FINDING A BOUNDARY FOR AN OPEN MANIFOLD

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1 The result

This chapter is based on a paper by W. Browder, J. Levine and G.R. Livesay [BLL65]. The aim is to (partially) answer the following question:

When is an open manifold the interior of a compact manifold with boundary?

In this chapter all manifolds are PL or smooth. Therefore by isomorphism we will mean an isomorphism in the appropriate category.

Definition 1.1. A topological space X is said to be *simply connected at* ∞ if for any compact $C \subseteq X$ there exists a compact D, $C \subseteq D \subseteq X$ such that $X \setminus D$ is simply connected.

Theorem 1.2. *Let W be a connected, orientable, non-compact n-manifold without boundary, with $n \geq 6$. Then there exists a compact manifold U with simply connected boundary such that $W = \operatorname{Int} U$ if and only if $H_*(W)$ is finitely generated and W is simply connected at ∞. Moreover such a U is unique up to isomorphism.*

Remark 1.3. Notice that if W is the interior of a compact manifold with boundary U then $H_*(W)$ is finitely generated. Moreover if the boundary ∂U is simply connected then of course W is simply connected at ∞ as a consequence of the collaring theorem. In fact for every compact $C \subseteq W$, one can always find an open collar V of the boundary of U which does not intersect C. Let $V' \subsetneq V$ be a subcollar of V such that V' corresponds to $\partial U \times (\frac{1}{2}, 1]$ inside of $V \cong \partial U \times (0, 1]$. Notice that that there is an isomorphism $U \xrightarrow{\sim} U \setminus V'$ that is the identity on $U \setminus V$ and shrinks the collar V inside V'. Then $U \setminus V'$ is compact and is contained in W. Set $D = U \setminus V'$, then $C \subset D$ and $W \setminus D \cong \partial U \times (0, +\infty)$ is simply connected, hence W is simply connected at ∞.

2 Proof of uniqueness

Theorem 2.1. *Let U_1 and U_2 be compact oriented n-manifolds with simply connected boundaries. Suppose that U_1 is embedded in $\operatorname{Int} U_2$ and the inclusion is a homology isomorphism. Suppose as well that $V \coloneqq U_2 \setminus \operatorname{Int} U_1$ is simply connected. Then V is a h-cobordism between ∂U_1 and ∂U_2.*

Proof. By excision $H_*(V, \partial U_1) \cong H_*(U_2, U_1)$ and both are trivial since $H_*(U_1) \xrightarrow{\sim} H_*(U_2)$ by hypothesis. Since $\pi_1(V, \partial U_1) = 0$ and ∂U_1 is simply connected, Hurewicz theorem in the relative form implies that $\pi_i(V, \partial U_1) \cong H_i(V, \partial U_1) = 0$ for all i. Hence By Whitehead's theorem it follows that the inclusion of ∂U_1 in V is a homotopy equivalence. By relative Poincaré duality

$$H_j(V, \partial U_2) \cong H^{n-j}(V, \partial U_1) = 0$$

and therefore with a similar process we obtain that the inclusion of ∂U_2 in V is a homotopy equivalence. □

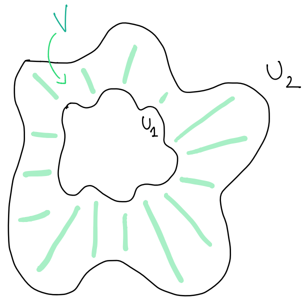

FIGURE 1. The manifolds U_1, U_2 and V in Theorem 2.1.

Corollary 2.2. *If $W \cong \operatorname{Int} U_1 \cong \operatorname{Int} U_2$, where U_1 and U_2 are compact manifolds of dimension $n \geq 6$, with simply connected boundaries, then U_1 and U_2 are isomorphic.*

Proof. We can embed U_1 in its interior using a collar of the boundary $A \cong \partial U_1 \times [0,1]$. Let A' be the subcollar corresponding to $\partial U_1 \times [\frac{1}{2}, 1]$ inside A. Then there is an embedding $U_1 \to \operatorname{Int} U_1$ that is the identity on $U_1 \setminus A$ and that shrinks the collar A inside A'. Moreover notice that this embedding is homotopic to the identity. Since $\operatorname{Int} U_1 \cong W$, we obtain an embedding $U_1 \hookrightarrow W$. Then $U_1 \hookrightarrow W \hookrightarrow U_2$, where the second map is the embedding induced by $W \cong \operatorname{Int} U_2 \subseteq U_2$. Notice that both maps are homotopy equivalences. If we identify U_1 with its image in U_2, it follows that $V := U_2 \setminus \operatorname{Int} U_1$ is homotopy equivalent to a collar of ∂U_1 and hence is simply connected. Hence by Theorem 2.1, V is an h-cobordism and therefore $V \cong \partial U_1 \times [0,1]$ and $U_1 \cong U_2$ by the h-cobordism theorem [Sma62]. □

3 Proof of Theorem 1.2

Theorem 1.2 is a direct consequence of the following proposition:

Proposition 3.1. *Let W be an oriented open n-manifold, with $n \geq 6$. Suppose $H_*(W)$ is finitely generated and W is simply connected at ∞. Then given a compact set C there is a connected compact n-manifold U, with simply connected boundary, such that $U \subseteq W$, $C \subseteq \operatorname{Int} U$ and the inclusion induced map*

$$H_*(U) \to H_*(W)$$

is an isomorphism.

Proof of Theorem 1.2. Let $C_1 \subsetneq C_2 \subsetneq \ldots \subsetneq W$ be a sequence of compact sets such that $W = \bigcup_{i=1}^{\infty} C_i$. Since W is simply connected at ∞ we may suppose that $W \setminus C_i$ is simply connected. By Proposition 3.1 for every i we can find a manifold with boundary U_i such that $U_{i-1} \cup C_i \subseteq \operatorname{Int} U_i$, ∂U_i is simply connected and the inclusion induced map in homology is an isomorphism.

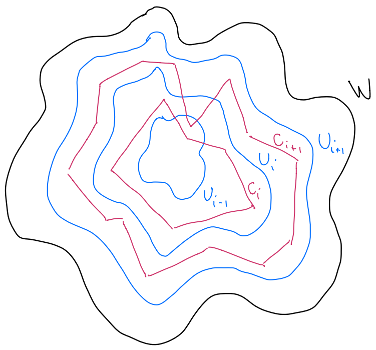

FIGURE 2. The compact sets C_i and the compact manifolds U_i in the proof of Theorem 1.2.

