha^n}$, or equivalently, if and only if $(\Phi_\alpha^n)^{-1}(A)$ is closed (open) in D_α^n for every characteristic map.

Example 1.12. An example of cellular complex that is closure finite but doesn't have the weak topology is S^1 with the trivial cellular structure.



Example 1.13. An example of cellular complex that has the weak the weak topology but isn't closure finite is D^2 with a single 2-cell and the trivial cellular complex on ∂D^2.



Definition 1.14. Given a cellular complex X, we say it is a *CW complex* if it is closure finite and X has the weak topology relative to its cells.

1.2. **Subcomplexes.**

Proposition 1.15. *Given a CW complex X, $F \subset X$ is closed if and only if $F \cap A$ is closed in A for every finite subcomplex $A \subset X$.*

Remark 1.16. Since a finite subcomplex is compact (being a finite reunion of compact subspaces) in a Hausdorff space X, it is a closed subspace of X. Therefore, the preceding characterization of the weak topology says that $A \subset X$ is closed in X relative to the weak topology if and only if its intersection with the closure of each cell is a closed subspace of X.

Proof. Let $F \subset X$ be a closed subset of X. By definition of weak topology on X, we have that $F \cap \overline{e}$ is closed in X, for every cell e. In particular, for a finite subcomplex A, we have that $F \cap A = \bigcup_{e_\alpha^n \in A} F \cap \overline{e_\alpha^n}$, where the reunion is finite. Hence, it is a closed subset, being the finite reunion of closed subsets. Conversely, suppose $F \subset X$ is such that $F \cap A$ is closed, for all finite subcomplex $A \subset X$. Since X is closure finite, we have that $X(e)$ is a finite subcomplex, for all cell e. Therefore, we have that $(F \cap X(e)) \cap \overline{e} = F \cap \overline{e}$ is closed, for all cell e. □

Corollary 1.17. *Given a CW complex X, $U \subset X$ is open if and only if $U \cap A$ is relatively open for every finite subcomplex $A \subset X$.*

Corollary 1.18. *The 0-skeleton X^0, considered as a subspace of X, has the discrete topology.*

Corollary 1.19. *Let X be a CW complex and $A \subset X$ a subcomplex.*

(1) A is closure finite;
(2) The subspace topology of A coincide with its weak topology.

Proof. (1) Let $A \subset X$ be a subcomplex. Let e be a cell of A. Then we obviously have that $A(e) \subset X(e)$, since $A \cap X(e)$ is a subcomplex of A containing e. Therefore, since X is closure finite, we deduce that $A(e)$ is a finite subcomplex.
(2) Suppose $F \subset A$ is such that $F \cap A_0$ is closed in X for every finite subcomplex $A_0 \subset A$. Let $X_0 \subset X$ be a finite subcomplex of X. Then $A_0 := A \cap X_0$ is a finite subcomplex of A and we have that

$$F \cap A_0 = F \cap (X_0 \cap A) = F \cap X_0$$

is closed in X. By generality of the finite subcomplex X_0, F is closed in X and therefore $F \cap A = F$ is closed in the subspace topology of A. Conversely, suppose $F \subset A$ is closed in A. Then there exist $\tilde{F} \subset X$ such that $\tilde{F} \cap A = F$ and $\tilde{F} \cap X_0$ is closed for every finite subcomplex $X_0 \subset X$. Since every subcomplex of A is in particular a subcomplex of X, we have that $\tilde{F} \cap A_0 = F \cap A_0$ is closed in X for every finite subcomplex $A_0 \subset A$. □

In particular, part (2) of the proof above shows that every subcomplex of a CW complex is a closed subspace of X. Therefore, this gives a characterization of subcomplexes in a CW complex.

