Arf Invariant

Content of the talk

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Aim.

1. Define the Arf invariant;
2. Prove it is well defined;
3. Show that $L_2(\mathbb{Z}) \cong \mathbb{Z}/2$.

Remark. Every fact used are a result of Lemme 4.80, p.152 in the Surgery Book [1], except if said otherwise.

Let P be a finitely generated projective $\mathbb{Z}$-module, let (P, λ) be a nonsingular (-1)-symmetric form over $\mathbb{Z}$ and let $\mu \colon P \to Q_{(-1)}(\mathbb{Z}) = \mathbb{Z}/2$ be a quadratic refinement of (P, λ). For every $x \in P$, we have that $\lambda(x, x) = -\lambda(x, x)$. This allows us to state the following fact:

Fact 1. Every nonsingular (-1)-symmetric from (P, λ) admits a symplectic basis $\{e_1, f_1, ..., e_m, f_m\}$, i.e. $\lambda(e_i, f_i) = 1$, $\lambda(e_i, f_j) = 0$ for $i \neq j$ and $\lambda(e_i, e_i) = 0 = \lambda(f_i, f_i)$.

Definition. The Arf invariant of (P, λ, μ), denoted by $Arf(P, \lambda, \mu)$, is defined by

$$Arf(P, \lambda, \mu) := \sum_{i=1}^{m} \mu(e_i)\mu(f_i) \quad \in \mathbb{Z}/2.$$

Remark. The Arf invariant is additive, i.e. for another quadratic form (P_1, λ_1, μ_1) aver $\mathbb{Z}$, we have that

$$Arf(P \oplus P_1, \lambda \oplus \lambda_1, \mu \oplus \mu_1) := Arf(P, \lambda, \mu) + Arf(P_1, \lambda_1, \mu_1).$$

Proposition. *The Arf invariant is well defined.*

Proof. This proof will be separated in two parts. We will first prove that the Arf invariant is well defined on nonsingular (-1)-quadratic forms over $\mathbb{Z}^2$ and then prove that it is well defined in general.

Part 1. The Arf invariant is well defined on forms over $\mathbb{Z}^2$. Let's prove that there are two isomorphisms classes of nonsingular (-1)-quadratic forms over $\mathbb{Z}^2$. The first is $H_-(\mathbb{Z})$. The second is defined as follow: let $\{e_1, f_1\}$ be a the basis for $\mathbb{Z}^2$ and define the second isomorphisms class to be $A(\mathbb{Z}) := (\mathbb{Z}^2, \lambda_A, \mu_A)$, where $\lambda_A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and μ_A is such that $\mu_A(e_1) = 1 = \mu_a(f_1)$.

Fact 2. Every nonsingular (-1)-symmetric form $(\mathbb{Z}^2, \lambda)$ is isomorphic to $H^-(\mathbb{Z})$.

Now let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$. We have that $det(X) = \pm 1$. However,

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} -c & a \\ -d & b \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} 0 & ad-bc \\ -ad+bc & 0 \end{pmatrix} \\ &= (ad-bc) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

Therefore, $det(X) = 1$. This and Fact 2 give us that $Aut(H^-(\mathbb{Z})) = SL_2(\mathbb{Z})$.

Fact 3. The following statement is not due to lemme 4.80 in the Surgery Book [1]: $Aut(H_-(\mathbb{Z})) \subset Aut(H^-(\mathbb{Z}))$ and it is a subgroup of index 3.

Since $Aut(H_-(\mathbb{Z}))$ is a subgroup of order 3, $H_-(\mathbb{Z})$ has 3 quadratic refinements. Adding the one of $A(\mathbb{Z})$, we have 4 quadratic refinements which we expose in the following table:

$x =$	(0,0)	(1,0)	(0,1)	(1,1)
$\mu_{H_1(x)}$	0	0	0	1
$\mu_{H_2(x)}$	0	0	1	0
$\mu_{H_3(x)}$	0	1	0	0
$\mu_{A(x)}$	0	1	1	1

The $\mu_{H_i(x)}$s are the quadratic refinements of $H_-(\mathbb{Z})$ and correspond to $Arf(H_-(\mathbb{Z})) = 0$, while $\mu_{A(x)}$ is the quadratic refinement of $A(\mathbb{Z})$ and corresponds to $Arf(A(\mathbb{Z})) = 1$. Since those four are the only quadratic refinements, the Arf invariant is well defined on nonsingular (-1)-quadratic forms on $\mathbb{Z}^2$.

We will need the following fact later in the proof, but stating it now is more convenient.

