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HOMOTOPY IS NOT ISOTOPY FOR HOMEOMORPHISMS OF 3-MANIFOLDS

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INTRODUCTION

THE existence of a homeomorphism that is homotopic but not isotopic to the identity has remained an open question for closed 3-manifolds [1, 2]. We consider here homeotopy groups‡ of spherical spaces, finding as a by-product of our work an example of such a homeomorphism for a closed 3-manifold whose prime factors include certain spherical spaces.

The homeotopy groups of a composite 3-manifold have as subgroups the disk-fixing or point-fixing homeotopy groups of each prime factor [3, 4]. In the work reported here our primary aim has been to calculate, for spherical spaces the corresponding 0th homeotopy groups, the groups of path connected components of the spaces of disk-fixing and point-fixing homeomorphisms.

Homeomorphism groups of spherical spaces have been considered recently by Rubinstein *et al.* [5–7], Asano [8], Bonahon [9] and Ivanov [10]. Their results are consistent with Hatcher's conjecture [11] that for each spherical space the group of homeomorphisms has the same homotopy type as the group of isometries. Homotopy classes of the groups $\mathscr{H}_D$ and $\mathscr{H}_x$ of homeomorphisms that fix respectively a disk and a point do not generally have this character (for spherical spaces): in particular, nonzero elements of $\pi_0(\mathscr{H}_D)$ and $\pi_0(\mathscr{H}_x)$ are commonly not represented by isometries. For several spherical spaces of the form S^3/H, with $H \subset SU(2)$, however, we find that each class of homeomorphisms in $\mathscr{H}_x$ *is* represented by an isometry; $\pi_0(\mathscr{H}_x) = \pi_0(\mathscr{I}_x)$, where $\mathscr{I}_x$ is the group of isometries that fix a point. Each of these spherical spaces (with $H \subset SU(2)$) can be constructed by identifying opposite faces of a polyhedron, and $\pi_0(\mathscr{H}_x)$ can then be regarded as the group of orientation preserving symmetries of the polyhedron. Thus $\pi_0(\mathscr{H}_x)$ is isomorphic to a subgroup of $SO(3)$, and (except for the case of lens spaces) the corresponding group $\pi_0(\mathscr{H}_D)$ is isomorphic to the double covering in $SU(2) \approx \overline{SO(3)}$ of $\pi_0(\mathscr{H}_x)$. The additional generator in $\pi_0(\mathscr{H}_D)$ is a rotation parallel to a sphere enclosing the fixed disk.

The homeotopy groups of 3-manifolds appear to play a role in quantum gravity. In classical general relativity, two metrics or two sets of tensor fields are physically equivalent if they differ by the action of a diffeomorphism. The analogue in the quantum theory of this "general covariance" is the invariance of state vectors under diffeomorphisms in the component of the identity. Diffeos *not* connected to the identity, however, can act nontrivially on the vector space of quantum states associated with a fixed 3-manifold [13–16]. In

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‡ The homeotopy groups of a manifold are the homotopy groups of its space of homeomorphisms [12]. By "disk-fixing" ("point-fixing") homeotopy groups we will mean the homotopy groups of the space of homeomorphisms that fix a disk (or a point) of the manifold.

canonical quantum gravity, the zeroth homeotopy groups of 3-manifolds are then dynamical symmetry groups, groups which act on the state space but which leave the Hamiltonian invariant.

A surprising feature of the work, mentioned above, is the result (section 2) that a rotation parallel to a sphere is not *isotopic* to the identity for certain 3-manifolds where Hendriks [17] has shown it to be *homotopic* to the identity. This appears to be the first known instance of a homeomorphism that is homotopic but not isotopic to the identity for a closed 3-manifold.

§1. PRELIMINARIES

(a) *Construction of the spaces from polyhedra*

Each spherical space S^3/H, with H a finite subgroup of $SU(2)$, can be constructed from a polyhedron by identifying opposite faces [18]. The orientation-preserving symmetry group of the polyhedron respects these identifications and acts as a group of isometries on S^3/H (with its natural metric). Apart from Z_p, the finite subgroups of $SU(2)$ are double coverings of the finite subgroups of $SO(3)$, namely T^*, the 24 element covering of the tetrahedral group; O^*, the 48 element covering of the octahedral group; I^*, the 120 element covering of the icosahedral group; and the family D^*_{4m}, the $4m$-element coverings of the dihedral groups.