Corollary 1.20. *Given a CW complex X, a subspace $A \subset X$ is a subcomplex if and only if it is a closed subspace and the disjoint reunion of cells of X.*

1.3. **Product of CW complexes.** Given $(X, (\{e^n_\alpha\}, \{\Phi^n_\alpha\}))$ and $(Y, (\{\tilde{e}^n_\alpha\}, \{\tilde{\Phi}^n_\alpha\}))$ two cellular complexes, we define a cellular structure on $X \times Y$ by setting

$$(X \times Y)^n := \{e^k_\alpha \times \tilde{e}^l_\beta | 0 \leq k + l \leq n\},$$

for all n, and

$$\begin{array}{rcl} \Psi^{k+l}_{\alpha,\beta} \colon D^{k+l}_{\alpha,\beta} & \to & X \times Y \\ (x, y) & \mapsto & (\Phi^k_\alpha(x), \tilde{\Phi}^l_\beta(y)) \end{array}$$

for all, k, l, α and β. We easily see that it is a cellular structure, since we have $D^{k+l}_{\alpha,\beta} \equiv D^k_\alpha \times D^l_\beta$ topologically.

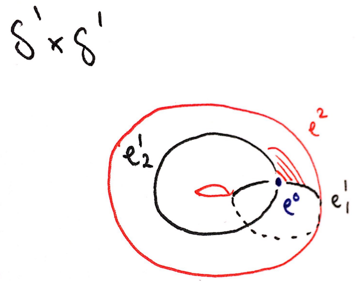
Moreover, since $X(e^k_\alpha) \times Y(\tilde{e}^l_\beta)$ is a subcomplex of $X \times Y$ containing $e^k_\alpha \times \tilde{e}^l_\beta$ (for any cell $e^k_\alpha \times \tilde{e}^l_\beta$ of $X \times Y$), we have

$$X \times Y(e^k_\alpha \times \tilde{e}^l_\beta) \subset X(e^k_\alpha) \times Y(\tilde{e}^l_\beta).$$

Therefore, if X and Y are closure finite, then $X \times Y$ with the above cellular structure is closure finite.

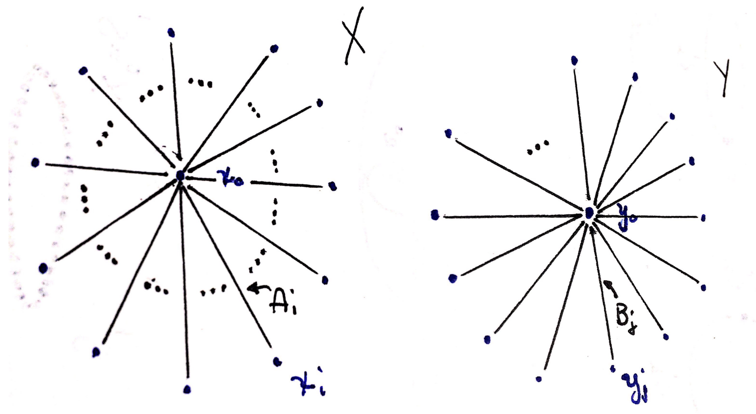
Remark 1.21. Given two cellular complex X and Y, we will always assume $X \times Y$ has the above cellular structure. We will call it the *product cellular structure* of $X \times Y$.

Example 1.22. The torus with the product cellular structure induced by S^1 as in Example 1.4.



Unfortunately, it is not necessarily true that $X \times Y$ has the weak topology if X and Y have it (meaning that the product topology on $X \times Y$ doesn't necessarily coincide with the weak topology of $X \times Y$ (relative to its product cellular structure), even if X and Y are CW complexes). The following counter-example is due to C. H. Dowker.

Counter-Example 1.23. Consider an uncountable collection of distinct points together with an uncountable collection of closed intervals, say $\{x_0\} \cup \{x_i\}_{i \in \mathbb{N}^{*\mathbb{N}}}$ and $\{A_i\}_{i \in \mathbb{N}^{*\mathbb{N}}}$ respectively, and a countable collection of distinct points with a countable collection of closed intervals, say $\{y_0\} \cup \{y_j\}_{j \in \mathbb{N}}$ and $\{B_j\}_{j \in \mathbb{N}}$ respectively. Consider the space X obtained by identifying one end of A_i to x_0 and the other to x_i, for all i, and put the obvious cellular structure on the set X. It can be shown that this is a CW complex (try to picture it for the moment, we shall see later that a space constructed in such a way is always a CW complex). Consider also the CW complex Y similarly constructed from the countable set of cells. Since each A_i and B_j are homeomorphic to $[0, 1]$, we can specify a point of A_i and a point of B_j, by specifying its value in $[0, 1]$ under some choosen homeomorphism (namely, under the implicit characteristic map or its reverse, so that x_0 and y_0 always correspond to 0).



