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ON THE (A,B)-SLICE PROBLEM

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THE REMAINING questions regarding 4-dimensional topological surgery seem to be well expressed in terms of the link-slice problem for certain links L' which are Whitehead doubles, $L' = \mathrm{Wh}(L)$. The first author introduced in [3] and [4] a method involving the theory of decomposition spaces for "undoubling" the problem and expressing surgery in terms of a condition, "(A,B)-slice"‡, on L. A complete set of ("atomic") surgery problems may be constructed from links L' of the form $L' = \mathrm{Wh}(L)$ where L is an iterated ramified, Bing double of the Hopf links; we call such L a GBR, a generalized Borromean rings. For L a GBR we know:

$$\mathrm{Wh}(L) \text{ is free-slice§} \Leftrightarrow L \text{ is } (A,B)\text{-slice}$$

The full four-dimensional (topological) surgery conjecture is equivalent to all GBRs are (A,B)-slice. The definition of (A,B)-slice and a slightly less stringent condition "weakly (A,B)-slice" are given at the beginning of Section 1. The subsets A and B are complementary pieces of a 4-ball. Our concern—though not rigorously formulated in these terms—is with non-Abelian extensions of the Alexander duality relating A and B. If one thinks of the "slice problem" as a relative imbedding problem for 2-handles, one should think of our (A,B)-versions as a relative imbedding problem for fragments (A and B) of 2-handles. If one fragment is diminished the other is increased and one theme here is the surprisingly subtle problem of deciding which piece (or that possibly both or neither?) carries obstructions to imbedding . This leads to admittedly provisional definitions ("strong" in Section 2 and "robust" in Section 5) which attempt to isolate notions—different from the usual homological ones—of a cycle (in our case the core of $\partial^+ A$ or $\partial^+ B$) dying or not dying in a space.

We begin the analysis of the (A,B)-slice problem using the methods of: handle body theory, combinatorial group theory (the "Magnus expansion"), and secondary operations (in the form of lower central series and Massey product calculations). We find a constraint on the (A,B)-decompositions that can arise in an (A,B)-slicing of any homotopically essential link L. This result applies to the case of most interest since every GBR is homotopically essential. We are unable to go beyond this to show that homotopically essential links are not (A,B)-slice, in fact, we still do not know if the Borromean rings are (A,B)-slice.

The ultimate goal is to develop a method of analyzing relative imbedding problems for all possible (A,B) decompositions. This is not achieved. We reduce the general case to the

† Supported in part by NSF DMS83-3126 and the MacArthur Foundation.
‡ Defined in Section 1.
§ In this paper, L' slice will mean L' bounds disjoint topologically flat 2-disks in B^4. L' is "free-slice" if this extra condition holds: $\pi_1(B^4\text{-disks})$ has a free basis consisting of meridians to L'.

case that A and B each have (relative) handle decompositions containing handles of indices one and two and our methods determine obstructions to (weak) (A, B)-slicing when either side (say the A-side) lacks 1-handles. We are able to formulate an obstruction (see Section 5) in the case of general decompositions but do not know how to evaluate it. Thus the results of this paper are consistent with the surgery conjecture although the program has been to find a (nontrivial) obstruction—specifically an obstruction to the Borromean rings being (A, B)-slice. Since the outcome is not definitive, perhaps the main interest for the reader lies in seeing the classical techniques of link theory (relying on Milnor, Stallings, and Massey) arrayed against a novel problem.

SECTION 1. DEFINITIONS AND EXAMPLES

Define a *decomposition* of B^4 to be a pair of smooth, compact codimension-0 submanifolds with boundary $A, B \subset B^4$ satisfying: (1) $A \cup B = B^4$, (2) $A \cap B = \partial^- A = \partial^- B$, (3) $\partial A = \partial^+ A \cup \partial^- A$, $\partial^+ A = \partial A \cap \partial B^4$, $\partial^+ A \cap \partial^- A =$ the Clifford torus $S^1 \times S^1 \subset S^3 = \partial B^4$, and similarly (3′) $\partial B = \partial^+ B \cup \partial^- B$, $\partial^+ B = \partial B \cap \partial B^4$, $\partial^+ B \cap \partial^- B = \partial^+ A \cap \partial^- A$. Roughly put, a decomposition of B^4 is some extension to B^4 of the standard genus one Heegaard decomposition of $S^3 = \partial^+ A \cup \partial^+ B$.

Suppose $L \subset S^3$ is a smooth link of l components. Let $D(L)$ be the $2l$-component link obtained by pushing off an untwisted (i.e., the linking number is equal to zero) parallel to L. We say L is *(A, B)-slice* if there exist l decompositions: $(A_1, B_1), \ldots, (A_l, B_l)$ of B^4 and $2l$ self-homeomorphisms of B^4, say $\alpha_1, \ldots, \alpha_l$, $\beta_1, \ldots, \beta_l$, such that the entire collection: $\alpha_1 A_1, \ldots, \alpha_l A_l$, $\beta_1 B_1, \ldots, \beta_l B_l$, are pairwise disjoint and satisfy the boundary data: $\alpha_i \partial^+ A_i$ is a tubular neighborhood of the ith component of L and $\beta_i \partial^+ B_i$ is a tubular neighborhood of the ith component of the parallel copy of L, $1 \leq i \leq l$.

Suppose $L \subset S^3$ is a tame link of l components. We say L is *weakly (A, B)-slice* if there exist k decompositions (A_i, B_i) $(i = 1, 2, \ldots, k)$ of B^4 such that for all choices $C_i =$ either A_i or B_i $(i = 1, 2, \ldots, k)$, there exists a topological imbedding $s: \coprod_{i=1}^{k} C_i \to B^4$ with

$$s\left\{\coprod_{i=1}^{k} \partial^+ C_i\right\} = n(L).$$

Example 1.1 If L is slice (in the topologically flat sense) then L is weakly (A, B)-slice. Simply choose the decompositions (A_i, B_i) so that $(A_i, \partial^+ A_i)$ is a 2-handle and $(B_i, \partial^+ B_i)$ is a product collar $= (\partial^+ B_i \times I, \partial^+ B_i)$.

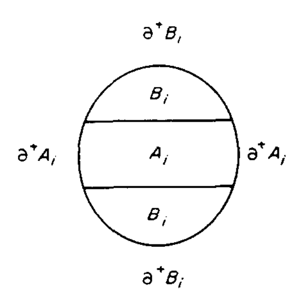

Fig. 1.1.

This is the simplest (A, B)-decomposition and somewhat degenerate in that the entire problem is concentrated in A.

Example 1.2. The next simplest example occurs when $(A, \partial^+ A)$ has an unknotted punctured torus (T^2_-, ∂) as its (relative) spine. As a handle body rel $\partial^+ A$, A is described by attaching two 1-handles and one 2-handle to the inner boundary of a collar on $\partial^+ A$:

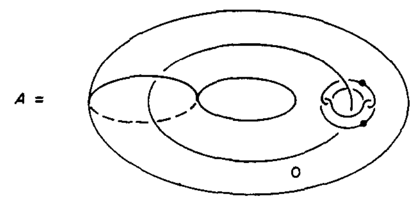

Fig. 1.2.

The complement B consists of two 2-handles attached to the inner boundary of a collar on $\partial^+ B$:

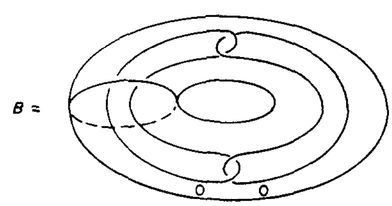

Fig. 1.3.

To see this, consider $T^2_- \subset B^3$:

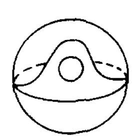

Fig. 1.4.

The complement $B^3 - T^2_-$ has the spine:

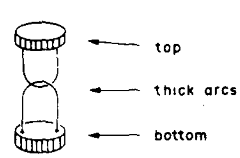

Fig. 1.5.