The octahedral space, S^3/T^*, is constructed from a solid octahedron as in Fig. 1. The identification of a pair of (shaded) faces is shown, and identification of the other pairs of faces are implied by the octahedral symmetry. The spaces S^3/O^* and S^3/I^* are similarly obtained from a truncated cube and a solid dodecahedron as shown in Figs. 2 and 3, respectively. Each space S^3/D^*_{4m} is constructed from a $2m$-sided prism: The top and bottom are identified after a relative rotation by π/m, and opposite rectangular faces are identified after a rotation by $\pi/2$ as in Fig. 4.

The lens spaces $L(p, q)$ are constructed from a disk by identifying the top and bottom hemispherical surfaces after a relative rotation by $2\pi(q/p)$.

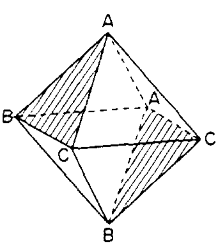

Fig. 1. Octahedron space, S^3/T^*. In Figs. 1–4 opposite faces are identified so that vertices labelled by the same letter coincide.

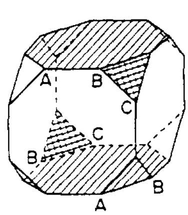

Fig. 2. Truncated cube space, S^3/O^*.

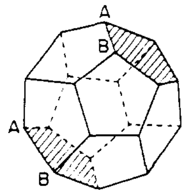

Fig. 3. Dodecahedron space, S^3/I^*.

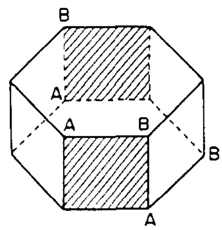

Fig. 4. A prism manifold: S^3/D^*_{12}.

(b) *Exact sequences for homeomorphism groups*

The group $\mathscr{H}$ of homeomorphisms of a compact manifold M is a fibration (in fact a fiber bundle) over M with fibers isomorphic to the group $\mathscr{H}_x$ of homeomorphisms that fix a point $x \in M$. (We assume throughout the compact-open topology for spaces of homeomorphisms.) The projection map $p: \mathscr{H} \to M$ is given by $p(\psi) = \psi(x)$, $\psi \in \mathscr{H}$. Because $\mathscr{H}$ is a topological group, the homotopy exact sequence associated with this fibration ends in

$$\to \pi_1(\mathscr{H}) \xrightarrow{p_*} \pi_1(M) \xrightarrow{\partial_*} \pi_0(\mathscr{H}_x) \xrightarrow{i_*} \pi_0(\mathscr{H}) \to 1. \tag{1}$$

For a spherical space $M = S^3/H$, $\pi_1(M) = H$. The sequence (1) implies

$$o(H) = o(\operatorname{Im} p_*)o(\operatorname{Im} \partial_*)$$

and

$$o[\pi_0(\mathscr{H}_x)] = o(\operatorname{Im} i_*)o(\operatorname{Im} \partial_*).$$

Thus

$$o[\pi_0(\mathscr{H}_x)] = o(H)o[\pi_0(\mathscr{H})]/o(\operatorname{Im} p_*). \tag{2}$$

There is an analogous relation between $\pi_0(\mathscr{H}_D)$ and $\pi_0(\mathscr{H})$, where, as before, $\mathscr{H}_D$ is the group of homeomorphisms fixing an embedded disk $D \subset M$. Again $\mathscr{H}$ is regarded as a fibration, now over the space $\mathscr{E}$ of embeddings in M of a disk $D_0 \subset R^3$. The projection $\hat{p}: \mathscr{H} \to \mathscr{E}$ is given in terms of a fixed embedding $e: D_0 \to M$ by $\hat{p}(e) = \psi \circ e$. The corresponding exact sequence ends in

$$\to \pi_1(\mathscr{H}) \xrightarrow{\hat{p}_*} \pi_1(\mathscr{E}) \xrightarrow{\hat{\partial}_*} \pi_0(\mathscr{H}_D) \to \pi_0(\mathscr{H}) \to 1. \tag{$\hat{1}$}$$