Then

$$W = \bigcup_{i=1}^{\infty} C_i \subseteq \bigcup_{i=1}^{\infty} U_i = W.$$

Set $V_i = \overline{U_{i+1} \setminus U_i}$. Since ∂V_i consists of ∂U_i and ∂U_{i+1} which are simply connected, by the Seifert-van Kampen theorem

$$\pi_1(W \setminus C_i) \cong \pi_1(U_i \setminus C_i) * \pi_1(V_i) * \pi_1(W \setminus U_{i+1})$$

for each i.

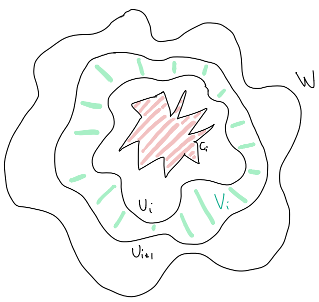

FIGURE 3. The compact manifold V_i.

Since $W \setminus C_i$ is simply connected, it follows that $\pi_1(V_i)$ is trivial. In fact the free product of non-trivial groups is always non-trivial. By Theorem 2.1, V_i is an h-cobordism between ∂U_i and ∂U_{i+1}, which are simply connected and of dimension bigger or equal to 5. By the h-cobordism theorem [Sma62] there are isomorphisms $f_i : V_i \xrightarrow{\sim} \partial U_i \times [0,1]$ that are the identity on ∂U_i. Call

$$\varphi_i : \partial U_{i+1} \xrightarrow{\sim} \partial U_i$$

the isomorphism induced by $f_i(\_, 1)$ and let

$$F_i : \partial U_{i+1} \times [0,1] \to \partial U_i \times [1,2]$$

send (x,t) in $(\varphi_i(x), t+1)$. By uniqueness up to isotopy of collars, we can suppose that $f_i \cup F_{i+1}$ is an isomorphism between $\overline{U_{i+1} \setminus U_{i-1}}$ and $\partial U_{i-1} \times [0,2]$. Hence for every i there are isomorphisms

$$U_i \xrightarrow{\sim} U_1 \cup \partial U_1 \times [0, i-1]$$

obtained by gluing the at each step the maps as shown above. Therefore

$$W = \bigcup_{i=1}^{\infty} U_i = \bigcup_{i=1}^{\infty} U_1 \cup \partial U_1 \times [0, i-1] = U_1 \cup \partial U_1 \times [0, +\infty)$$

and W is isomorphic to the interior of U_1. □

4 Proof of Proposition 3.1

The following lemma allows us to find a compact n-manifold $U \subseteq W$ with simply connected boundary and such that $W \setminus U$ is simply connected as well.

Lemma 4.1. *Let W be a connected manifold of dimension $n \geq 5$, simply connected at ∞ and such that $H_*(W)$ is finitely generated. Then for $C \subseteq W$ a compact subset there exists a compact n-manifold U with simply connected boundary such that $C \subseteq \operatorname{Int} U$ and $W \setminus U$ is simply connected and the inclusion induced map in homology*

$$H_*(U) \to H_*(W)$$

is surjective.

Proof. Since $H_*(W)$ is finitely generated we can always find a compact set $K \subseteq W$ such that

$$H_*(K) \to H_*(W)$$

is onto. In fact, we just need to take a representative for each element of a finite set of generators of $H_*(W)$. Therefore if O is any subset of W such that $K \subseteq O \subseteq W$, the following diagram commutes:

$$\begin{array}{ccc} H_*(K) & \twoheadrightarrow & H_*(W) \\ & \searrow \quad \nearrow & \\ & H_*(O) & \end{array}$$

and hence $H_*(O) \to H_*(W)$ is surjective too.

Let D be compact so that $C \cup K \subseteq D \subseteq W$ and $W \setminus D$ is simply connected. Such a D always exists because W is simply connected at ∞. We can find a compact manifold with boundary U^1 with $D \subseteq \operatorname{Int} U^1$:

- In the smooth case by choosing a proper smooth function $f : D \to \mathbb{R}$ such that $f_{|D} \equiv 0$. We can pick a regular value ε and fix $U^1 := f^{-1}([0, \varepsilon])$;
- In the PL case D lies in a finite subcomplex of W: we take U^1 to be a regular neighbourhood of K in W.

We can assume U^1 to be connected by taking ambient connected sums along the boundary: in fact we can join the connected components by arcs and then add to U^1 a regular neighbourhood for each arc.

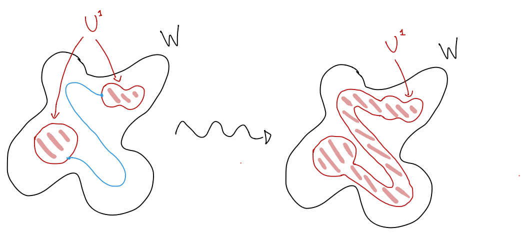

FIGURE 4. Ambient connected sum along the boundary.

By the fact that W is connected at ∞ it follows that all but one of the connected components of $W \setminus U^1$ are compact. Let U^2 be the union of U^1 and the compact components of $W \setminus U^1$, so that both U^2 and $W \setminus U^2$ are connected.

Since $W \setminus U^2$ is a connected manifold, then it is path connected and we can join the components of ∂U^2 by disjoint arcs with their interiors contained in $W \setminus U^2$. We define U^3 to be the union of U^2 and closed regular neighbourhoods of these arcs. Notice that U^3 and ∂U^3 are connected. Observe that $W \setminus U^3$ is connected as well: in fact $W \setminus U^2$, which is connected, can be obtained by gluing back the regular neighbourhoods of the arcs, which are isomorphic to $D^{n-1} \times D^1$ along a piece of the boundary isomorphic to $\partial D^{n-1} \times D^1$ which is also connected because n is strictly bigger than 2 and this easily implies by a Mayer-Vietoris argument that $W \setminus U^3$ is connected.