Now the complement $B^3 \times [-1, 1] - T^2_- \times [-\frac{1}{2}, \frac{1}{2}]$ has the spine

$$B^3 \times ([-1, -\tfrac{1}{2}] \cup [\tfrac{1}{2}, 1]) \cup (\text{top} \cup \text{bottom}) \times [-1, 1] \cup \text{thick arcs} \times [-\tfrac{1}{2}, \tfrac{1}{2}].$$

This is simply: collar $(\partial^+ B) \cup$ thick arcs $\times$ interval $=$ collar$(\partial^+ B) \cup$ two 2-handles. The attaching curves for the 2-handles are ∂(arcs $\times$ interval) which may be constructed by doubling the arcs:

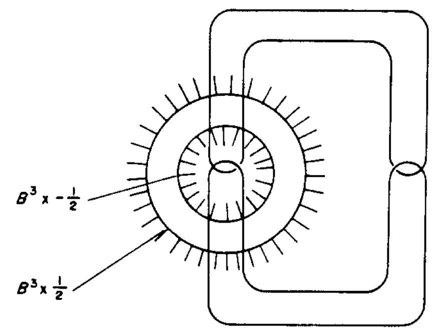

Fig. 1.6.

Let us show that the Borromean rings (BR) are not (A,B)-slice with decompositions $(A_i,B_i) \cong (A,B)$ above, $i = 1,2,3$. Notice that the Borromean rings bound three disjoint (oriented) surfaces of genus $\leqslant 1$ in B^4 (any link with pairwise linking numbers zero bounds disjoint surfaces in B^4) so the A_is provide no obstruction. On the other hand, we suppose that $\coprod_{i=1}^{3} B_i$ were imbedded with $\partial^+ B_i = n$ (ith component of BR) and quickly find a contradiction. Here n will denote the tubular neighborhood. Extend the 2-handles of each B_i through the collar to obtain a slicing of the six component link Bing (BR), the Bing double of the Borromean rings. Schematically we have:

$$\mathrm{BR} \subset S^3$$

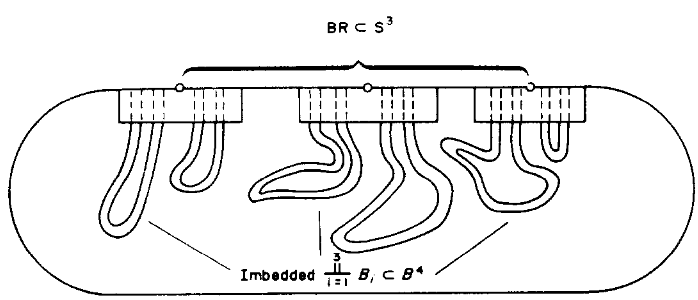

Fig. 1.7.

One may calculate that Bing (BR) has a non-zero $\bar{\mu}$-invariant [10], $\bar{\mu}(1,2,3,4,5,6) = 1$ and therefore [13] is not slice, a contradiction.

The lesson to be drawn from this example is that if the essentiality of L is first detected by n-ary operation the elimination of a particular (A,B)-slicing scenario may require m-ary operations, m arbitrarily large. This is a recurrent difficulty.

Example 1.3. The A and B of Example 1.2 can be concatenated in the various ways. Schematically we have (this is Example 1.2):

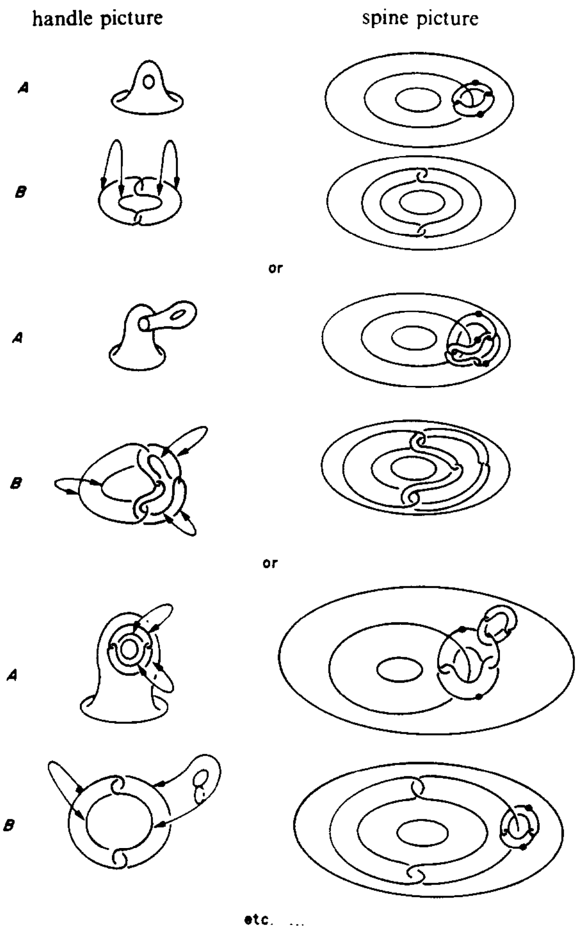

Fig. 1.8.

With patience, this infinite class of examples can be eliminated as possible (A, B)-decompositions for an (A, B)-slicing of any GBR. Unfortunately these examples are quite restricted, in particular $\partial^- A = \partial^- B$ is always a "graph-manifold".

The following proposition shows that at least in one respect the preceding examples are sufficiently general.

PROPOSITION 1.4. *If a link L is (A, B)-slice, then it admits another (A, B)-slicing with decompositions (A'_i, B'_i) where A'_i and B'_i are handle bodies of index* 1 *and* 2 *on $\partial^+ A_i$ and ∂B_i respectively.*

Proof. Let (A_i, B_i) be the initial (smooth) decompositions, by standard Morse theory these are handle bodies of index 1, 2, and 3 on their positive boundaries. In the abstract decompositions, trade all 3-handles to the opposite side (where they become 1-handles) to obtain (A'_i, B'_i) with only 1- and 2-handles. The imbeddings (α'_i, β'_i) are obtained from (α_i, β_i)

by restricting where 3-handles are deleted and extending where new 1-handles are attached. The necessary disjointness is only a matter of general position since a 1-handle must be made to miss a 2-complex. □

The full interplay of 1-handle and 2-handles is studied at the end of this paper (Section 5). We are led to a generalization of Milnor's theory of link homotopy which is as yet inadequately analyzed. In the next section we make a restrictive assumption on the handle structure of allowable pairs (A_i, B_i) and in this context complete the analysis for an essential L. A separate more geometric argument for this case results from the analysis in Section 5.

SECTION 2. A RESTRICTED (A-B)-SLICE PROBLEM

In this section we show that no generalized Borromean rings (and in fact no homotopically essential link) can be (A, B)-slice if the A_i's have a special feature, namely, for each decomposition (A_i, B_i) of B^4, A_i is a handlebody on the solid torus $\partial^+ A_i$ with only 2-handles. With this restriction on the A_i we prove that a homotopically essential link is not "weakly (A, B)-slice". Now we state the main theorem of this section.

THEOREM 2.2. *Let $L \subset S^3$ be a homotopically essential of k components with tubular neighborhood n. For any k decompositions (A_i, B_i) $(i = 1,2,\ldots,k)$ of B^4, if each A_i (say) is handlebody on $\partial^+ A_i$ with only 2-handles, then there exists a choice $C_i = A_i$ or B_i $(i = 1,2,\ldots,k)$ such that $\coprod_{i=1}^{k} C_i$ can not be imbedded into B^4 with $\coprod_{i=1}^{k} \partial^+ C_i = n(L)$.*

Roughly speaking, a homotopically essential link cannot be weakly (A, B)-sliced if each A_i is restricted to have only 2-handles.

The main ingredients in the proof of Theorem 2.2 are so called "link composition lemma" and "half-grope lemma". We first give some definitions which are needed in the statement of these two lemmas. Since the proofs of these two lemmas are somehow technical, they are postponed to the next two sections. After stating these two lemmas, we will use them to prove Theorem 2.2.

Let $S^1 \times B^2 \subset S^3$ be a standard solid torus in S^3. Q is a link in $S^1 \times B^2$. Denote $\hat{Q} = Q \cup 1 \times \partial B^2$. It is a link in S^3. Let $L = (L_1, L_2, \ldots, L_k)$ be a link in S^3. Suppose ϕ: $S^1 \times B^2 \to S^3$ is a 0-framing imbedding such that $\phi(S^1 \times B^2) = n(L_k)$. Let $L^k = (L_1, \ldots, L_{k-1})$.

THEOREM 2.3. (*Link composition lemma*). *In L and $\hat{Q}$ are both homotopically essential, then the link $L^k \cup \phi(Q)$ is also homotopically essential.*

Definition 2.4. A *k-stage half-grope* G is a 2-complex obtained in the following way: Let $\{F_{ij}, j = 1,2,\ldots,g_i, i = 1,2,\ldots,k,\ g_1 = 1\}$ be a collection of disks-with-handles. Identify $\partial F_{i1}, \ldots, \partial F_{ig_i}$ with an $\frac{1}{2}$-sympletic basis of $\bigcup_{j=1}^{g_i - 1} F_{i-1,j}$, $i = 2,\ldots,k$, the resulting space is then G. By ∂G we mean ∂F_{11}, the boundary of the first stage of G.