For $M = S^3/H$, $\pi_1(\mathscr{E}) = Z_2 \times H$ and Eq. (2) is replaced by

$$o[\pi_0(\mathscr{H}_D)] = 2o(H)o[\pi_0(\mathscr{H})]/o(\operatorname{Im} \hat{p}_*). \tag{3}$$

(c) *Notation*

Elements of $SU(2)$ can be written in the form

$$u(\theta\hat{n}) = \exp(\theta\hat{n}), \tag{4}$$

where $\hat{n}$ is a unit vector in the Lie algebra $su(2)$. We will denote by $R(\theta\hat{n})$ the rotation by θ about a unit vector $\hat{n}$ in R^3. Then the projection $SU(2) \to SO(3)$ has the form

$$u\left(\frac{\theta}{2}\hat{n}\right) \mapsto R(\theta\hat{n}).$$

If x is a vector in R^3, we will denote the rotated vector by $R(\theta\hat{n})x$.

§2. ROTATIONS PARALLEL TO A SPHERE

Denote by $\mathscr{R}$ a rotation parallel to a sphere of a 3-manifold. If the manifold is prime with the sphere enclosing the prime factor, one would like to know when $\mathscr{R}$ is isotopic to the identity via a path of homeomorphisms that fix a disk outside the sphere.

If the manifold is not prime, one can ask the question without restricting consideration to maps that fix a disk: If $\mathscr{R}$ is parallel to a sphere that separates prime factors, then $\mathscr{R}$ is isotopic to the identity iff $\mathscr{R}$ is isotopic to the identity rel. D for one of the prime factors.

A result due to Hendriks [17] decides the related question of when $\mathscr{R}$ is homotopic to the identity:

THEOREM 2.1 (Hendriks). *Let S be a sphere embedded in a closed 3-manifold M that separates M into submanifolds M_1 and M_2 and let $\mathscr{R}$ be a rotation parallel to S. Then $\mathscr{R}$ is homotopic to the identity iff M_1 or M_2 is the connected sum of a 3-sphere and manifolds homotopy equivalent to $S^2 \underset{\sim}{\times} S^1$, $S^2 \times S^1$, $P^2 \times S^1$, or to a closed manifold whose fundamental group is finite and has a cyclic 2-Sylow subgroup ($S^2 \underset{\sim}{\times} S^1$ denotes the non-orientable handle). Equivalently, for a prime manifold M, $\mathscr{R}$ is homotopic rel. D to the identity iff M is homotopy equivalent to $S^2 \times S^1$, $S^2 \underset{\sim}{\times} S^1$, $P^2 \times S^1$ or to a closed manifold whose fundamental group is finite and has a cyclic 2-Sylow subgroup.*

This characterization of manifolds for which $\mathscr{R}$ is homotopic to the identity can be made more explicit as follows.

COROLLARY 2.1. *For a prime 3-manifold M, $\mathscr{R}$ is homotopic rel. D to the identity iff M is homeomorphic to $S^2 \times S^1$ or $S^2 \underset{\sim}{\times} S^1$, is homotopy equivalent to $P^2 \times S^1$, or is homeomorphic to Σ^3/H, where Σ^3 is a homotopy sphere and $H \approx Z_p$, $D^*_{4(2n+1)}$, $D'_{2^k(2n+1)}$, $Z_p \times D^*_{4(2n+1)}$, or $Z_p \times D'_{2^k(2n+1)}$.*

Proof. "Homotopy equivalent to $S^2 \times S^1$, $S^2 \underset{\sim}{\times} S^1$" was strengthened to "homeomorphic" because there are no prime fake handles [19].

The only prime closed 3-manifolds with finite fundamental group H are homeomorphic to Σ^3/H, where H acts freely on Σ^3. For $\Sigma^3 \approx S^3$, H is a subgroup of $SO(4)$ in the following list [20]; Z_p, T^*, O^*, I^*. D^*_{4m} (these are the finite subgroups of $SU(2)$); $D'_{2^k(2n+1)}$, an extension of $Z'_{2^{k-1}}$ by $D^*_{4(2n+1)}$; $T'_{8.3^k}$, $k \geq 2$, an extension of $Z_{3^{k-1}}$ by T^*; and $Z_p \times H$, where H is any of the above groups and Z_p a cyclic group of relatively prime order. Of these groups, T^*, O^*, I^*, $T'_{8.3^k}$, and D^*_{8m} have as a subgroup the 8-element quaternion group D^*_8. Consequently their 2-Sylow subgroups contain D^*_8 and cannot be cyclic. Moreover, the 2-Sylow subgroup of $Z_p \times H$ is cyclic only if it is cyclic for H. Thus only Z_p, $D^*_{4(2n+1)}$, $D'_{2^k(2n+1)}$, $Z_p \times D^*_{4(2n+1)}$, and