Now, for each $(i, j) \in \mathbb{N}^{*\mathbb{N}} \times \mathbb{N}$, let p_{ij} be the point corresponding to $(\frac{1}{i_j}, \frac{1}{i_j})$ in $A_i \times B_j$ and define $P := \{p_{ij} | (i, j) \in \mathbb{N}^{*\mathbb{N}} \times \mathbb{N}\}$. We will show that P is closed in the weak topology of $X \times Y$, but not in the product topology. First of all, since $P \cap A_i \times B_j = \{p_{ij}\}$, for all $(i, j) \in \mathbb{N}^{*\mathbb{N}} \times \mathbb{N}$, we have that P is closed in the weak topology of $X \times Y$. Now, let us suppose that $U \times V$ is some subbasic open neighbourhood of (x_0, y_0) in the product topology. Since X has the weak topology, there exists $a_i > 0$ such that $\{t \in [0, 1] | t < a_i\}$ is an open subset of $U \cap A_i$, for all $i \in \mathbb{N}^{*\mathbb{N}}$. In other words, there exist a family $\{a_i\}_i \subset (0, 1]$ such that the reunion $\bigcup_i \{t \in [0, 1] | t < a_i\} \subset U$ is an open neighbourhood of x_0 in the weak topology of X. Similarly, there exists $b_j > 0$, for all $j \in \mathbb{N}$, such that $\bigcup_j \{t \in [0, 1] | t < b_j\} \subset V$ is an open neighbourhood of y_0 in the weak topology of Y. Hence, we have that

$$W := \bigcup_i \{t \in [0, 1] | t < a_i\} \times \bigcup_j \{t \in [0, 1] | t < b_j\} \subset U \times V$$

is an open neighbourhood of (x_0, y_0) in the product topology of $X \times Y$. If we choose $\hat{i} := (\hat{i}_1, \cdots, \hat{i}_k, \cdots) \in \mathbb{N}^{*\mathbb{N}}$ such that $\hat{i}_j > j$ and $\hat{i}_j > \frac{1}{b_j}$, for all $j \in \mathbb{N}$, and if we choose $\hat{j}$ such that $\hat{j} > \frac{1}{a_{\hat{i}}}$, then we have that $p_{\hat{i}\hat{j}} \in W$. In other words, we have just shown that, for every subbasic open neighbourhood $U \times V$ of (x_0, y_0), the intersection $U \times V \cap P$ is not empty. Hence, (x_0, y_0) is an accumulation point of P. Since (x_0, y_0) is not in P, P is not closed in the product topology. □

In what follows, we will show that *if X and Y are CW complexes and that at least one of them is locally compact, then $X \times Y$ has the weak topology (and is therefore a CW complex).* The result will follow from the theorem below (shown in class).

Theorem 1.24. *If X is compactly generated and Y is locally compact and Hausdorff, then $X \times Y$ is compactly generated.*

In order to deduce our result from this one, all we have to do is

(1) Show that if X and Y are CW complexes and $X \times Y$ is compactly generated, then $X \times Y$ has the weak topology.

(2) Show that all CW complexes are compactly generated.

Proposition 1.25. *If X is a CW complex, X is compactly generated.*

Proof. Suppose $F \subset X$ is closed in X. Since X is Hausdorff, its intersection with all compact subset of X is closed. Conversely, suppose $F \subset X$ is such that $F \cap C$ is closed for every compact subset $C \subset X$. Since every finite subcomplex is compact, F is closed in X. □

Now let us focus on the first of the two assertions above.

Lemma 1.26. *If X is a CW complex and $C \subset X$ is compact, then $X(C)$ is a finite subcomplex.*

Proof. Since X is closure finite, it suffices to show that every compact subset of X is contained in a reunion of cells.