Notice if γ is a loop in a space X, then $[\gamma]$ belongs to kth term of the lower central series of $\pi_1(X)$ iff there exists a $(k-1)$-stage half-grope G and a map $F: G \to X$ such that $f|\partial G = \gamma$.

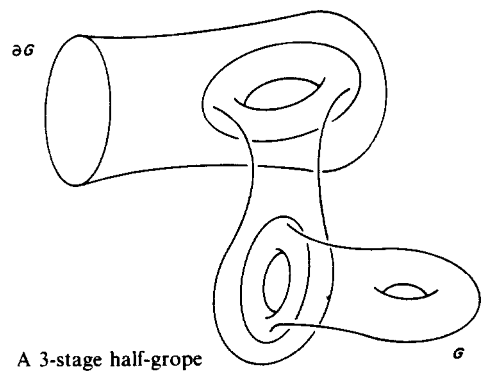

Fig. 2.1.

THEOREM 2.5. *(Half-grope lemma). Suppose* $L = (L_1, \ldots, L_k)$ *is a link in* $S^3 = \partial B^4$. *If there exist* $(k-1)$*-stage half-gropes* $G_1, G_2, \ldots, G_k$ *and maps* $f_i: G_i \to B^4$, $i = 1, 2, \ldots, k$ *such that* $f_i(G_i) \cap f_i(G_j) = \phi$ *for* $i \neq j$ *and* $f_i(\partial G_i) = L_i$, $i = 1, 2, \ldots, k$. *Then the link L is homotopically trivial.*

Proof of the Theorem 2.2. Consider a decomposition (A, B) of B^4 such that A is a handlebody on the solid torus $\partial^+ A$ with only 2-handles. $A = \partial^+ A \times I \cup$ 2-handles and $\partial^+ A = \partial^+ A \times 0$. Let b be the core of $\partial^+ B$. We call A is *strong* if $[b] \notin (\pi_1(B))_\omega$, and B is *strong* if $[b] \in (\pi_1(B))_\omega$. Here, for any group π, we denote $\pi_n =$ the nth term of the lower central series of π, and $\pi_\omega = \cap \pi_n$. By definition, either A or B is strong and they can not be both strong.

The attaching region of the 2-handles of A consists of disjoint solid tori in $\partial^+ A \times 1$. Let Q_A be the cores of these solid tori. We may think Q_A is a link in $\partial^+ A \subset S^3$. Note Q_A is a slice link and the slice complement is B. Therefore by Stalling [13] A is strong if and only if $[b] \notin (\pi_1(S^3 - Q_A))_\omega$.

LEMMA 2.6. *If* $[b] \notin (\pi_1(S^3 - Q_A))_\omega$, *then there exists a link* Q'_A *which is obtained by replacing each component of* Q_A *by a collection of parallel copies, so that the link* $Q'_A \cup b$ *is homotopically essential.*

Notice Q_A is still a slice link.

Proof. Since Q_A is a slice link, we have

$$\pi_1(S^3 - Q_A)/(\pi_1(S^3 - Q_A))_k \cong F/F_k$$

for any positive integer k. Where F is a free group whose rank is equal to $|Q_A|$, the number of the components of Q_A. Since $b \notin (\pi_1(S^3 - Q_A))_w$, we have $1 \neq b \in F/F_k$ for k sufficiently large. We may consider the Magnus expansion of b in the ring of all formal polynomials with $|Q_A|$ variables. Using the same argument as in the proof of Theorem 7 in [10], we can obtain a link Q'_A as in our Lemma such that the Magnus expansion of b with respect to the slice link Q'_A has a nonconstant monomial with distinct variables. This implies the expansion of b in $R(|Q'_A|)$-variables) is not equal to 1. Thus the link $Q'_A \cup b$ is homotopically trivial. See Section 3 for the definition and properties of the ring R. □

Now suppose L is a link in S^3 with k components, and L is homotopically essential. Let $(A_1, B_1), \ldots, (A_k, B_k)$ be decompositions of B^4 such that each A_i has only 2-handles. Choose $C_i = A_i$ if A_i is strong and $C_i = B_i$ if B_i is strong. Then there does not exist an imbedding $s\colon \coprod_{i=1}^{k} C_i \to B^4$ so that $s\left[\coprod_{i=1}^{k} \partial^+ C_i\right] = n(L)$. If there were, we would obtain a homotopically essential link (Lemma 2.6, Theorem 2.3) which bounds disjoint singular half-gropes of any number of stages in B^4. By a singular half-grope, we mean the image of a half-grope under a continuous map. (Note we may consider a (singular) disk to carry nearby singular half-gropes of any number of stages.) This will contradict to the Theorem 2.5. □

SECTION 3. LINK COMPOSITION LEMMA

To discuss the link composition lemma, let us first review Milnor's link homotopy theory. See [9] for reference. Let L be a link in S^3. The Milnor's link group of L, $G(L)$, is a nilpotent quotient group of $\pi_1(S^3 - L)$. If two links L and L' are homotopic, then $G(L)$ and $G(L')$ are isomorphic. Moreover, the isomorphism between $G(L)$ and $G(L')$ preserves the conjugate classes of the longitude-meridian pairs. Conversely, if a link L satisfies $G(L) \cong G$ (the trivial link) and the isomorphism preserves the longitude-meridian pairs, then L is homotopically trivial.

The ring $R = R(x_1, \ldots, x_k)$ is defined in the following way: First, it is a free abelian group generated by 1 and $\{x_{i_1} x_{i_2} \ldots x_{i_r} | i_1, i_2, \ldots i_r$ are distinct integers among $1, 2, \ldots, k, r \geqslant 1\}$. The ring multiplication is then given by extending the following multiplication rule linearly to the whole $R(x_i, \ldots, x_k)$. The rule is

$$(x_{i_1} \ldots x_{i_r}) \cdot (x_{j_1} \ldots x_{j_s}) = \begin{cases} 0 & \text{if some } i_\alpha = \text{some } j_\beta, \\ x_{i_1} \ldots x_{i_r} x_{j_1} \ldots x_{j_s} & \text{otherwise} \end{cases}$$

Let L be a homotopically trivial link in S^3 with k components. Let $m_1, \ldots, m_k$ be the meridians of L. Milnor showed that the Magnus expansion

$$\begin{aligned} G(L) &\to R(x_1 \ldots x_k) \\ m_i &\mapsto 1 + x_i, \quad i = 1, 2, \ldots, k \end{aligned}$$

is a monomorphism. Note the meridians are only well defined up to conjugacy. The ambiguity here is clarified by the fact that the endomorphism

$$\begin{aligned} R(x_1, \ldots, x_k) &\to R(x_1, \ldots, x_k) \\ x_i &\mapsto x_i, \; i = 1, 2, \ldots, k-1 \\ x_k &\mapsto (1 + x_j) \cdot x_k \cdot (1 - x_j) \quad \text{some } j \end{aligned}$$

is actually an automorphism.

Notice if L is a trivial link, then $\pi_1(S^3 - L)$ is a free group. So the Magnus expansion of $G(L)$ in R is a monomorphism is a simple fact from the algebraic viewpoint. But the following lemma shows that this simple fact is sometimes quite useful.

LEMMA 3.1. *Let L be a (homotopically) trivial link in S^3, γ is a loop in $S^3 - L$. Then the link $L \cup \gamma$ is homotopically trivial iff $[\gamma] = 1$ in $G(L)$.*

Proof. See [9]. □

So to ask whether or not $L \cup \gamma$ is homotopically trivial, we only need to ask whether or not $[\gamma] = 1$ in $G(L)$ which is equivalent to ask whether or not the Magnus expansion of $[\gamma]$ in R is 1. Usually, it is easier to check whether or not an element in R is 1 than to check whether or not an element in $G(L)$ is 1.

To prove the link composition lemma (Theorem 2.3), we need another technical lemma. Let $S^1 \times B^2 \subset S^3$ be a standard torus in S^3, Q is a link in $S^1 \times B^2$. Denote $\hat{Q} = Q \cup 1 \times \partial B^2$, a link in S^3.