$Z_p \times D'_{2^k(2n+1)}$ can have cyclic 2-Sylow subgroups; and they all do: For $D^*_{4(2n+1)}$ and $Z_p \times D^*_{4(2n+1)}$ it is isomorphic to Z_4; for $D'_{2^k(2n+1)}$ and $Z_p \times D'_{2^k(2n+1)}$ it is isomorphic to Z_{2^k}.

The only additional groups that could act freely on a fake 3-sphere are [21] $Q(8n, k, l) = \langle P, Q, A : P^2 = (PQ)^2 = Q^{2n}, PAP^{-1} = A^r, QAQ^{-1} = A^{-1}, A^{kl} = 1 \rangle$, $r \equiv -1 \bmod k, r \equiv 1 \bmod l$, where $8n$, k and l are relatively prime, and n is odd. Each of these groups has 2-Sylow subgroup isomorphic to D^*_8 (generated by P and Q^n). Thus the only spaces Σ^3/H for which H has cyclic 2-Sylow subgroup are those listed in the Corollary. □

Hendrick's Theorem implies that $\mathscr{R}$ is *homotopic* rel. D to the identity for $S^3/D'_{2^k m}$, $S^3/(D'_{2^k m} \times Z_p)$, and $S^3/(D^*_{4m} \times Z_p)$, m odd. We now show that it is not isotopic to the identity.

LEMMA 2.1. *If $\mathscr{R}$ is isotopic to the identity rel.D then $Im(\hat{p}_*)$ contains the element $(1, -1) \in \pi_1(\mathscr{E}) \approx \pi_1(M) \times Z_2$. Here, as in the sequence $(\hat{1})$, $\hat{p}_*$: $\pi_1(\mathscr{H}) \to \pi_1(\mathscr{E})$ corresponds to the projection $\hat{p}: \mathscr{H} \to \mathscr{E}$.*

Proof. Given a chart x on $U \subset M$ with $U = \{p: |x(p)| < R\}$ we may choose for $\mathscr{R}$ the homeomorphism $\mathscr{R}(x) = R(\hat{z}2\pi f)x$, where $f = f(|x|)$ is continuous with $f = 0$ in a neighborhood of ∂U and $f = 1$ in an embedded disk D about $x = 0$. ($\mathscr{R}$ is to be the identity outside U.) Define a path of homeomorphisms $\theta \mapsto \mathscr{R}_\theta, 0 \le \theta \le 2\pi$ by $\mathscr{R}_\theta(x) = R(\hat{z}\theta f)x$. Then $\mathscr{R}_{2\pi} = \mathscr{R}$ and the orbit of the disk D is a sequence of rotated disks, to a closed loop in $\mathscr{E}$ belonging to $(1, -1)$ in $\pi_1(\mathscr{E}) \approx \pi_1(M) \times Z_2$.

If $\mathscr{R}$ is isotopic to the identity rel. D and I_λ, $0 \le \lambda \le 1$ is such an isotopy, then $\mathscr{R}_\theta \cdot I_\lambda$ is a closed loop in $\mathscr{H}$ for which $\hat{p}_*(\mathscr{R}_\theta \cdot I_\lambda) = (1, -1) \in \pi_1(\mathscr{E})$. □

LEMMA 2.2. *For the spaces S^3/G, with $G = D'_{2^k m}$, $D'_{2^k m} \times Z_p$ or $D^*_{4m} \times Z_p$, the map p_*: $\pi_1(\mathscr{H}) \to D$ has image isomorphic to $Z_{2^{k-1}p}$.*

Proof. The group $D'_{2^k m}$, $D'_{2^k m} \times Z_p$, or $D^*_{4m} \times Z_p$ with m odd has the presentation $G = \langle a, b | a^{2^k p} = 1, b^m = 1, aba^{-1}b = 1 \rangle$ where $k > 2$ with $p = 1$, $k > 2$ with $p > 1$, or $k = 2$ with $p > 1$ respectively.