Let us proceed by contradiction and suppose there exist a compact subset $C \subset X$ that intersect an infinite number of cells of X, a subset of which is $\{e_i\}_{i\in\mathbb{N}}$. Let x_i be a point in $C \cap e_i$, for all i, and let $P := \{x_i\}_i$. For every finite subcomplex $X_0 \subset X$, the intersection $P \cap X_0$ consist of a finite number of point and is therefore closed (since X is Hausdorff). Hence, P is a closed subset of X. Similarly, we can show that every subset of P is a closed subset. It follows that P has the discrete topology. Since it is closed and included in the compact subset C, it is also compact. Therefore P is finite. This is a contradiction. □

Proposition 1.27. *If X and Y are CW complexes and $X \times Y$ is compactly generated, then $X \times Y$ has the weak topology.*

Proof. Let $F \subset X \times Y$ be such that $F \cap A$ is closed in $X \times Y$ for all finite subcomplex $A \subset X \times Y$. Let $C \subset X \times Y$ be a compact subset and $C_i := pr_i(C)$, for $i = 1, 2$. Since the projections pr_i are continuous, the subsets C_i are compact. Hence, by the above lemma, $X(C_1)$ and $Y(C_2)$ are finite subcomplexes. Since $C \subset C_1 \times C_2$, we have that $(X \times Y)(C) \subset X(C_1) \times Y(C_2)$ and therefore $(X \times Y)(C)$ is a finite subcomplex of $X \times Y$. It follows by hypothesis on F that $F \cap (X \times Y)(C)$ is closed and therefore, since C is closed (X is Hausdorff), that

$$F \cap C = (F \cap (X \times Y)(C)) \cap C$$

is closed. By generality of the compact subset C and since X is compactly generated, F is closed. □

Hence, we have shown the desired theorem.

Theorem 1.28. *If X and Y are CW complexes and that at least one of them is locally compact, then $X \times Y$ has the weak topology and is therefore a CW complex.*

Example 1.29. For any CW complex X, we have that $X \times I$ is a CW complex, since I is a locally compact CW complex.

Corollary 1.30. *Let X and Y be CW complexes. If one of them is locally finite, $X \times Y$ is a CW complex.*

Proof. By the above theorem, it suffices to show that if Y a locally finite CW complex (meaning each point of Y has a neighbourhood intersecting only a finite number of cells of Y) then Y is locally compact. Suppose we are given a point $p \in Y$ and a neighbourhood V of p intersecting only a finite number of cells. Since the closure of V (in X) is contained in the reunion of the closure (in X) of the cells intersecting it, we have that $\overline{V}$ is a closed subset of this compact set. Hence, $\overline{V}$ will be a compact neighbourhood of p. □

1.4. **CW complexes are normal.** Next, we would like to show that every CW complex X is normal. In order to do that, we will show how to construct inductively an ϵ-neighbourhood of any closed subset of X. First, we fix $B \subset X$, any closed subset of X.

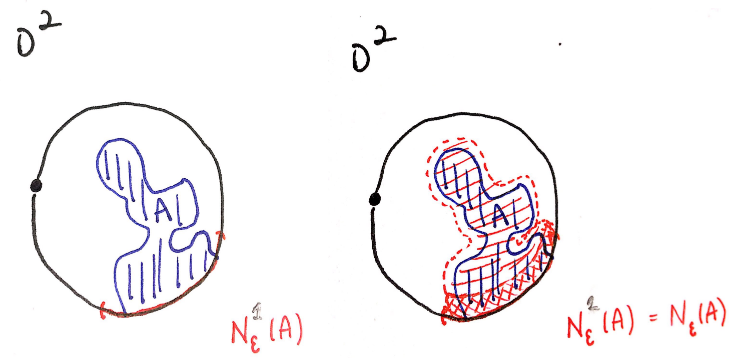
We start the construction by defining $N_\epsilon^0(B) := X^0$ (the set of 0-cells), which is open in X^0 (since it has the discrete topology). Supposing that we have defined an open neighbourhood of $B \cap X^{n-1}$ in X^{n-1}, say $N_\epsilon^{n-1}(B)$, we define $N_\epsilon^n(B)$ by specifying its pre-image relative to each charateristic map associated to a n-cell. Namely, for Φ_α^n, we want $(\Phi_\alpha^n)^{-1}(N_\epsilon^n(B))$ to be an ϵ-neighbourhood of $(\Phi_\alpha^n)^{-1}(B)\backslash\partial D_\alpha^n$ in $D_\alpha^n\backslash\partial D_\alpha^n$ together with

$$(1-\epsilon, 1] \times ((\Phi_\alpha^n)^{-1}(N_\epsilon^{n-1}(B)),$$

where we have assumed the 'spherical' coordinates on D_α^n (i.e. every points of $D_\alpha^n\backslash\{0\}$ express itself as $(r, x) \in (0, 1] \times \partial D_\alpha^n$). We then define

$$N_\epsilon(B) := \bigcup_n N_\epsilon^n(B).$$

This is an open set in the weak topology since it is relatively open in each cell by construction.