Now we want to do surgery on the boundary of U^3 to make it simply connected. Let γ be a generator of $\pi_1(\partial U^3)$. We can suppose that γ is a simple closed curve and that it is smooth (respectively PL) if we are in the smooth case (respectively PL) since $\dim(\partial U^3) \geq 3$.

Since $W \setminus D$ is simply connected, there is a map $f : D^2 \to W \setminus D$ that restricts to γ on the boundary. Since the dimension $n \geq 5$ we can approximate this map relative to the boundary with a smooth embedding and we can also suppose it is transverse to ∂U^3. Consider the inverse image of $f(D^2) \cap \partial U^3$ in D^2: this is a collection of simple closed curves in the interior of D^2. Take an innermost one δ; this curve bounds a disc $\Delta \subseteq D^2$ whose image is either contained in U^3 or in $\overline{W \setminus U^3}$. If it is contained in U^3 we carve a regular neighbourhood of $f(\Delta)$ out of U^3, otherwise, when the image of the disc Δ is contained in $\overline{W \setminus U^3}$ we add a regular neighbourhood $N \cong D^2 \times D^{n-2}$ of $f(\Delta)$ to U^3. Call the new manifold U^4. Suppose now $f(\Delta) \subseteq U^3$, the other case being analogous. Observe that $U^3 \setminus \operatorname{Int} U^4$ is homotopy equivalent to $\partial U^3 \cup f(\Delta)$ and therefore

$$\pi_1(U^3 \setminus \operatorname{Int} U^4) \cong \pi_1(\partial U^3)/\langle\delta\rangle.$$

Call $B = \{0\} \times D^{n-2} \subset N$ the cocore of the regular neighbourhood and notice that $U^3 \setminus \operatorname{Int} U^4$ is homotopy equivalent to

$$\partial U^4 \cup B.$$

Since $n \geq 5$, the boundary of this disc is simply connected and Seifert-van Kampen theorem implies

$$\pi_1(\partial U^4) \cong \pi_1(\partial U^4 \cup B) \cong \pi_1(U^3 \setminus \operatorname{Int} U^4) \cong \pi_1(\partial U^3)/\langle\delta\rangle$$

We can keep on carving out or adding a regular neighbourhood of the disc bounded by the innermost curve until we reach and kill γ. Repeating the same process for a finite set of generators of $\pi_1(\partial U^3)$, which exists because ∂U^3 is compact, we obtain a compact manifold U with simply connected boundary. Notice that

$$\pi_1(W \setminus U) * \pi_1(U \setminus D) = \pi_1(W \setminus D) = 1$$

and therefore $W \setminus U$ is simply connected as well. Since $K \subseteq U$, then $H_*(U) \to H_*(W)$ is onto. □

We now prove a weaker version of Proposition 3.1.

Proposition 4.2. *Let W be a connected and orientable open n-manifold, with $n \geq 6$. Suppose $H_*(W)$ is finitely generated and W is simply connected at ∞. Then given a compact $C \subseteq W$ and $k \leq n-3$, there is a compact n-manifold with boundary U, with $C \subseteq \operatorname{Int} U$, such that ∂U and $W \setminus U$ are simply connected and such that the inclusion induced homomorphisms*

$$H_i(U) \to H_i(W)$$

are isomorphisms when $i < k$ and are onto for all i.

We will prove Proposition 4.2 by induction and we shall show now the base case of the induction.

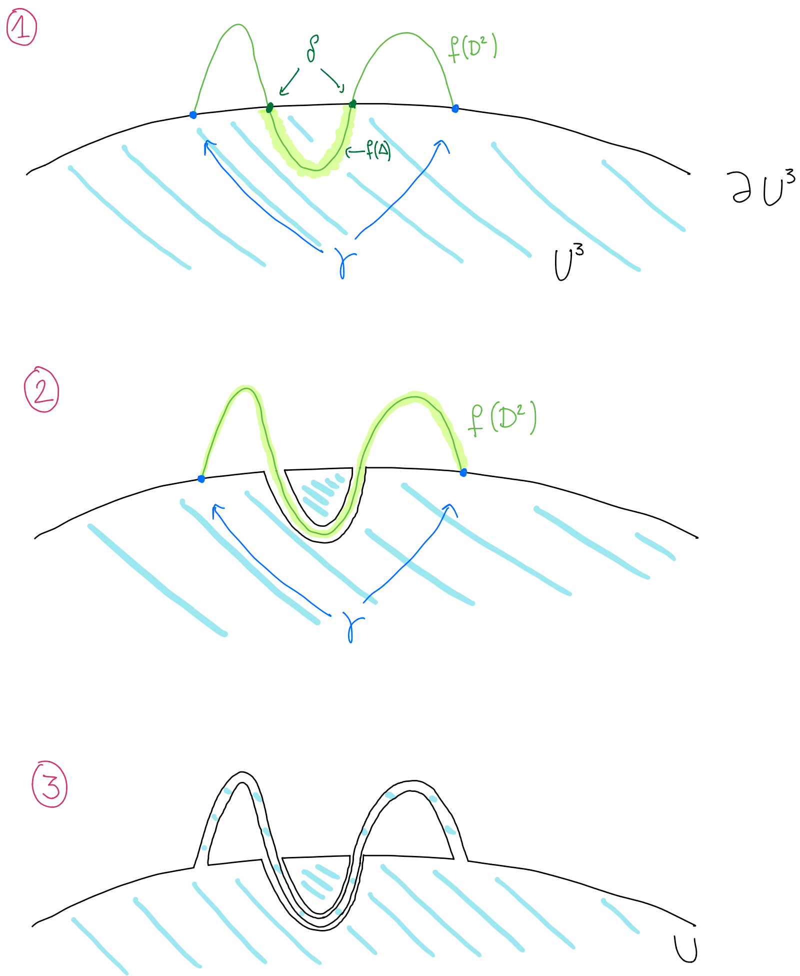

FIGURE 5. The procedure described in the proof of Theorem 4.1: at first there is a curve γ which is non-trivial in $\pi_1(\partial U^3)$, in the second step we carved out a regular neighbourhood of $f(\Delta)$. At last, after applying repeatedly the described procedure, we obtain U with simply connected boundary.

Proof of Proposition 4.2.