LEMMA 3.2. *Let $\phi: S^1 \times B^2 \to S^3$ be any 0-framing imbedding. Then $\hat{Q}$ and $\phi(\hat{Q})$ are homotopically equivalent.*

Proof. Let $D \subset S^1 \times \mathring{I}$ be a small disk so that if $Q \times I \subset S^1 \times B^2 \times I = S^1 \times I \times B^2$, then $Q \times I \cap \partial D \times B^2$ consists of parallel copies of the core of $\partial D \times B^2$.

We can imbed $\overline{S^1 \times I - D} \times B^2$ into $S^3 \times I$ such that $S^1 \times \{0\} \times B^2 \subset S^3 \times \{0\}$ is the standard solid torus, $S^1 \times \{1\} \times B^2 = \phi(S^1 \times B^2) \subset S^3 \times \{1\}$, and $\partial D \times B^2 \subset S^3 \times \{0\}$ is separated from $S^1 \times \{0\} \times B^2$. Now $Q \times I \cap \partial D \times B^2 \subset S^3 \times \{0\}$ is a boundary link, it is homotopically trivial. Therefore, $\hat{Q}$ and $\phi(\hat{Q})$ are "singular concordant" in $S^3 \times I$ (in the terminology of [8], this means joined by disjoint singular annuli). This, by the result of [8], implies $\hat{Q}$ and $\phi(\hat{Q})$ are homotopically equivalent. □

Now we are ready to prove the link composition lemma.

Proof of the Theorem 2.3. Let $L = (L_1, \ldots, L_k)$ be a homotopically essential link, $Q = (L'_1, \ldots, L'_m)$ be a link in $S^1 \times B^2$ so that $\hat{Q}$ is homotopically essential in S^3. Let $\phi: S^1 \times B^2 \to S^3$ be a 0-framing imbedding with $\phi(S^1 \times B^2) = n(L_k)$. We will show the link $(L_1, \ldots, L_{k-1}) \cup \phi(Q)$ is homotopically essential.

By the proof of Lemma 3.2, if Q is homotopically essential, so is $\phi(Q)$. Thus we need only consider the case that Q is homotopically trivial in S^3. Inductively, we may assume the link L is almost homotopically trivial, i.e., L becomes a homotopically trivial link when one omits any component of L. Thus, the Magnus expansion of L_1 in $R(x_2, \ldots, x_k)$ is of the form

$$1 + \sum \mu_{i_2 \ldots i_k} x_{i_2} \ldots x_{i_k},$$

where the summation extends over all permutations $i_2, \ldots, i_k$ of the integers $2, \ldots, k$ and some coefficients $\mu_{i_2 \ldots i_k}$ are not zero.

Suppose the link $(L_2, \ldots, L_{k-1}) \cup \phi(Q)$ is homotopically trivial. Consider the Magnus expansion of $\phi(1 \times \partial B^2)$ in $R(x_2, \ldots, x_{k-1}, x'_1, \ldots, x'_m)$ which is denoted by σ. The image of σ under

$$R(x_2, \ldots, x_{k-1}, x'_1, \ldots, x'_m) \to R(x'_1, \ldots, x'_m)$$
$$\text{let } x_2 = \ldots = x_{k-1} = 0$$

is equal to the Magnus expansion of $1 \times \partial B^2$ in $R(x'_1, \ldots, x'_m)$ by Lemma 3.2. It is not equal to 1 by our assumption that $\hat{Q}$ is homotopically essential. So σ is of the form

$$1 + \text{terms not containing } x_2, \ldots, x_{k-1} + \ldots$$

Replace x_k in the expansion of L_1 in $R(x_2, \ldots, x_k)$ by σ, we get the expansion of L_1 in $R(x_2, \ldots, x_{k-1}, x'_1, \ldots, x'_m)$. It is easy to see that this expansion of L_1 in $R(x_2, \ldots, x_{k-1}, x'_1, \ldots, x'_m)$ is not equal to 1. Thus $(L_1, \ldots, L_{k-1}) \cup \phi(Q)$ is homotopically essential, completing the proof of Theorem 2.3. □

SECTION 4. HALF-GROPE LEMMA

Beside the application to the (A, B)-slice problem, the result of this section is of independent interest. Before the work of Casson and the first author ([1, 2]), there was a

Japanese school engaged in exploring the failure of the Whitney's trick in dimension 4. In particular, the work [11] and [7] showed that the separability of homology classes in a compact, 1-connected 4-manifold with vanishing intersection form will put some strong restriction on the Massey products on the boundary of the 4-manifold. For example, we have:

Theorem 4.1. ([7], Theorem 1). Let W be a compact, 1-connected smooth 4-manifold with connected boundary $\partial W = M$. The intersection form on $H_2(W)$ vanishes. Let $x_1, \ldots, x_k$ be a basis of $H_2(W)$, $\alpha_i \in H^1(M)$ is dual to the boundary reduction of x_i, $i = 1, \ldots, k$. Furthermore, we assume any $r(< k-1)$-tuple Massey product on $H^1(M)$ vanishes. If $x_1, \ldots, x_k$ can be represented by disjoint singular 2-spheres, then we have

$$\langle \alpha_1, \ldots, \alpha_{k-1} \rangle \cup \alpha_k = 0$$

Here $\langle \alpha_1, \ldots, \alpha_{k-1} \rangle$ is the $(k-1)$-tuple Massey product of $\alpha_1, \ldots, \alpha_{k-1}$.

Remark. The statement of [7], Theorem 1, may not be precise. In his proof, Kojima used the assumption that any $r(< k-1)$-tuple Massey product on M vanishes. But in his statement of the theorem, he only assume any $(k-2)$-tuple Massey product on M vanishes.

A corollary of this theorem is that if a link in S^3 bounds disjoint singular disks in B^4, then it is homotopically trivial. See the proof of Theorem 3.5 in the end of this section. Also see [8] for a geometric argument.

We are interested in separating homology classes by "closed half-gropes". Let G be a finite stage half-grope. We call $\hat{G} = G \cup_\partial B^2$ a *closed half-grope.* Note $H_2(\hat{G}) \cong \mathbb{Z}$. A homology class $x \in H_2(W)$ is represented by a singular closed half-grope $\hat{G}$ if there is a map $\hat{G} \to W$ such that the induced map $H_2(\hat{G}) \to H_2(W)$ maps a generator of $H_2(\hat{G})$ to x. The main theorem of this section is the following generalization of the Theorem 4.1.

THEOREM 4.2. *The setup is the same as in the theorem* 4.1. *If* $x_1, \ldots, x_k$ *can be represented disjointly by singular closed* $(k-1)$*-stage half-gropes, then we have*

$$\langle \alpha_1, \ldots, \alpha_{k-1} \rangle \cup \alpha_k = 0.$$

Except some interesting technical details, the proof of theorem 5.2 is similar to the proof of the theorem 4.1 in [7]. That is, first construct a 4-manifold P with $\partial P = \partial^+ P \cup \partial^- P$ and $\partial^+ P = M$, and then relate the Massey products on $\partial^+ P$ to the Massey products on $\partial^- P$ via P. Since we know up to some slight ambiguity what $\partial^- P$ is, we can study the Massey products on $\partial^+ P$ in this way.

A key technical point in the proof of Theorem 4.2 is the calculation of the Massey products on the following 3-manifolds. Let $L(k)$ be the link drawn in the following picture.

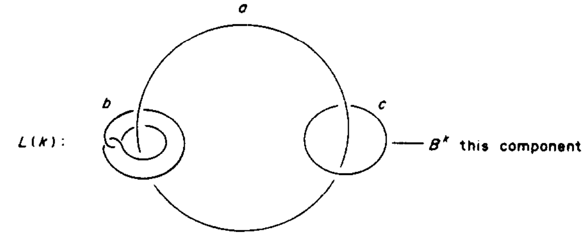

Fig. 4.1.

where B (a link) denotes the link with one component replaced by its (untwisted) Bing double, and $B^k = B \cdot B^{k-1}$. Let $X = S_0(L(k))$ (0-framing surgery on $L(k)$). The homology of X is quite easy to visualize. For example, $H_1(X)$ is freely generated by meridians of each component of $L(k)$, and $H_2(X)$ is freely generated by imbedded surfaces dual to these meridians respectively. Since Massey products are defined for cohomology classes, we should work in the cochain level. But if one keeps the above homological picture of X in mind, it will aid in understanding the following discussion.

Suppose K is a finite simplicial space and $C_*(K)$ is the simplicial chain complex of K. We use the Alexander–Whitney diagonal approximation to define the cup product on the cochain complex $C^*(K)$. The cup product is associate and bilinear.