Remark 1.31. The notation used above is a bit unfortunate (but practical) since we could have chosen different $\epsilon = \epsilon_\alpha^n$ for every cell e_α^n and the resulting neighbourhood would still be open, by the same argument. We will exploit this construction in the next proposition.

Proposition 1.32. *If X is a CW complex, then X is a normal space.*

Proof. Let A, $B \subset X$ be two disjoint closed subsets. Suppose we have constructed disjoint ϵ-neighbourhoods $N^n_\epsilon(A)$ and $N^n_\epsilon(B)$ in X^n.

Let e^{n+1}_α be a $(n+1)$-cell of characteristic map $\Phi^{n+1}_\alpha\colon D^{n+1}_\alpha \to X^{n+1}$. Since $(\Phi^{n+1}_\alpha)^{-1}(A)$ and $(\Phi^{n+1}_\alpha)^{-1}(B)$ are disjoint compact subsets of the metric space D^{n+1}_α, we can find $\epsilon_A > 0$ and $\epsilon_B > 0$ such that the ϵ_A-neighbourhood of $(\Phi^{n+1}_\alpha)^{-1}(A)\backslash\partial D^{n+1}_\alpha$ in $D^{n+1}_\alpha\backslash\partial D^{n+1}_\alpha$, say V_{ϵ_A}, is disjoint from its homonym for $(\Phi^{n+1}_\alpha)^{-1}(B)\backslash\partial D^{n+1}_\alpha$, say V_{ϵ_B}. The problem that may occur is that the

$$(1-\epsilon_A, 1] \times ((\Phi^{n+1}_\alpha)^{-1}(N^n_\epsilon(A)))$$

part that we want to add (to proceed as in the construction above) may intersect V_{ϵ_B} (and vice versa changing the role of A and B). In order to avoid that, we only have to see that $(\Phi^{n+1}_\alpha)^{-1}(N^n_\epsilon(A))$ and B are a positive distance appart (and same thing for $(\Phi^{n+1}_\alpha)^{-1}(N^n_\epsilon(B))$ and A) so that, choosen small enough, $\epsilon_A > 0$ and $\epsilon_B > 0$ are such that $N^{n+1}_{\epsilon_A}$ and $N^{n+1}_{\epsilon_B}$ misses each other. If it wasn't the case (meaning that the distance from B to $(\Phi^{n+1}_\alpha)^{-1}(N^n_\epsilon(A))$ is zero), then we could find a sequence in $(\Phi^{n+1}_\alpha)^{-1}(N^n_\epsilon(A))$ converging to some point of $B \cap \partial D^{n+1}_\alpha$ (by compactness of ∂D^{n+1}_α). Therefore, we would have that $(\Phi^{n+1}_\alpha)^{-1}(N^n_\epsilon(A))$ intersect $(\Phi^{n+1}_\alpha)^{-1}(N^n_\epsilon(B))$, which is a contradiction of the induction hypothesis. □

1.5. **Construction of CW complex.**

Proposition 1.33. *Let X be a CW complex with cellular structure given by $(\{e^n_\alpha\}, \{\Phi^n_\alpha\})$. If we define $\phi^n_\alpha\colon \partial D^n_\alpha \to X^{n-1}$, for all n,α, by restricting Φ^n_α to ∂D^n_α, then we have*

$$X \cong (\bigsqcup_{\alpha,n} D^n_\alpha)/(x \sim \phi^n_\alpha),$$

where the space on the right has the quotient topology.