Base case: $\mathbf{i = 0, 1, 2}$

Thanks to Lemma 4.1 we may find a compact manifold $U_1 \subseteq W$ with simply connected boundary, such that $C \subseteq \operatorname{Int} U_1$, the manifold $W \setminus U_1$ is simply connected and $H_*(U_1) \to H_*(W)$ is onto. Notice that by construction both U_1 and W are connected and that $\pi_1(U_1) \cong \pi_1(W)$, hence

$$H_i(U_1) \xrightarrow{\sim} H_i(W) \text{ for i=0,1.}$$

Let $V_1 \coloneqq \overline{W \setminus U_1}$ and consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H_{k+1}(V_1) & \xrightarrow{j'} & H_{k+1}(V_1, \partial U_1) & \xrightarrow{\partial'} & H_k(\partial U_1) & \xrightarrow{i'} & H_k(V_1) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow \cong & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H_{k+1}(W) & \xrightarrow{j} & H_{k+1}(W, U_1) & \xrightarrow{\partial} & H_k(U_1) & \xrightarrow{i} & H_k(W) & \longrightarrow & \cdots \end{array}$$

The second vertical arrow is an isomorphism for all k due to excision. Therefore, since i is onto for all k, the map j is trivial and also j' needs to be trivial for all k. Similarly if i' is injective then ∂' is trivial and ∂ must be trivial too, hence i is injective for all k. Therefore we just need to kill the kernel of i', and this will kill the kernel of i. Let $x \in H_2(\partial U_1)$ be a generator of $\ker(i')$.

Note that, since ∂U_1 is simply connected, the Hurewicz theorem implies that an element $x \in H_2(\partial U_1)$ can be represented by a map $f : S^2 \to \partial U_1$.
Moreover, since V_1 is simply connected too, if $i'x = 0$ in $H_2(V_1)$ then f is homotopic to a constant in V_1.
In the smooth case, since the dimension of ∂U_1 is $n - 1 \geq 5$, by a general position argument f is homotopic to an embedding $g : S^2 \to \partial U_1$. If $n > 6$, since $i'x = 0$ this map extends to an embedding $\overline{g} : D^3 \to V_1$ which meets ∂U_1 transversally in $\partial D^3 = S^2$ only. When $n = 6$ we can suppose $\overline{g}$ is an immersion with only transverse double points: these intersections can be removed by applications of the Whitney trick and therefore we can suppose $\overline{g}$ is an embedding. The PL case can be handled similarly using analogous results of Irwin [Irw62].

Define U_1' as $U_1 \cup \mathcal{N}$, where $\mathcal{N} \cong D^3 \times D^{n-3}$ is a regular neighbourhood of $\overline{g}D^3$. Notice that the intersection of $V_1 \setminus \overline{g}D^3$ and the regular neighbourhood $\mathcal{N}$ is homotopy equivalent to S^{n-4}. Since $n \geq 6$

$$1 \cong \pi_1(V_1) \cong \pi_1(V_1 \setminus \overline{g}D^3) * \pi_1(\mathcal{N}).$$

Notice that $W \setminus U_1'$ and $V_1 \setminus \overline{g}D^3$ are homotopy equivalent and hence $W \setminus U_1'$ is simply connected. Similarly $\partial U_1'$ is homotopy equivalent to $\partial U_1 \cup \mathcal{N} \setminus \overline{g}D^3$ and

$$1 \cong \pi_1(\partial U_1) * \pi_1(\mathcal{N}) \cong \pi_1(\partial U_1 \cup \mathcal{N}) \cong \pi_1(\partial U_1 \cup \mathcal{N} \setminus \overline{g}D^3) * \pi_1(\mathcal{N}).$$

Therefore $\partial U_1'$ is simply connected as well. Let $V_1' = \overline{W \setminus U_1'}$ and $k' : H_*(\partial U_1') \to H_*(V_1')$ be the inclusion induced homomorphism. Notice that

$$H_j(\partial U_1') \cong H_j(\partial U_1)$$

for $j \neq 2, n-3$ and

$$H_2(\partial U_1') \cong H_2(U_1)/(x)$$

and by Poincaré duality a similar result holds for $j = n - 3$.

Then $\ker(k')_2 \cong \ker(i')_2/(x)$ and we did not increase the number of generators of $\ker(i')_j$ for $j \neq 2$. Iterating this procedure we arrive at $U_2 \supset U_1$ such that $H_2(U_2) \to H_2(W)$ is an isomorphism, and both ∂U_2 and $V_2 \coloneqq \overline{W \setminus U_2}$ are simply connected, proving the statement of Proposition 4.2 for $k = 2$.

Inductive step: $\mathbf{k} \implies \mathbf{k+1}$

We will need the following:

Lemma 4.3. *Let X be an n-manifold with boundary with $n \geq 6$, $\partial X = M \sqcup N$, where M, N and X are simply connected. Suppose $\pi_j(X, M) = 0$ for $2 \leq j < k - 1 < n - 4$. Then any element $w \in H_{k+1}(X, M)$ can be represented by a properly embedded disc $D^{k+1} \subseteq X$.*

Proof. We will just prove the theorem in the smooth case, using the handlebody theory of Smale [Sma62]; the PL case follows from analogous facts proven by Stallings [Sta62].

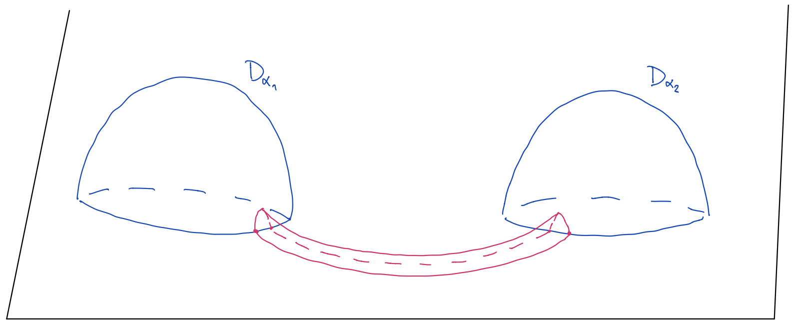

FIGURE 6. Two discs D_{α_1} and D_{α_2} connected by a tube $D^k \times I$.