For a cochain $u \in C^*(K)$, the *support* of u is defined to be a subspace of K

$$\operatorname{supp}(u) = U\{\sigma; \sigma \text{ is a simplex of } K, \langle u,\sigma\rangle \neq 0\}.$$

The following facts are obvious from the definition:

(1) If $\operatorname{supp}(u) \cup \operatorname{supp}(v) = \phi$, then $u \cup v = 0$;

(2) If K' is a subspace of K and $\operatorname{supp}(u) \cap K' = \phi$ or $\operatorname{supp}(v) \cap K' = \phi$, then $\operatorname{supp}(u \cup v) \cap K' = \phi$.

Back to our link $L(k)$. Draw a big 2-sphere S dividing $L(k)$ into two parts:

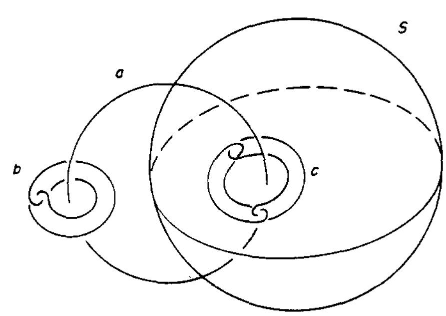

Fig. 4.2.

Denote these two components of $S^3 \backslash$open neighborhood $(L(k) \cup S)$ by X_b and X_c so that $X_b \subset$ the component of $S^3 \backslash S$ which contains b and $X_c \subset$ the component of $S^3 \backslash S$ which contains $B^k(c)$. Triangulate X so that X_b and X_c are subcomplexes. If u is a 2-coboundary in $C^*(X)$ with $\operatorname{supp}(u) \subset X_b$ (or X_c), then there is 1-cochain v with $\operatorname{supp}(v) \subset X_b$ (or X_c) and $\delta v = u$. To see this, think of u as a 1-cycle in the dual cell structure. Because $H_1(X_b) \to H_1(X)$ is an injection u bounds (cellularly) in X_b as well as in X.

We will choose 1-cocycles in $C^*(X)$ which are dual to the meridians of $L(k)$. For simplicity, we call a^* the 1-cocycle dual to the meridian of the component a, b^* the 1-cocycle dual to the meridian of the component b, and c^* those 1-cocycles which are dual to the meridians of the components in $B^k(C)$ respectively. Let A^*, B^*, C^* be the subgroup of $H^1(X)$ generated by a^*, b^*, c^* respectively, then we have $H^1(X) = A^* \oplus B^* \oplus C^*$. Notice the longitudes of $L(k)$ bound surfaces in $S^3 \backslash L(k)$. We will choose these surfaces to be those we usually draw in the picture, so that the surfaces bounded by the component b or the components in $B^k(C)$ are disjoint with S. Close these surfaces up in X and let them be transversal to the 1-skeleton of the triangulation of X. The 1-cocycles given by these closed surfaces will be a^*, b^*, and c^* respectively.

LEMMA 4.3. *For any* $u, v \in \{a^*, b^*, c^*\}$, *if* $u \neq v$, *then either* $supp(u \cup v) \subset X_b$ *or* $supp(u \cup v) \subset X_c$; *if* $u = v$ *or* $u = b^*$ *and* $v \in c^*$, *then* $u \cup v = 0$.

Proof. Obvious. □

Let A, B, C be the subgroups of $H_1(X)$ generated by meridians of the component a, the component b and the components in $B^k(c)$ respectively. We have $H_1(X) = A \oplus B \oplus C$.

Let $\alpha_1, \ldots, \alpha_n \in H^1(X)$. Suppose there is only one α_i not belonging to $B^* \oplus C^*$, all other α_i's are in $B^* \oplus C^*$. A defining system $\{\Gamma_{ij}\}$ of the Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ is called *separated* if

(1) Γ_{ij} is a linear combination of a^*, b^*, and c^*;
(2) $\Gamma_{ij} = \Gamma_{ij}^b + \Gamma_{ij}^c$ such that $\Gamma_{ij}^b = u_{ij}^b + \gamma_{ij} b^*$ with $\mathrm{supp}(u_{ij}^b) \subset X_b$ and $\Gamma_{ij}^c = u_{ij}^c + s_{ij} c^*$ with $\mathrm{supp}(u_{ij}^c) \subset X_c$ for $i \neq j$, r_{ij}, s_{ij} are integers. (We have abused notation here. Since there are many components corresponding to c, $\Gamma_{ij}^c = u_{ij}^c + \sum_k S_{ijk} c_k^*$ would be more precise.)

LEMMA 4.4. *For a separated defining system within* $C^*(X)$ *for* $\alpha_{ij}, \ldots, \alpha_n$ *as above and* $n \leq k-1$, $\left[\sum_{r=1}^{n-1} \Gamma_{1r} \cup \Gamma_{r+1n}\right]$ *is Poincaré dual to an element in* A.

Proof. Since the defining system $\{\Gamma_{ij}\}$ is separated, the calculation of $\langle \alpha_1, \ldots, \alpha_n \rangle$ is reduced to the calculation of an n-fold Massey product in S_0 (Whitehead link) and an n-fold Massey product in S_0 (B^k (Hopf link)). We can use the relation between Massey products and Milnor's $\bar{\mu}$-invariants (see Theorem 4.6, for example) and the linearity of Massey products to calculate these Massey products. These suffice to prove the lemma. □

Remark. If all α_i's are in $B^* \oplus C^*$, then

$$\left[\sum_{r=1}^{n-1} \Gamma_r \cup \Gamma_{r+1n}\right] = 0.$$

We will consider a generalized ($L(k)$ which may have Whitehead components and ramified Bing components. Let G be a k-stage half grope, $(N, \partial G \times D^2)$ be the 4-dimensional thickening of $(G, \partial G)$. Let Y be a 4-manifold obtained from attaching a kinky 2-handle to N along $\partial G \times D^2$ with 0-framing. Then we should have $\partial Y = S_0(L(k))$. Use the same notation for the cohomology and homology classes of $S_0(L(k))$ as in the case when $L(k)$ as in the case when $L(k)$ is the simplest one. Then, Lemma 4.4 is still true for a generalized $L(k)$.

Proof of the Theorem 4.2. By assumption, W is compact, 1-connected 4-mainfold with connected boundary $\partial W = M$. The intersection form on $H_2(W)$ vanishes. We have a basis $\{x_1, \ldots, x_k\}$ for $H_2(W)$ which is represented by disjoint singular closed $(k-1)$-stage half-gropes. Namely, we have closed $(k-1)$-stage half gropes $\hat{G}_1, \ldots, \hat{G}_k$ and maps $\hat{G}_i \to W$ with disjoint images such that x_i is carried by the first stage of $\hat{G}_i$. Suppose these maps are in general position.

Claim. We may assume the only singularities of the map $\hat{G}_i \to W$ are the self-intersections of the first stage and all framings are correct (i.e., we can extend the map $\hat{G}_i \to W$ to an immersion: the 4-dimensional thickening of $\hat{G}_i \to W$).

Proof of the claim. First notice that all framings can be corrected at the cost of introducing new singularities. Next, we will use the trick—"pushing a singularity down to the next stage" to arrange the singularities of the map $\hat{G}_i \to W$ as we desired. We use the following picture to show how we change the map $\hat{G}_i \to W$ by this trick:

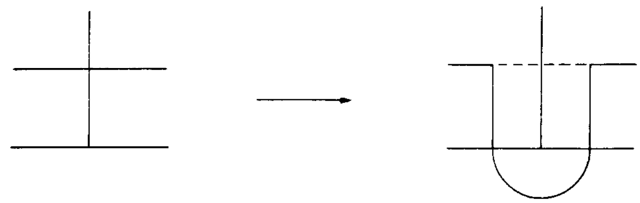

Fig. 4.3.

Note this is a regular homotopy, so the framings will not be changed. We change the map $\hat{G}_1 \to W$ in several steps.

Step 1. Push the self-intersections of the rth-stage ($r > 1$) down to the first stage. So we may assume the rth-stage is embedded for any $r > 1$.

Step 2. Push the intersections of the rth-stage and the sth-stage ($r > s$) down to the sth-stage. So we may assume there is not singularity on the top stage.

Step 3. Repeat Step 1 and Step 2. Finally, there will be only singularities on the first stage.