Proof. This is a simple application of the fact that X has the weak topology relative to its cells (Remember that if $q\colon \bigsqcup_{\alpha,n} D^n_\alpha \to (\bigsqcup_{\alpha,n} D^n_\alpha)/(x \sim \phi^n_\alpha)$ is the quotient map, then U is open in $(\bigsqcup_{\alpha,n} D^n_\alpha)/(x \sim \phi^n_\alpha)$ if and only if its pre-image is open in $\bigsqcup_{\alpha,n} D^n_\alpha$, or equivalently U is open if and only if $q^{-1}(U) \cap D^n_\alpha$ is open in D^n_α, for all n, α). □

For the sake of seing the concept of CW complex from a different perspective we would like to give a constructive, yet equivalent, way to define a CW complex. In the paragraph that follows, we describe the said construction and we establish that every cellular complex constructed in this way is indeed a CW complex. The converse is given by Proposition 1.33.

We start by defining X^0 to be any set of points with the discrete topology. Its underlying cellular structure is the trivial one. Then we suppose we already have defined its $n-1$-skeleton X^{n-1} together with its underlying cellular structure and that we are given a set $\{e^n_\alpha\}_\alpha$ of n-disks and a set of continuous maps (called attaching maps) $\{\phi_\alpha\}_\alpha$ of the form $\phi^n_\alpha : \partial D^n_\alpha \to X^{n-1}$. We then define

$$X^n := (X^{n-1} \bigsqcup_\alpha D^n_\alpha)/(x \sim \phi_\alpha(x))$$

with the quotient topology. The cells of X^n are the ones in X^{n-1} together with the image of $Int(D^n_\alpha)$ (which we shall denote e^n_α) under the characteristic map Φ^n_α defined as the following composition :

$$\Phi^n_\alpha : D^n_\alpha \longrightarrow X^{n-1} \bigsqcup_\alpha D^n_\alpha \longrightarrow X^n$$

[To actually be able to talk about cells, we need to show that Φ^n_α is a homeomorphism of $Int(D^n_\alpha)$ onto its image in X^n. Lets investigate the characteristic maps defined above. The first map is the inclusion. It is obviously a homeomorphism onto its image when restricted to D^n_α (and in particular when restricted to $Int(D^n_\alpha)$). The second map is the quotient map which is injective on $Int(D^n_\alpha)$ and therefore a homeomorphism onto its image by definition of the quotient topology.] We can continue this process of *adjunction of cells* as long as we want and in the end we define $X := \bigcup_n X^n$ with the weak topology relative to the subsets $\Phi^n_\alpha(D^n_\alpha) = \overline{e^n_\alpha}$.

We would like to see that a space constructed in such a way is indeed a CW complex, but first, to be able to speak of cellular structure on the space X, we have to see that X is Hausdorff. This is simple since the proof given above that *CW* complexes are normal spaces also give that the space X is normal (and Hausdorff). Therefore, to show that X is a CW complex, it suffices to show that such a space X is closure finite.

Proposition 1.34. *The cellular complex X is closure finite.*

Proof. Let us show by induction on the dimension that any cell is contained in a finite subcomplex. The case of the 0-cells is trivial. Suppose that every $n-1$-cell is contained in a finite subcomplex. By an argument similar to the one given above, we can show that every compact subset of X is contained in a finite number of cells. Therefore, for any cell e^n_α, since ∂e^n_α is compact, it is contained in a finite number of cells of dimension $n-1$. By the induction hypothesis, we have our result. □

1.6. **Homotopy Extension Property.**

Proposition 1.35. *Let X be a CW complex. A map $f\colon X \to Y$ is continuous if and only if $f \restriction_{\overline{e^n_\alpha}}$ is continuous for every cell e^n_α.*

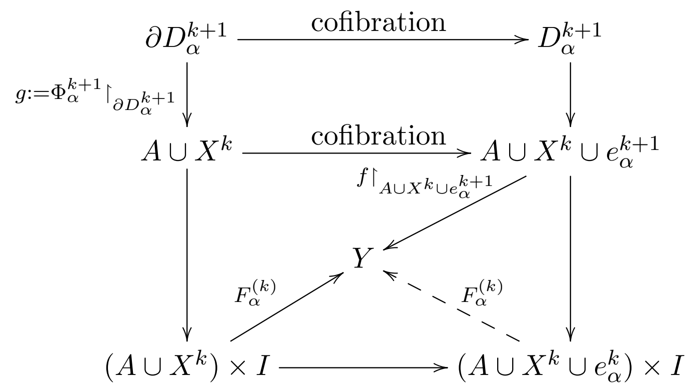
Corollary 1.36. *Let X be a CW complex. A map $F\colon X \times I \to Y$ is continuous if and only if $F \restriction_{\overline{e^n_\alpha}\times I}$ is continuous for every cell e^n_α.*