Fact 4. There are isomorphisms describing the following: $A(\mathbb{Z}) \oplus A(\mathbb{Z}) \cong A(\mathbb{Z}) \oplus -A(\mathbb{Z}) \cong H_-(\mathbb{Z}^2)$, where $-A(\mathbb{Z}) = (\mathbb{Z}^2, -\lambda_A, \mu_A)$.

Part 2. We need to prove that the Arf invariant is well defined on nonsingular (-1)-quadratic forms over $\mathbb{Z}$.

Fact 5. We have the equality $Arf(P, \lambda, \mu) = Arf(\mathbb{F}_2 \otimes (P, \lambda, \mu))$ and therefore, we can apply Fact 4 to $\mathbb{F}_2$:

$$A(\mathbb{F}_2) \oplus A(\mathbb{F}_2) \cong A(\mathbb{F}_2) \oplus -A(\mathbb{F}_2) \cong H_-(\mathbb{F}_2^2).$$

Let (Q, λ, μ) be a nonsingular (-1)-quadratic form over $\mathbb{F}_2$. We define another invariant of (Q, λ, μ), which we denote $Arf'((Q, \lambda, \mu))$, by:

$$Arf'((Q, \lambda, \mu)) := \begin{cases} 0 & \text{if } |\mu^{-1}(0)| > |\mu^{-1}(1)| \\ 1 & \text{if } |\mu^{-1}(1)| > |\mu^{-1}(0)| \end{cases}.$$

By looking at the table on last page, we see that the case $|\mu^{-1}(1)| = |\mu^{-1}(0)|$ cannot happen, therefore it is not treated. Note that Fact 5 allows us to see the previous listing of quadratic refinements as refinements of nonsingular (-1)-symmetric forms over $\mathbb{F}_2^2$ instead of over $\mathbb{Z}^2$. Thus, we have two isomorphism classes of nonsingular (-1)-quadratic forms over $\mathbb{F}_2^2$: $H_-(\mathbb{F}_2)$ and $A(\mathbb{F}_2)$. By looking once again at the table, we get that $Arf'(H_-(\mathbb{F}_2)) = 0$ and $Arf'(A(\mathbb{F}_2)) = 1$.

We now need to prove that for any nonsingular (-1)-quadratic form (Q, λ, μ), $Arf((Q, \lambda, \mu)) = 1$ if and only if $Arf'(Q, \mu) = 1$.

By looking at the table one last time, we see that

$$\begin{aligned} &|(\mu \oplus \mu_{H_i})^{-1}(1)| = 3|\mu^{-1}(0)| + |\mu^{-1}(1)| \quad \text{for } i = 1, 2, 3 \\ \text{and} \quad &|(\mu \oplus \mu_A)^{-1}(1)| = 3|\mu^{-1}(1)| + |\mu^{-1}(0)|. \end{aligned}$$

Hence, Arf' is additive, i.e. $Arf(Q \oplus Q_1, \lambda \oplus \lambda_1, \mu \oplus \mu_1) := Arf(Q, \lambda, \mu) + Arf(Q_1, \lambda_1, \mu_1)$, with $Q_1 = H_-(\mathbb{F}_2)$ or $A(\mathbb{F}_2)$.

Recall Fact 2: $(\mathbb{Z}^2, \lambda) \cong H^-(\mathbb{Z})$ for $(\mathbb{Z}^2, \lambda)$ a nonsingular (-1)-symmetric form. Using it on nonsingular (-1)-quadratic forms over $\mathbb{F}_2$, we get $(Q, \lambda) \cong H^-(\mathbb{Z}^r)$ for some r. Hence, $(Q, \lambda, \mu) \cong \bigoplus_{i=1}^{s} H_-(\mathbb{F}_2) \oplus \bigoplus_{i=1}^{t} A(\mathbb{F}_2)$, where $r = s + t$. Therefore, by the additivity of Arf' , we get $Arf'((Q, \lambda, \mu)) \cong t \bmod 2$. But, by Fact 5, we have that $A(\mathbb{F}_2) \oplus A(\mathbb{F}_2) \cong H_-(\mathbb{F}_2^2)$. Thus,

$$(Q, \lambda, \mu) \cong \begin{cases} H_-(\mathbb{F}_2^r) & \text{if } Arf'((Q, \lambda, \mu)) = 0 \\ A(\mathbb{F}_2) \oplus H_-(\mathbb{F}_2^{r-1}) & \text{if } Arf'((Q, \lambda, \mu)) = 1 \end{cases}.$$

Since Arf is also additive, we also have that $Arf((Q, \lambda, \mu)) \cong t \bmod 2$.