The image of $\hat{G}_i$ is still denoted by $\hat{G}_i$. Let $N_i = N(\hat{G}_i)$, the regular neighborhood of $\hat{G}_i$ in W, and $N = \coprod_{i=1}^{k} N_i$. Then $\partial N_i = S_0$ (some generalized $L(k)$).

Let $Q = \overline{W - N}$. Then $\partial Q = \partial^+ Q \cup \partial^- Q$, $\partial^+ Q = \partial W = M$ and $\partial^- Q = \coprod_{i=1}^{k} \partial N_i$. Use the previous notational conversions for the cohomology and homology of $S_0(L(k))$, we have $H^1(\partial^- Q) = A^* \oplus B^* \oplus C^*$, and $H_1(\partial^- Q) = A \oplus B \oplus C$. Note $H^2(\partial^- Q)$ is Poincaré dual to $H_1(\partial^- Q)$, we also denote $H^2(\partial^- Q) = A \oplus B \oplus C$.

Let $i^{\pm}$: $H^*(Q) \to H^*(\partial^{\pm} Q)$ be induced by inclusions. We have:

(1) i^-: $H^{2,3}(Q) \to H^{2,3}(\partial^- Q)$ is an isomorphism;
(2) i^+: $H^1(Q) \to H^1(\partial^+ Q)$ is an isomorphism;
(3) $H^1(\partial^- Q) = B^* \oplus C^* \oplus i^-(H^1 Q)$
(4) $H^2(\partial^- Q) = A \oplus i^-$ $(\ker i^+ \colon H^2(Q) \to H^2(\partial^+ Q))$.

These cohomological properties are not hard to check.

Now we have $\alpha_1^+, \ldots, \alpha_k^+ \in H^1(M) = H^1(\partial^+ Q)$ dual to the boundary reduction of $x_1, \ldots, x_k$ respectively. According to (2), there are cohomology classes $\alpha_1, \ldots, \alpha_k \in H'(Q)$ such that $i^+(\alpha_i) = \alpha_i^+$, $i = 1, \ldots, k$. Let $\alpha_i^- = i^-(\alpha_i) \in H'(\partial^- Q)$, $i = 1, \ldots, k$. For each component of $\partial^- Q$, there is only one α_i^- restricted on this component not belonging to $B^* \oplus C^*$, all other α_i^-'s restricted on this component are in $B^* \oplus C^*$. From now on, any statement about the properties of $\partial^- Q$ should be understood as applying to each component of $\partial^- Q$.

Claim. There is a defining system $\{\Gamma_{ij}\}$ for the Massey product $\langle \alpha_1, \ldots, \alpha_{k-1} \rangle$ such that $\{i^- \Gamma_{ij}\}$ is a separated defining system for the Massey product $\langle \alpha_1^-, \ldots, \alpha_{k-1}^- \rangle$. We use double brackets $\langle\langle \quad \rangle\rangle$ to denote any element of such a subset of the usual Massey product coset.

We first show this claim will finish the proof of the Theorem 4.2. By Lemma 4.4, we have $\langle\langle\alpha_1^-, \ldots, \alpha_{k-1}^-\rangle\rangle \in A$. At most one $\alpha_i^- \notin B^* \oplus C^*$, if this α_i^- is not among the first $k-1$ cocycles, $\langle\langle\alpha_1^-, \ldots, \alpha_{k-1}^-\rangle\rangle = 0$. If it is, the cup product $\langle\langle\alpha_1^-, \ldots, \alpha_{k-1}^-\rangle\rangle \cup \alpha_k^-$ is between A and $B^* \oplus C^*$ and hence zero. Consequently $\langle\langle\alpha_1^-, \ldots, \alpha_{k-1}^-\rangle\rangle \cup \alpha_k^- = 0$. According to (1), we have $\langle\langle\alpha_1, \ldots, \alpha_{k-1}\rangle\rangle \cup \alpha_k = 0$. Since any $r(<k-1)$-tuple Massey product on $M = \partial^+ Q$ vanishes, $\langle\alpha_1^+, \ldots, \alpha_{k-1}^+\rangle$ contains only one element which is $i^+(\langle\langle\alpha_1, \ldots, \alpha_{k-1}\rangle\rangle)$. Thus $\langle\alpha_1^+, \ldots, \alpha_{k-1}^+\rangle \cup \alpha_k^+ = 0$.

Now let us prove the claim. Suppose we already have a defining system $\{\Gamma_{ij}\}$ for $\langle\alpha_1, \ldots, \alpha_{k-2}\rangle$ such that $\{i^-(\Gamma_{ij})\}$ is separated. Then $i^-(\langle\langle\alpha_1, \ldots, \alpha_{k-2}\rangle\rangle) = \langle\langle\alpha_1^-, \ldots, \alpha_{k-1}^-\rangle\rangle \in A$. But $i^+(\langle\langle\alpha_1, \ldots, \alpha_{k-2}\rangle\rangle) = \langle\langle\alpha_1^+, \ldots, \alpha_{k-2}^+\rangle\rangle = 0$. According to (4) and (1), we have $\langle\langle\alpha_1, \ldots, \alpha_{k-2}\rangle\rangle = 0$. So we have a 1-cochain Γ_{1k-2} such that

$$\delta\Gamma_{1k-2} = \sum_{r=1}^{k-3} \Gamma_{1r} \cup \Gamma_{r+1k-2}.$$

Consider $i^-(\Gamma_{1k-2})$. Note

$$\sum_{r=1}^{k-3} i^-(\Gamma_{1,r}) \cup i^-(\Gamma_{r+1k-2}) = \delta u^b + \delta u^c$$

such that $\operatorname{supp}(u^b) \subset X_b$ and $\operatorname{supp}(u^c) \subset X_c$ since the defining system $\{i^-(\Gamma_{ij})\}$ for $\langle\alpha_1^-, \ldots, \alpha_{k-2}^-\rangle$ is separated and $\langle\langle\alpha_1^-, \ldots, \alpha_{k-2}^-\rangle\rangle = 0$. Remember we have arranged that if u is a 2-coboundary with $\operatorname{supp}(u) \subset X_b$ (or X_c), then there is an 1-cochain v with $\operatorname{supp}(V) \subset X_b$ (or X_c) and $\delta v = u$. We can use (3) to adjust Γ_{1k-2} so that $[i^-\Gamma_{1k-2} - (u^b + u^c)] \in B^* \oplus C^*$. So

$$i^-\Gamma_{1k-2} = u^b + u^c + rb^* + sc^* + \delta\Omega,$$

where r and s are integers and Ω is a 0-cochain in $\partial^- Q$. Now extend the 0-cochain Ω to Q, we can adjust Γ_{1k-2} so that

$$i^-\Gamma_{1k-2} = u^b + u^c + rb^* + sc^*.$$

Inductively, this proves the claim. So, the theorem 4.2 has also been proved. □

A final remark to the cautious is in order.

We have computed Massey products within the simplicial cochain complex whereas some of the standard results are derived in the context of singular cochains. However, as both are provided with multiplications derived from the Alexander–Whitney map the inclusion of the first into the second commutes—on the nose—with multiplication. It is immediate that Massey products in the two complexes commute with inclusion. For completeness we note that even for maps of multiplicative cochain complexes which do not preserve cochain level cup-products, Massey products may be corresponded via the double cochain complexes of Cěch theory with coefficients in the respective complexes. This was done in H. Schulman's thesis [12].

Now we can prove the half-grope lemma.

Proof of the Theorem 2.5. We have a link L with k components in $S^3 = \partial B^4$. It bounds disjoint singular $(k-1)$-stage half-gropes in B^4. We want to show L is homotopically trivial. Notice if L' is homotopic to L, L' has the same property as L. So we can simplify L by homotopy. Inductively, we may assume L is homotopically almost trivial, i.e., any link obtained from L by omitting one component of L is homotopically trivial.

LEMMA 4.5. *If L is a homotopically almost trivial link with k components, then L is homotopic to a link where all $\bar{\mu}$-invariants of length $\leq k-1$ are zero.*

Proof. We may assume $L=(L_1,\ldots,L_k)$, where $(L_1,\ldots,L_{k-1})$ is a trivial link. Let a_i be the meridian of L_i, $i=1,2,\ldots,k$. After a link homotopy, we may arrange that as a loop in the complement of $(L_1,\ldots,L_{k-1})$, L_k is homotopic to a product of $\beta_{k-1}(a_{i_1},\ldots,a_{i_{k-1}})$'s, where β_{k-1} is the basic $(k-1)$-fold commutator and $(i_1,\ldots,i_{k-1})$ is a permutation of $(1,2,\ldots,k-1)$. We may further arrange that the link

$$(L_1,\ldots,L_{k-1},\beta_{k-1}(a_{i_1},\ldots,a_{i_{k-1}}))$$

is isotopically almost trivial. So the desired vanishing of $\bar{\mu}$-invariants is obtained. □

The relation between Massey products and Milnor's $\bar{\mu}$-invariants allow us to derive the half-grope lemma from the theorem 4.2. Let $M=S_0(L)$ and $\{\alpha_1,\ldots,\alpha_k\}$ be a basis of $H^1(M)$ dual to the meridians of L. We will use the following theorem.