Proof. Since I is locally compact and Hausdorff, F is continuous if and only if the adjoint map $\overline{F}\colon X \to Y^I$ is continuous. Since X has the weak topology, this means that $\overline{F}$ is continuous if and only if $\overline{F} \restriction_{\overline{e^n_\alpha}}$ is continuous for every cell e^n_α. Finally, since I is locally compact and Hausdorff, $\overline{F} \restriction_{\overline{e^n_\alpha}}$ is continuous if and only if $F \restriction_{\overline{e^n_\alpha}\times I}$ is continuous, for every e^n_α. □

Theorem 1.37. *Let X be a CW complex and $A \subset X$ be a subcomplex. Then the inclusion $i\colon A \to X$ is a cofibration.*

Proof. Let $f\colon X \to Y$ and $F\colon X \times I \to Y$ be continuous maps such that $f \restriction_A = F \restriction_{A\times\{0\}}$. Suppose we have constructed a continuous map $F^{(k)}\colon (A \cup X^k) \times I \to Y$, for some $k \geq -1$, where $F^{(-1)} := f \restriction_A$ and $X^{-1} := \emptyset$, and $F^{(k)} \restriction_{(A \cup X^k) \times \{0\}} = f \restriction_{A \cup X^k}$, for $k > -1$.

Let e_α^{k+1} be a random $k+1$-cell of X. Since the inclusion $\partial D_\alpha^{k+1} \to D_\alpha^{k+1}$ is a cofibration (by an exercice of the problem sheet 1) and since the push-outs of a cofibration is a cofibration (by a proposition in the course), we can extend $F^{(k)}$ to a continuous map, say $F_\alpha^{(k)}$, on $((A \cup X^k) \cup e_\alpha^{k+1}) \times I$ such that $F_\alpha^{(k)} \restriction_{(A \cup X^k) \times I} = F^{(k)}$ and $f \restriction_{A \cup X^k \cup e_\alpha^{k+1}} = F_\alpha^{(k)} \restriction_{(A \cup X^k \cup e_\alpha^{k+1}) \times \{0\}}$.



[Let us note that we used the fact that there is an homeomorphism between $(A \cup X^k) \cup_g D_\alpha^{k+1}$ and $A \cup X^k \cup e_\alpha^{k+1}$ in the diagram above.] Hence, we can take these extensions F_α^k, for all e_α^{k+1}, and put them together to define

$$\begin{array}{rcl} F^{k+1} \colon (A \cup X^{k+1}) \times I & \to & Y \\ (x,t) & \mapsto & F_\alpha^k(x,t) \end{array},$$

for $x \in A \cup X^k \cup e_\alpha^{k+1}$. This is a continuous map by corollary 1.36 and it satisfies $F^{k+1} \restriction_{(A \cup X^k) \times I} = F^k$ and $F^{k+1} \restriction_{(A \cup X^{k+1}) \times \{0\}} = f \restriction_{A \cup X^{k+1}}$, by construction. Therefore, we can suppose we have such extensions of F for all $k \geq -1$ and define $\tilde{F} \colon X \times I \to Y$ has the map $\tilde{F}(x,t) := F^k(x,t)$, for $x \in X^k$. This is the extension we were looking for. $\square$

Remark 1.38. For an alternate proof, Hatcher shows in his book that $X \times I$ (strongly) deformation retracts on $A \times I \cup X \times \{0\}$.

1.7. **Quotient of CW Pairs.** Let (X, A) be a CW pair (meaning that X is a CW complex and $A \subset X$ a subcomplex) where the cellular structure of X is given by $(\{e_\alpha^n\}, \{\Phi_\alpha^n\})$. We will define a cellular on X/A called the *quotient cellular structure* such that X/A together with this cellular structure is a CW complex.