By a theorem of Smale [Sma62] we can say that X has a handle decomposition relative to M

$$X = \bigcup_{i=k-1}^{n} X_i$$

where $X_{k-1} = M \times I$ and X_j is obtained from X_{j-1} attaching j-handles on $\partial X_{j-1} \setminus M \times \{0\}$.

Since X_j has the homotopy type of X_{j-1} with some j-discs attached, it follows that

$$H_i(X_j, M) \to H_i(X, M)$$

is an isomorphism for $i < j$ and surjective for $i = j$. Therefore there exists $w' \in H_{k+1}(X_{k+1}, M)$ such that w' is sent to w in $H_{k+1}(X, M)$. Consider the long exact sequence in homology of the triple (X_{k+1}, X_k, M):

$$\cdots \to H_{k+1}(X_{k+1}, M) \xrightarrow{k_*} H_{k+1}(X_{k+1}, X_k) \xrightarrow{\partial} H_k(X_k, M) \to \cdots$$

Let $y = k_* w'$. Notice that $H_{k+1}(X_{k+1}, X_k) \cong \mathbb{Z}^p$ where p is the number of $(k+1)$-handles and it is freely generated by the cores of the $(k+1)$-handles.

Recall that our goal is to represent w by a properly embedded disc D^{k+1}. We start by representing y as an embedded disc. It is a theorem of Smale [Sma62] (see also Wallace [Wal61]) that if we are given any basis for $H_{k+1}(X_{k+1}, X_k)$ we may find some handles $H_1, \dots, H_r$ in X_{k+1} attached to X_k so that $X_{k+1} = X_k \cup \bigcup_{i=1}^r H_i$ and the cores of the H_i's yield the given basis of $H_{k+1}(X_{k+1}, X_k)$.
Hence we may assume that $y = mz$, z being the core of one of the handles of X_{k+1}. Since the codimension is strictly bigger than one, y can also be represented as a properly embedded disc. In fact, let z be the core of a handle $H_i \cong D^{k+1} \times D^{n-k-1}$. Pick m different points $p_\alpha \in D^{n-k-1}$, and let $D_\alpha = D^{k+1} \times \{p_\alpha\} \subseteq H_i$. Since $n - k - 1 > 1$ the boundaries of the D_α's do not separate $\partial D^{k+1} \times D^{n-k-1}$. Hence we can join $S^k_\alpha = \partial D_\alpha$ by tubes $S^{k-1} \times I$ in $S^k \times D^{n-k-1}$ to form the connected sum of the S^k_α's and the D_α's can be connected by tubes $D^k \times I$ in $H_i \cong D^{k+1} \times D^{n-k-1}$ to form the connected sum along the boundaries of the D_α with the proper orientation and we can call the resulting disc D.

Then D has the homology class of y in $H_{k+1}(X_{k+1}, X_k)$. This disc is attached to ∂X_k rather than $M \times \{1\}$ so it remains to show that it can be chosen to miss the handles of X_k.

If the boundary on the disc does not meet the belt sphere of any k-handle in ∂X_k by handle sliding it can be moved off these handles by an isotopy. Suppose now that the algebraic intersection of ∂D with one belt sphere is zero. Then, taken two intersection points with opposite sign we can apply the Whitney trick to lower the number of intersection points: iterating this procedure we obtain that ∂D does not intersect the belt spheres of the k-handles.

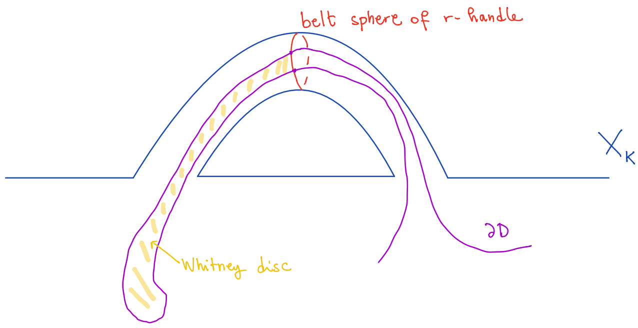

FIGURE 7. If the algebraic intersection of boundary of the disc D and the belt sphere of a k-handle is zero we can find a Whitney disc Δ.

In fact, take a loop γ that goes to one intersection point to the other inside ∂D and then goes back to the first intersection point inside the belt sphere. Notice that $\pi_1(\partial X_k) = 1$, in fact M was simply connected and we attached handles of order bigger than 2. Then we can find a disc Δ in ∂X_k that is bounded by γ. Since the dimension of ∂X_k is at least 5 we can approximate Δ relative to the boundary with a smooth embedded disc. Since the codimensions of ∂D and the belt sphere in ∂X_k are both strictly bigger than 2, by a general position argument we can suppose that Δ does not intersect them and is a Whitney disc, therefore we can use the disc Δ to move the belt sphere by an isotopy to remove the two intersection points.

To conclude, just notice that $\partial y = \sum \alpha_j h_j \in H_k(X_k, M)$, where h_j is the homology class of the core of the k-handles which freely generate $H_k(X_k, M)$ and α_j is the intersection number of ∂D and the belt sphere of the j-th k-handle. Therefore, since $\partial y = 0$, we deduce that $\alpha_j = 0$ for all j, which concludes the proof. □

Recall that we want to prove Proposition 4.2 by induction and we are left to prove the inductive step.

Assume now that Proposition 4.2 holds for some $k < n-3$, that is for any compact C one can find $U \subseteq W$, U compact manifold with boundary, with ∂U and $V = \overline{W \setminus U}$ simply connected such that $C \subseteq \operatorname{Int} U$ and

$$i_* : H_j(U) \to H_j(W)$$

is an isomorphism for $j < k$ and surjective for all j. Suppose $x \in \ker(i_*)_k$. Then there is a compact set $D \supset U$ such that, if $j : U \hookrightarrow D$ is the inclusion, $j_* x = 0$. By assumption, we can find U' with all the required properties and such that $D \subseteq \operatorname{Int} U'$. Notice that the image of x in $H_k(U')$ must be zero by functoriality. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
H_{k+1}(X, \partial U) & & & & & & \\
\downarrow & & & & & & \\
H_k(\partial U) & \xleftarrow[\partial]{} & H_{k+1}(V, \partial U) & \xrightarrow{\cong} & H_{k+1}(W, U) & \xrightarrow{\partial} & H_k(U) \\
\downarrow{\scriptstyle h_*} & & \downarrow & & \downarrow & & \downarrow \\
H_k(X) & \xleftarrow[\partial']{} & H_{k+1}(V, X) & \xrightarrow{\cong} & H_{k+1}(W, U') & \xrightarrow{\partial'} & H_k(U')
\end{array}$$