THEOREM 4.6. ([7], Theorem 3). *If every $\bar{\mu}$-invariants of L of length $\leq p$ vanishes, then, for any sequence $1\leq j_1,\ldots,j_p,\ i\leq k$, the p-tuple Massey product $\langle\alpha_{j_1},\ldots,\alpha_{j_p}\rangle$ on $H^1(M)$ contains only one element and*

$$\langle\alpha_{j_1},\ldots,\alpha_{j_p}\rangle\cup\alpha_i=(-1)^p\bar{\mu}(j_1,\ldots,j_p,i)[M^*],$$

where $[M^*]$ is the generator of $H^3(M)$.

By Lemma 4.5, we can assume L is isotopically almost trivial. So $\bar{\mu}$-invariants of L of length $\leq k-1$ are all zero. This by Theorem 4.6 implies that any $r(<k-1)$-tuple Massey product on $M=S_0(L)$ vanishes. Let W be the 4-manifold obtained from attaching 2-handles along L with 0-framing. Then this W satisfies the assumption of Theorem 4.2. Thus

$$\langle\alpha_1,\ldots,\alpha_{k-1}\rangle\cup\alpha_k=0.$$

This, by Theorem 4.6 again, implies $\bar{\mu}(1,2,\ldots,k)=0$ for the link L. Therefore, L is homotopically trivial. □

SECTION 5. HOMOTOPY OF LINK PAIRS—A POSSIBLE OBSTRUCTION TO SURGERY

We return to the general setting where both A_i and B_i are handle bodies with handles of index one and two.

The reader may observe that in the examples in Section 1, Alexander duality between A and B is explicitly manifested in the handle diagrams. In general, a handle structure of $A=\partial^+A\times[0,1]\cup\mathbf{H}_1\cup\mathbf{H}_2$ determines a somewhat ambiguous description of B described below.

Regard (see [3] for example) the 1-handles $\mathbf{H}_1$ as unknotted 2-handles $\mathbf{H}_1^*$ removed from a collar $\partial^+A\times[0,2]$, so $A=(\partial^+A\times[0,2]-\mathbf{H}_1^*)\cup\mathbf{H}_2$. For example:

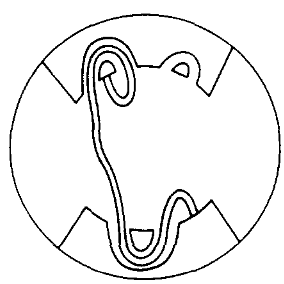

Fig. 5.1. $A\subset B^4$ (Three 1-handles and one 2-handles drawn).

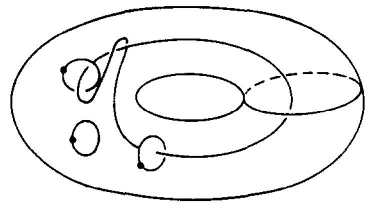

Fig. 5.2. A as a handle body.

Stabilize the handle structure of A by adding a trivial handle pair as indicated in the abstract and ambient diagram (Figs 5.4 and 5.3) for A. Denote the stabilizing 1-handle (or more precisely the corresponding unknotted 2-handle) by $\mathbf{SH}_1^*$.

Regarding the shaded region in Fig. 5.3 as a collar on $\partial^+ B$, we see that $B = (\text{collar} \cup \mathbf{H}_1^* \cup \mathbf{SH}_1^*) - \mathbf{H}_2$. If a link L is (A, B)-slice, then using this description of B, one may construct a slicing of an associated link J in a handle body: 4-ball $\bigcup_{K,\text{ some framing}}$ 2-handles. The link J consists of the attaching circles of $\coprod_{1 \le i \le l} (\alpha_i \mathbf{H}_2^i \cup \beta_i \mathbf{H}_1^{i*} \cup \beta_i \mathbf{SH}_1^{i*})$ and the link K consists of the attaching circles of $\coprod_{1 \le i \ge l} (\alpha_i \mathbf{H}_1^{i*} \cup \beta_i \mathbf{H}_2^i)$. The 4-ball in question is $X = B^4 -$ appropriate collars of $\coprod_{1 \le i \ge l} (\alpha_i \partial^+ A_i \cup \beta_i \partial^+ B_i)$. Since the 1-handle of $\beta_i \mathbf{A}_i$ do not pass through the 2-handles of $\beta_i \mathbf{A}_i$ the slices $\beta_i \mathbf{H}_1^{i*} \cup \beta_i \mathbf{SH}_1^{i*}$ do not pass through the 2-handles attached to $\beta_i \mathbf{H}_2^i$ for any fixed value of i. Call this condition on the slicing condition $(*)$.

In the (A, B)-slicing of L the 2-handles $\beta_i \mathbf{H}_2^i$ are not cleanly attached to ∂X, but have intersections disjoint from their attaching regions. For a given subscript i the 2-handles $\beta_i \mathbf{H}_2^i$ are disjoint so working sequentially for $i = 1, 2, \ldots, l$ we may replace the imbedding $\beta_i \mathbf{H}_2^i$ with abstract attachments to X agreeing with β_i on the attaching regions (and consequently having framings not necessarily equal to zero). The replacement of $\beta_1 \mathbf{H}_2^i$

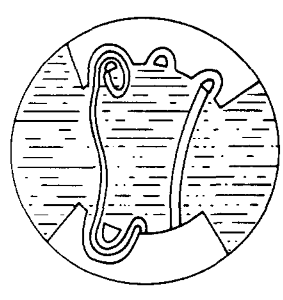

Fig. 5.3. $A \subset B^4$.

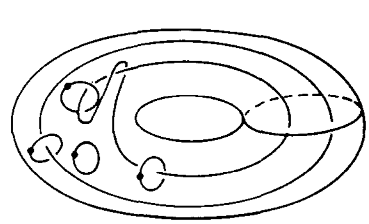

Fig. 5.4. Stabilized handle body structure for A.

changes $\beta_2\mathbf{H}_2^2, \ldots, \beta_l\mathbf{H}_2^l$ to new imbeddings $\beta_i'\mathbf{H}_2^2, \ldots, \beta_l'\mathbf{H}_2^l$ in the resulting handle body on X. However, the attaching regions are unchanged and disjointness (over a single subscript) is preserved so $\beta_2'\mathbf{H}_2^2$ may be replaced by abstractly attached handles. This in turn yields imbeddings $\beta_3'\mathbf{H}_2^3, \ldots, \beta_l'\mathbf{H}_2^l$. Proceeding in this way we finally replace all $\beta_i\mathbf{H}_2^i$ with abstractly attached handles. The above process yields the handle-body $X \bigcup_K$ 2-handles containing the 2-handle slices for the link J satisfying condition (*).

As in Fig. 5.4, the (stabilized) handle structure of A_i is determined by disjoint links M_1^i and M_2^i (disjoint links $\tilde{M}_1^i$ and $\tilde{M}_2^i$) representing the 1-handles and 2-handles respectively. Referring to the same figure, observe the surgery along the stabilizing 2-handle's attaching circle converts the inner boundary of collar $(\partial^+ A_i)$ to the inner boundary of collar $(\partial^+ B_i)$ and sends $\tilde{M}_1^i$ to N_2^i and M_2^i to N_1^i where N_2^i represents the attaching circles for the 2-handles $(\mathrm{H}_i^{i*} \cup Sh_1^{i*})$ added, and N_1^i the attaching circles for the 2-handles subtracted $(\mathbf{H}_2^i)$ in the description of B_i. This verifies the following proposition.

PROPOSITION 5.1. *The link pair* (N_2^i, N_1^i) in $S^1 \times D^2$ *has the form*:

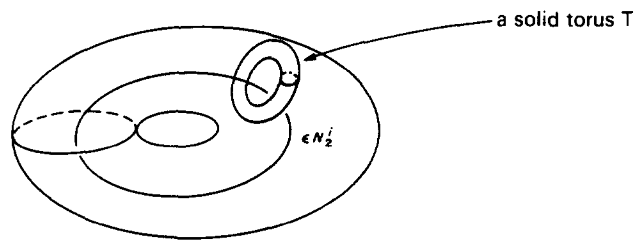

Fig. 5.5.