First, consider the quotient map $q \colon X \to X/A$. It is obvious that q is a bijection (and hence an homeomorphism) of $X \backslash A$ onto X/A. Hence, if we compose Φ_α^n with q, where the associated n-cell e_α^n is not in A, it gives us a map $\tilde{\Phi}_\alpha^n \colon D_\alpha^n \to X/A$ that is still a homeomorphism when restricted to $Int(D_\alpha^n)$. Therefore, we say that the following cells are the cells of the quotient cellular structure of X/A

$$\{\tilde{\Phi}_\alpha^n(Int(D_\alpha^n))\}_{(n,\alpha) \in \{n,\alpha | e_\alpha^n \subset X \backslash A\}} \sqcup e^0,$$

where $e^0 := q(A)$, and that the associated characteristic maps are

$$\{\tilde{\Phi}^n_\alpha\}_{(n,\alpha)\in\{n,\alpha|e^n_\alpha\subset X\setminus A\}} \sqcup \{\tilde{\Phi}^0 : \{*\} \mapsto q(A)\}.$$

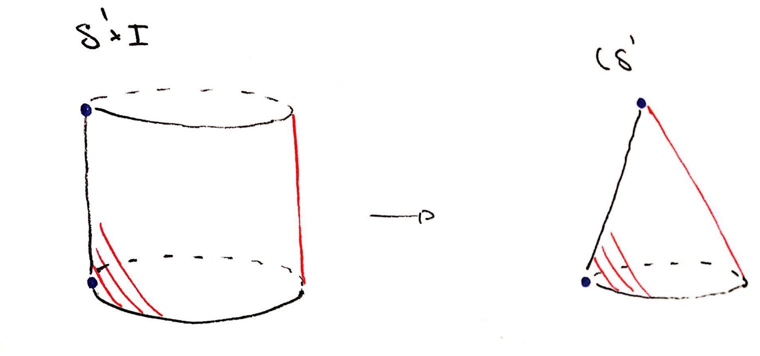
If we see X/A as the quotient of the quotient space $X = (\bigsqcup_{n,\alpha} D^n_\alpha)(x \sim \phi^n_\alpha)$, we see that X/A is indeed a CW complex (Exercice).

Example 1.39. Consider the CW pair $(X, A) := (S^n \sqcup S^m, \{*_{S^n}, *_{S^m}\})$, where S^k has the cellular structure of Example 1.4 (for $k = n, m$) and $*_{S^k}$ denotes the unique 0-cell in the cellular structure of S^k (for $k = n, m$). The quotient space X/A is the wedge $S^n \vee S^m$. Therefore the wedge of the sphere can be seen as a CW complex with the cellular structure defined above.

Example 1.40. Consider the CW pair $(X, A) := (S^n \times S^m, S^n \vee S^m)$, where the cellular structure of S^k is the one of Example 1.4. The quotient space X/A is the smash product $S^m \wedge S^m$, and by looking closely to the cellular structure defined above, we see that is is S^{n+m}. More importantly, this also means that $S^n \wedge S^m$ is a CW complex with the cellular structure defined above.

By generalizing the argument of Example 1.39 we can actually show that the wedge of two CW complex is always a CW complex (when the wedge is done by identifying two 0-cells and when we put the quotient cellular structure on it). Similarly, if reconsider Example 1.40, when one of the space is locally compact and Hausdorff (since the product of two CW complex is a CW complex with its product cellular structure (defined above)) we can generalize the argument to show that the smash product is a CW complex (when the wedge is a subcomplex of the product and when the smash product has the quotient cellular structure). Another nice example is the following.

Example 1.41. Given a CW complex X, the product $X \times I$ (with its product cellular structure) is a CW complex since I is locally compact and Hausdorff. Hence, since $X \times \{1\}$ is a subcomplex of $X \times I$ (Exercice) we have that the pair $(X \times I, X \times \{1\})$ gives us a CW complex structure on the cone CX.



References

[Dow] C. H. Dowker, *Topology of Metric Complexes*, (1950).
[Hat] Allen Hatcher, *Algebraic Topology*, Cambridge University Press (2002).

[Wh] J. H. C. Whitehead, *Combinatorial Homotopy*, (1946).