where $X = \overline{U' \setminus U}$ and the isomorphisms are given by excision. Since $x \in \ker(i_*)_k$, there is $y \in H_{k+1}(W, U)$ such that $\partial y = x$. Notice that both ∂ and ∂' are injective. Therefore, since the inclusion of x in $H_k(U')$ is zero, it follows that y goes to zero via the map

$$H_{k+1}(W, U) \to H_{k+1}(W, U').$$

Call z the image of y in $H_k(U)$. Note that $h_* z = 0$, hence there is an element $w \in H_{k+1}(X, \partial U)$ that maps to z via the boundary map.

By Lemma 4.3, applied to X with $M = \partial U$, we can find a properly embedded disc in X attached to ∂U which represents w, and we can add a regular neighbourhood $\mathcal{N}$ of this disc to U. Therefore z maps to 0 in the homology of $\overline{U} := \partial U \cup \mathcal{N}$. Notice that since both $k+1$ and $n-k+1$ are strictly bigger than 2 both $\partial\overline{U}$ and $W \setminus \overline{U}$ are still simply connected. Then

$$\ker i_k^{\overline{U}} \cong \ker i_k^U/(x).$$

We can apply this procedure to a finite set of generators of $\ker i_k^U$ and obtain a manifold $\tilde{U}$ which satisfies the inductive hypothesis for $i = k+1$. □

Proof of Proposition 3.1. By Proposition 4.2 we can suppose that given a compact $C \subseteq W$ we can find a compact manifold $U \subseteq W$ with simply connected boundary and such that $V \setminus U$ is simply connected as well, $C \subseteq \operatorname{Int} U$ and

$$i_* : H_i(U) \to H_i(W)$$

is an isomorphism for $i < n-3$ and surjective for all i.

Consider the following diagram with exact rows. Recall that i_k onto implies j_k onto.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_{k+1}(V, \partial U) & \xrightarrow{\partial'} & H_k(\partial U) & \xrightarrow{j_k} & H_k(V) & \longrightarrow & 0 \\
 & & \downarrow{\cong} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_{k+1}(W, U) & \xrightarrow{\partial} & H_k(U) & \xrightarrow{i_k} & H_k(W) & \longrightarrow & 0
\end{array}$$

We see that $\ker i_k \cong \ker j_k$. Notice that $H_{k+1}(W, U) = H_{k+1}(V, \partial U) = 0$ for $k < n-3$ since we know i_k is an isomorphism in this case. Since ∂U is simply connected,

$$H_{n-2}(\partial U) \cong H^1(\partial U) = \operatorname{Hom}(H_1(\partial U), \mathbb{Z}) = 0.$$

Therefore $H_{k+1}(W, U) = H_{k+1}(V, \partial U) = 0$ for $k = n-2$ too. Since V is a non compact n-manifold and $\partial V = \partial U$, we also get $H_n(W, U) \cong H_n(V, \partial U) = 0$.

Hence the only potentially nontrivial one is for $k = n-3$. Since

$$H_{n-3}(\partial U) = H^2(\partial U) \cong \operatorname{Hom}(H_2(\partial U, \mathbb{Z}))$$

by the universal coefficient theorem, thanks to the fact ∂U is simply connected, we deduce that $H_{n-2}(V, \partial U)$ is free. There is a compact set D, such that $U \subseteq D \subseteq W$ and $(i_D)_*(\ker i_{n-3}) = 0$ where i_D is the inclusion of U in D. Let U' be a manifold as in Proposition 4.2 such that $D \subseteq \operatorname{Int} U'$. Then if h is the inclusion of U in U', $h_*(\ker i_{n-3}) = 0$. It follows from the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_{n-2}(W, U) & \longrightarrow & H_{n-3}(U) & \longrightarrow & H_{n-3}(W) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{h_*} & & \downarrow{\mathrm{Id}} & & \\
0 & \longrightarrow & H_{n-2}(W, U') & \longrightarrow & H_{n-3}(U') & \longrightarrow & H_{n-3}(W) & \longrightarrow & 0
\end{array}$$

that the first vertical map is the trivial map. Define $V' = \overline{W \setminus U'}$, $M = \partial U$, $N = \partial U'$ and $X = \overline{U' \setminus U}$, so that $\partial X = M \sqcup N$.

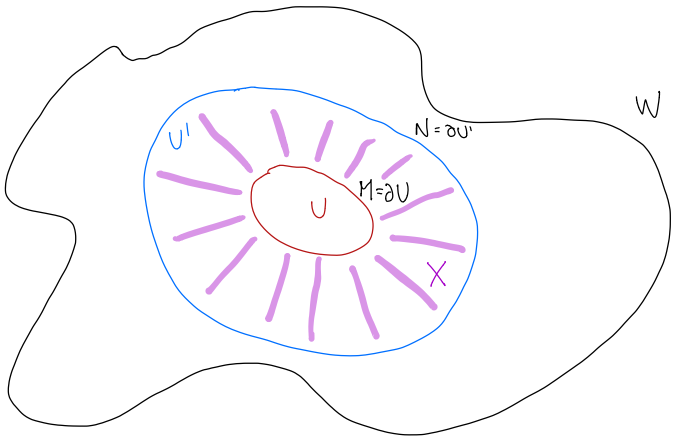

FIGURE 8. The manifold X in W.