In T, there is an (untwisted) copy of (M_1^i, M_2^i): *After composing with the pictured inclusion, the components of* M_1^i *become all but one component of* N_2^i *and the components of* M_2^i *constitute* N_1^i.

Thus the set of link pairs (J, K) which can arise from some (A, B)-slicing of L may be described as follows. Take disjoint link pairs $(M_1^i, M_2^i) \subset S^1 \times D^2$, $1 \le i \le l$, (these to arise must describe a handle body structure for A_i). Near each component l_i of L imbed a parallel (untwisted) $S^1 \times D^2$ containing (M_1^i, M_2^i) and continue to use the same symbols to denote the composed link pair. Then (without twisting) imbed a small linking $S^1 \times D^2$ very close to l_i and denote the image of (M_1^i, M_2^i) under this imbedding by $(N_{2,\text{minus}}^i, N_1^i)$. Now:

$$J = \bigcup_i [l_i \cup M_2^i \cup N_{2,\text{minus}}^i]$$

$$K = \bigcup_i (M_1^i \cup N_1^i)$$

The problem is to find a general obstruction to slicing J "relative to K". The 2-handles attached to K form bridges which can facilitate the slicing of J (although condition (*) prohibits some slices from passing over certain handles); in some sense one must measure the difference between J and K.

Questions about slicing links are traditionally very difficult; we will pose our problem somewhat weaker but more tractable setting of link homotopy (recall "concordance implies homotopy", see [5], [6], and [8]).

John Milnor's senior thesis (see [9]) which we have referred to in Section 3 introduced the notion of link homotopy in which two links in S^3 are equivalent if one can be homotopied to the other without *distinct* components intersecting. He showed that essentiality was detected by obstructions lying in nilpotent groups (of height equal to the number of components minus 1) and gave an algorithm for evaluating the first nonvanishing obstruction. He thus provided an algorithm to decide whether a link L in S^3 is essential or trivial (i.e., null homotopic).

Definition 5.2. We say a link L is *stably-trivial* if it has some associated link pair (J,K) as above (in the construction of (J,K) we now admit any link pair's $(M_1^i, M_2^i) \subset S^1 \times D^2$ without regard to whether they describe a handle body structure for some A_i) which admits a homotopically trivial derived link Q. In this case we say that the pair (J,K) is *homotopically trivial.*

Definition 5.3. A link Q is *derived* from a link pair (J,K) if Q can be formed from J as follows: first replace each component of K by several (possibly twisted) parallel copies then band connected sum each components of J to some of these copies (no two distinct components of J may be summed to the same copy). If a component of J comes from N_2^i it may not be summed to any copy of a component of K coming from N_1^i. The restriction embodies condition (∗) and is necessary to avoid trivial examples.

PROPOSITION 5.4. *If J, a link in $S^3 - K$, is slice in* $B^4 \bigcup_{K,\text{some framing}}$ *2-handles then the pair (J,K) is homotopically trivial.*

PROPOSITION 5.5. *If L is (A,B)-slice then L is stably-trivial.*

Proof of Proposition 5.5. If L is (A,B)-slice then associated to A_i, $i = 1, \dots, l$, there are links $(M_1^i, M_2^i) \subset S^1 \times D^2$ which defines an associated pair (J,K) satisfying the hypothesis of Proposition 5.4. □

What we have required above is not quite sufficient to exclude trivial examples. However, a more thorough analysis (to appear elsewhere) of the Morse theory of (A,B) — decompositions leads to a refinement of the stabilization operation. We require $M_1^i = S^1 \times D^2$ to become an unlink when $S^1 \times D^2$ is included into S^3 in the standard fashion, $M_2^i \subset S^1 \times D^2$ to be an unlink contained in a 3-ball in $S^1 \times D^2$, but the linking between M_1^i and M_2^i is still unrestricted. In the definition of "derived" all parallels to K should be taken untwisted. If a component of J comes from $M_2^i(N_2^i)$ it may not be summed to any copy of a component of K coming from $M_1^i(N_2^i)$. Proposition 5.5 continues to hold for this refinement of "stably-trivial" and no example is known where of a homotopically essential link which is "stably-trivial" in this sense.

Proof of Proposition 5.4. The argument is Morse theory plus the observation of the second author in [8]. Put the cores of the slices in smooth general position and arrange that they pass through the 2-handles as a parallel copes of their cores. Chop off the 2-handles to find a planar surface disjointly immersed in B^4.

We wish to remove all local maximum while preserving disjointness of the planar surfaces. By ordering critical points according to index it is sufficient to consider the case of a saddle and maximum which are in cancelling position in an annular region $S^3 \times [\rho_0, \rho_2] \subset B^4$ in which there are no other critical points. The group theoretic relation determined by the local maximum can be introduced at the level of the saddle point by self-"finger-moves" to the various planar surfaces [8]. Thus, after a homotopy the saddle can be eliminated by a different disk in its level. This cancels the pair.

Once local maxima are eliminated the saddles may float up to $S^3 = \partial B^4$. Here the ascending manifolds of the saddles become the cores of some band connected sums. The saddles in each connected planar surface now join one component of J and several parallel copies of components of K. (One may easily arrange each band to connect, at one of its ends, directly to a component of J.) Deleting the interior of the bands we find a classical null homotopy for some link Q derived from (J, K). □

We do not know if every stably-trivial link is in fact homotopically trivial.

We specialize this discussion to the case where A is a handle body on $\partial^+ A$ with only 2-handles. We obtain a proof of a variant of Theorem 2.2 which does not rely on Section 4.

Call A "robust" if there exist a link Q in $\partial^+ A$ so that: (1) the extended link $\hat{Q}$ in S^3 is homotopically essential, and (2) the components of Q bound disjoint singular disks in A. (To compare with Section 2, strong $\Rightarrow$ robust, but robust $\not\Rightarrow$ strong; both are provisional definitions. Note the definition of being robust does not depend on the handle structure of A. For the purpose of studying the general situation, this may be preferable.)

Assume L is A, B-slice with each A_i a 2-handle body as above. We will show that L is homotopically trivial.

Now M_1^i is empty for each i. The link $J = \bigcup_i (l_i \cup M_2^i)$ is slice in the handle body: 4-ball $\bigcup_{K_{0\text{-framed}}}$ 2-handles, $K = \bigcup_i N_1^i$. For each i where A_i is robust, we have a null homotopy in A_i of some $Q_i \subset \partial^+ A_i$ (where $\hat{Q}_i \subset S^3$ is homotopically essential.)

For each i where A_i is *not* robust it is possible to "cut off" disjoint singular disks that pass through 2-handles attached to $\bigcup_{A_i,\,\text{non-robust}} N_1^i$. That is disjoint singular planar surfaces which run through these copies of A_i are first replaced by disjoint singular disks and then by singular disks which do not enter the 2-handles. All disjointnesses may be maintained but to assure this self-finger moves must be introduced to l_i (these wind around N_1^i in some manner) for each i where A_i is not robust. This follows from the definition.

Return to those i where A_i is robust. The attaching of 2-handles along N_1^i is essentially the same as attaching a copy of A_i (to X) by identifying $\partial^+ A_i$ with a small linking solid torus to l_i. Extend this α solid torus to a 3-ball by spanning across l_i and extend $\partial^+ A$ to a 3-ball ball in ∂B^4. Finally, extend the attachment of A to X by the attachment of B^4 to X by identifying the 3-balls. This enlarges X to "engulf" the 2-handles or robust A_is whereas the cut off argument above pulls disjoint homotopies off the 2-handles of nonrobust A_is. Thus we may construct singular disks lying entirely in a 4-ball whose boundaries constitute a link $\bar{L}$ in ∂(4-ball) where $\bar{L} = \left(\bigcup_{A_i,\,\text{non-robust}} l_i\right) \cup \left(\bigcup_{A_i,\,\text{robust}} Q_i\right)$. As remarked earlier ([L]) a link which bounds disjoint singular disks in B^4 is homotopically trivial. Thus $\bar{L}$ is homotopically trivial. By the link composition lemma (Section 3) L is homotopically essential implies $\bar{L}$ is homotopically essential. It follows that L is homotopically trivial.

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