Let $B = \{e_1, f_1, ..., e_r, f_r\}$ be any symplectic basis for (Q, λ, μ).

$$\begin{aligned} Arf(Q, \lambda, \mu, B) = 1 &\Leftrightarrow t \cong 1 \bmod 2 \\ &\Leftrightarrow Arf'((Q, \lambda)) = 1. \end{aligned}$$

Hence, $Arf(Q, \lambda, \mu, B)$ is independent of B and Arf is well defined.

□

Theorem. $L_2(\mathbb{Z}) \cong \mathbb{Z}/2$

Proof. This proof will use many results of the one of last proposition.

Let (P, λ, μ) be a nonsingular (-1)-quadratic form over $\mathbb{Z}$. We have that $(P, \lambda) \cong H^-(\mathbb{Z}^r)$ for some r. Thus, $(P, \lambda, \mu) \cong \bigoplus_{i=1}^{s} H_-(\mathbb{Z}) \oplus \bigoplus_{i=1}^{t} A(\mathbb{Z})$, for $r = s + t$. Therefore,

$$(P, \lambda, \mu) \cong \begin{cases} H_-(\mathbb{Z}^r) & \text{if } Arf((Q, \lambda, \mu)) = 0 \\ A(\mathbb{Z}) \oplus H_-(\mathbb{Z}^{r-1}) & \text{if } Arf((Q, \lambda, \mu)) = 1 \end{cases}.$$

Hence, $L_2(\mathbb{Z}) \cong \mathbb{Z}/2$. □

Arf invariant of a knot

Let K be a knot and F its Seifert surface of genus g. Recall that a Seifert surface of a knot is an oriented connected surface with K as boundary.

It is possible to assign a sign to each crossing following their orientation: the crossing is either $+1$ or -1. The convention is represented on figure 1.

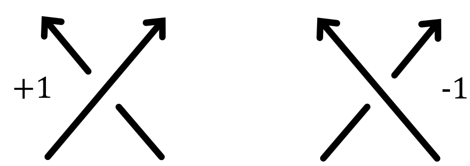


Figure 1

Definition. The linking number of K, denoted by $lk(K)$ is the sum of the signs of each crossing divided by 2.

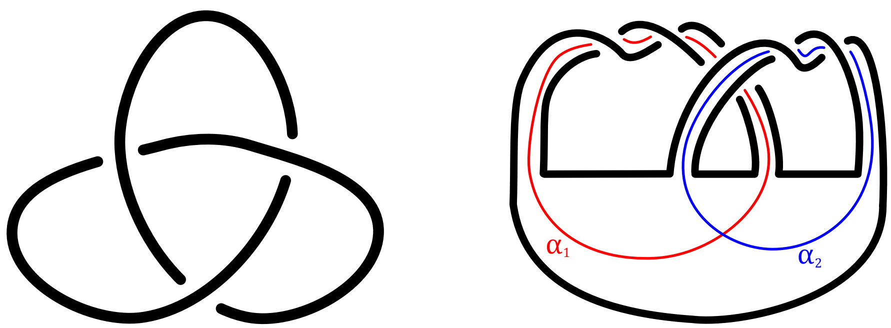
Definition. A Seifert Matrix V is a $2g \times 2g$ matrix for which $v_{i,j} = lk(a_i, a_j^+)$, where a_i is the i-th crossing and a_j^+ is the pushoff of a_j, the j-th crossing.

Definition. The Arf invariant of a knot K is

$$\begin{aligned} Arf(K) &= \sum_{i=1}^{2} g v_{2i-1,2i-1} v_{2i,2i} \bmod 2 \\ &= \sum_{i=1}^{m} lk(a_i, a_i^+) lk(b_i, b_i^+) \bmod 2 \end{aligned}$$

with $\{a_i, b_i\}$ a sympletic basis for $i = 1, ..., g$.

Example. The trefoil knot (figure 2a) has Seifert matrix $\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$. And therefore $Arf(\text{trefoil}) = 1$.



(a) Trefoil knot

(b) Its Seifert surface. α_1 and α_2 are the sympletic basis of the last definition.

Figure 2

However, the unknot has Seifert matrix 0. Thus, $Arf(\text{unknot}) = 0$. Hence the trefoil knot and the unknot are not the same.

References

[1] Tibor Macko Diarmuid Crowley Wolfgang Lück. *Surgery Theory: Foundations*. URL: `http://www.math.uni-bonn.de/people/macko/surgery-book.html`.

[2] Robion C. Kirby. *The Topology of 4-Manifolds*. Springer-Verlag, 1980. ISBN: 3-540-51 148-2.