Call $l_M : M \to X$ and $l_N : N \to X$ the inclusions. Then

$$\begin{array}{ccc} H_{n-2}(V,M) & \xrightarrow{\cong} & H_{n-2}(W,U) \\ \downarrow & & \downarrow \\ H_{n-2}(V,X) & \xrightarrow{\cong} & H_{n-2}(W,U') \end{array}$$

shows that the first vertical map is trivial too. Since as before $H_i(V,X)$ and $H_i(V,M)$ are either free (when $i = n-2$) or trivial (otherwise), this implies that $H^{n-2}(V,X) \cong \mathrm{Hom}(H_{n-2}(V,X),\mathbb{Z})$ and $H^{n-2}(V,M) \cong \mathrm{Hom}(H_{n-2}(V,M),\mathbb{Z})$ and therefore

$$\overline{h}^* : H^{n-2}(V,X) \to H^{n-2}(V,M)$$

is trivial too. The short exact sequence

$$0 \to H^{n-3}(V) \to H^{n-3}(X) \xrightarrow{\delta} H^{n-2}(V,X) \to 0$$

splits since $H^{n-2}(V,X)$ is free. Call $\alpha : H^{n-2}(V,X) \to H^{n-3}(X)$ the splitting morphism. Notice that $l_M^* \circ \alpha = 0$. The inclusion $h' : (V',N) \to (V,X)$ is an excision, and therefore

$$\beta := l_N^* \circ \alpha \circ (h'^*)^{-1} : H^{n-2}(V',N) \to H^{n-3}(N)$$

is defined. Call $\delta' : H^{n-3}(N) \to H^{n-2}(V',N)$ the boundary morphism. Then by construction $\delta' \circ \beta = \mathrm{Id}$, i.e. β is a section for the following short exact sequence:

$$0 \to H^{n-3}(V') \to H^{n-3}(N) \xrightarrow{\delta'} H^{n-2}(V',N) \to 0.$$

Moreover the image of β is contained in $l_N^*(\ker l_M^*)$ since the image of α is in the kernel of l_M^*.

Lemma 4.4. *Capping with the fundamental class:*

$$- \frown [N] : H^{n-k-1}(N) \to H_k(N)$$

sends $l_N^*(\ker(l_M^*)^{n-k-1})$ *isomorphically onto* $\ker((l_N)_*)_k$.

Proof. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccc} H^{n-k-1}(X) & \xrightarrow{l^*} & H^{n-k-1}(\partial X) & \xrightarrow{\delta} & H^{n-k}(X,\partial X) \\ \downarrow{\frown\nu} & & \downarrow{\frown\mu} & & \downarrow \\ H_{k+1}(X,\partial X) & \xrightarrow{\partial} & H_k(\partial X) & \xrightarrow{l_*} & H_k(X) \end{array}$$

where $\nu \in H_n(X,\partial X)$, $\mu \in H_{n-1}(\partial X)$ are the respective the fundamental classes. Notice that

$$H_*(\partial X) = H_*(N) \oplus H_*(M),$$

the fundamental classes μ and ν are related by:

$$\mu = \partial\nu = [N] - [M]$$

and

$$l_* = (l_N)_* - (l_M)_*$$
$$l^* = (l_N)^* - (l_M)^*.$$

Since $- \frown \nu$ is an isomorphism,

$$l^*(H^{n-k-1}(X)) \frown \mu = \ker(l_*).$$

Since the restriction of $- \frown \mu$ to N equals $- \frown [N]$, it follows that $l^*(H^{n-k-1}(X)) \cap H^{n-k-1}(N)$ is mapped isomorphically by $- \frown [N]$ onto $\ker(l_*) \cap H_k(N)$. But

$$l^*(H^{n-k-1}(X)) \cap H^{n-k-1}(N) = l_N^*(\ker l_M^*)$$

and similarly

$$\ker l_* \cap H_k(N) = \ker((l_N)_*)_k.$$

□

Since $\operatorname{Im}\beta$ is a free direct summand in $H^{n-3}(N)$, it follows that $B = \operatorname{Im}\beta \frown [N]$ is a free direct summand of $H_2(N)$ contained in $\ker((l_N)_*)_2$. Recall that $V = V' \cup X$, $X \cap V' = N$ and that V, V', N are simply connected. Then Seifert-Van Kampen theorem implies that X is simply conneted as well. By the Hurewicz theorem

$$\pi_2(N) \xrightarrow{\sim} H_2(N)$$

and

$$\pi_2(X) \xrightarrow{\sim} H_2(X).$$

Since $(l_N)_*B = 0$ this means that an element in B is represented by a map $f : S^2 \to N$ which is nullhomotopic in X. If $n > 6$ by a general position argument we can suppose there is a disc D^3 smoothly embedded in X such that the boundary of the disc is a representative for f in $\pi_2(N)$, while if $n = 6$ we need to use once again the Whitney trick. Taking a regular neighbourhood of the disc in X we find a 3-handle $D^3 \times D^{n-3}$. We can apply this procedure to a basis $\{b_j\}_{j=1}^h$ for B. If we add these handles $\{H_j\}_{j=1}^h$ to V' we obtain new manifolds $\overline{X} = X \setminus \bigcup_{j=1}^h \operatorname{Int}(H_j)$, $\overline{V} = V' \cup \bigcup_{j=1}^h H_j$ and $\overline{N} = \overline{X} \cap \overline{V}$.

Recall that $H_k(N) \cong H_k(V')$ for $k < n-3$ and that B is a free direct summand of $H_2(N)$, then it is possible to show that $H_k(\overline{V}) \cong H_k(V')$ for $k \neq 2, 3$ and

$$0 \to H_3(V') \to H_3(\overline{V}) \to \bigoplus_{j=1}^h \mathbb{Z}h_j \xrightarrow{\tilde{\partial}} H_2(V') \to H_2(\overline{V}) \to 0$$

where h_j is the attaching sphere of the handle H_j, hence $\overline{\partial}$ is injective. Therefore $H_k(V') \cong H_k(\overline{V})$ for $k \neq 2$ and

$$H_2(\overline{V}) \cong H_2(V')/r_*(B) \cong H_2(N)/B,$$

where $r_* : H_2(N) \to H_2(V')$. Recall that r_* is an isomorphism.

Notice that $\overline{N} \cup \{\text{cocores of } H_j\}$ is homotopically equivalent to N with some D^3 attached, which are the cores of the H_j's. By the Mayer-Vietoris sequence and the fact that B is free,

$$H_k(N) \cong H_k(N \cup \{\text{cores of the handles}\})$$

for $k \neq 2$ and

$$H_2(N \cup \{\text{cores of the handles}\}) \cong H_2(N)/B.$$

Since attaching the cocores of the H_j's to $\overline{N}$ can only modify the homology groups of $\overline{N}$ in dimension $n-3$ and $n-4$, when $n > 6$ it is a consequence of Poincaré duality and the universal coefficient theorem that $H_j(N) \cong H_j(\overline{N})$ for $j \neq 2, n-3$, and $H_2(\overline{N}) \cong H_2(N)/B$. The case $n = 6$ follows from Lemma 5.6 in [KM63]. By the same arguments we applied to $(V, \partial U)$, it follows that $H_i(\overline{V}, \overline{N}) = 0$ for $i \neq 2$ and $H^{n-2}(\overline{V}, \overline{N})$ is free and we have the following short exact sequence:

$$0 \to H^{n-3}(\overline{V}) \to H^{n-3}(\overline{N}) \to H^{n-2}(\overline{V}, \overline{N}) \to 0.$$

Notice that the image of B via the Poincaré duality isomorphism $H_2(N) \to H^{n-3}(N)$ is indeed the image of β. Hence

$$H^{n-3}(\overline{N}) \cong H^{n-3}(N)/\operatorname{Im}\beta.$$

Recall that $H^{n-3}(V') \cong H^{n-3}(N)/\operatorname{Im}\beta$ too, and

$$H^{n-3}(\overline{V}) \cong H^{n-3}(V').$$

Therefore $H^{n-3}(\overline{V})$ and $H^{n-3}(\overline{N})$ are isomorphic groups. Since they are finitely generated and we know that $H^{n-2}(\overline{V}, \overline{N})$ is free, it follows that $H^{n-2}(\overline{V}, \overline{N}) = 0$. By the universal coefficient theorem $H_i(\overline{V}, \overline{N}) = 0$ for all i and it follows that $\overline{U} = U \cup \overline{X}$ is a compact manifold with simply connected boundary and

$$H_*(\overline{U}) \to H_*(W)$$

is an isomorphism. Since the compact set C was contained in $\operatorname{Int} U$ it will be also contained in $\operatorname{Int} \overline{U}$. This proves Proposition 3.1. □

5 The h-cobordism theorem

As an interesting consequence of Theorem 1.2 we obtain an h-cobordism theorem for open manifolds.

Definition 5.1. Two oriented connected open manifolds M_1 and M_2 are called *h-cobordant* if there exists a manifold with boundary V with $\partial V = M_1 \sqcup (-M_2)$ such that the inclusions $M_i \hookrightarrow V$ are homotopy equivalences.

Theorem 5.2. *Let M_1, M_2 satisfy the hypothesis of Theorem 1.2 and let V be a h-cobordism between them which is simply connected at ∞. If N_1 and N_2 are the manifolds given by Theorem 1.2 for M_1 and M_2 respectively then they are h-cobordant.*

Proof. N_1 and N_2 are compact manifolds with boundary. Using a collar of the boundary of N_i we can embed N_i into M_i. Using now a collar C of the boundary of V we get embeddings of $N_i \times I \subseteq V$, with $N_i \times I \cap \partial V = N_i \times \{0\}$. We can join $N_1 \times \{1\}$ to $N_2 \times \{1\}$ by an arc in the interior of $V \setminus C$ and thickening the arc we get a compact manifold U, $\partial U = N_1 \cup W \cup N_2$ and $\partial W = \partial N_1 \sqcup \partial N_2$.

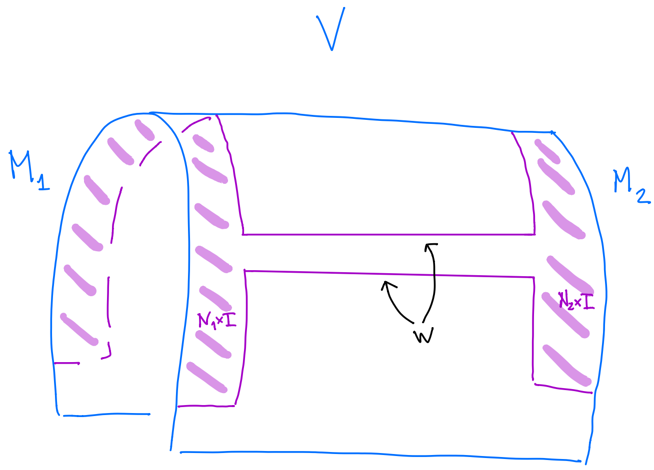

FIGURE 9. The manifold V.

Then, similarly to what we did for the proof of Theorem 1.2 we can enlarge U to get $\overline{V} \subseteq V$, $V \cong \operatorname{Int} \overline{V}$ just by adding handles far from N_1 and N_2. Therefore $\partial\overline{V} = N_1 \cup \overline{W} \cup N_2$, $\partial\overline{W} = \partial N_1 \sqcup \partial N_2$. From the diagram:

$$\begin{array}{ccc} N_i & \longrightarrow & \overline{V} \\ \downarrow & & \downarrow \\ M_i & \longrightarrow & V \end{array}$$

it follows that $N_i \to \overline{V}$ is a homotopy equivalence since all other three maps are. We are only left with showing that $\overline{W}$ is a h-cobordism between N_1 and N_2. Now Poincaré-Lefschetz duality gives

$$H^*(\overline{V}, N_1) \cong H_*(\overline{V}, N_2 \cup \overline{W})$$

and similarly exchanging N_1 and N_2. Since $N_i \to \overline{V}$ is a homotopy equivalence the left-hand side must be trivial. Notice that in

$$H_*(N_i) \xrightarrow{i_*} H_*(N_i \cup \overline{W}) \xrightarrow{j_*} H_*(\overline{V})$$

both $j_* i_*$ and j_* are isomorphisms, hence i_* is as well. Therefore $0 = H_*(N_i \cup \overline{W}, N_i) \cong H_*(\overline{W}, \partial N_i)$ by excision. Since both $\overline{W}$ and ∂N_i are simply connected it follows by the Hurewicz theorem that $\partial N_i \to \overline{W}$ is a homotopy equivalence and therefore $\overline{W}$ is an h-cobordism. $\square$

The following is a direct corollary of the above using the h-cobordism theorem [Sma62].

Corollary 5.3. *Let M_1, M_2, V as in Theorem 5.2 and suppose M_1 and M_2 are simply connected. Then M_1 and M_2 are isomorphic.*

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