Ivanov [10] has shown that the space Diff of diffeomorphisms of S^3/G has the same homotopy type as the space of isometries, namely $Z_2 \times Z_2 \times S^1$. Then Hatcher's proof [22] of the Smale conjecture together with Cerf's result [23] that, given the Smale conjecture, Diff and $\mathscr{H}$ have the same homotopy type imply that $\pi_1(\mathscr{H}) \approx \pi_1(Z_2 \times Z_2 \times S^1) \approx Z$. Now $\mathrm{Im}(p_*) \subset \mathrm{Center}(G) \approx Z_{2^{k-1}p}$ (McCarty, Remark 5.24 [12]). Thus we need only show that $\mathrm{Im}\, p_*$ includes a generator of a group isomorphic to $Z_{2^{k-1}p}$.

We construct as follows an explicit path of diffeos whose image is this generator. Regard G as a subgroup of $(SU(2) \times SU(2))/\{\pm(1, 1)\} \approx SO(4)$, and choose as generators

$$a = \left[\left(j, \exp\left(\frac{i\pi}{2^{k-1}p}\right)\right)\right] \quad b = \left[\left(\exp\left(\frac{i\pi}{m}\right), -1\right)\right] \quad \text{for } k > 2$$

$$a = \left[\left(j, \exp\left(\frac{i\pi}{p}\right)\right)\right] \quad b = \left[\left(\exp\left(\frac{i\pi}{m}\right), -1\right)\right] \quad \text{for } k = 2$$

Then $[(1, e^{i\pi/2^{k-1}p})]$ generates $Z_{2^{k-1}p}$. Let $u_0 = e^{i\theta}$, $0 \le \theta \le \pi/2^{k-1}p$, and define a family of automorphisms of S^3 by

$$x \mapsto x e^{i\theta},$$

for $x \in SU(2) \approx S^3$. Then because $[(1, e^{i\theta})]$ is in the normalizer in $SO(4)$ of G, the automorphism

$$\tilde{u}_\theta \colon [x] \mapsto [xe^{i\theta}]$$

of S^3/G is well defined for each θ.

Since the orbit in $SU(2)$ of a point x is a path $\theta \mapsto xe^{i\theta}$ from x to $x\, e^{i\pi/2^{k-2}p}$, we have

$$p_*(\tilde{u}_\theta) = e^{i\pi/2^{k-2}p}$$

the generator of $Z_{2^{k-1}p}$. Thus $\operatorname{Im} p_* \approx Z_{2^{k-1}p}$.

THEOREM 2.2. *$\mathscr{R}$ is not isotopic to the identity rel. D for S^3/G where $G = D'_{2^k m}, D'_{2^k m} \times Z_p$, or $D^*_{4m} \times Z_p$ with m odd.*

Proof. By Lemma 2.1, it suffices to show that $(1, -1)$ is not in $\operatorname{Im}(\hat{p})$. Since $\pi_1(\mathscr{H}) = Z$, $\operatorname{Im}(\hat{p}_*) = \hat{p}_*(Z)$ is a cyclic subgroup of $\pi_1(\mathscr{E}) = G \times Z_2$. From Lemma 2.2 and the commutative diagram

$$\begin{array}{ccc} \pi_1(\mathscr{H}) & \xrightarrow{\hat{p}_*} & \pi_1(\mathscr{E}) \\ & {\scriptstyle p_*} \searrow \quad \swarrow {\scriptstyle \rho_*} & \\ & \pi_1(M) & \end{array}$$

where $\rho\colon \mathscr{E} \to M$ is the projection, we have $Z_{2^{k-1}p} \subset \operatorname{Im} \hat{p}_*$. But the only cyclic subgroups in $G \times Z_2$ that project to $Z_{2^{k-1}p}$ are generated by $(a^2, 1)$ and $(a^2, -1)$, where a generates $Z_{2^k p}$ in G. Since $(1, -1)$ is in neither of these groups, $(1, -1) \notin \operatorname{Im}(\hat{p}_*)$. □

COROLLARY 2.2. *Let M be the connected sum of two factors of the form $S^3/D'_{2^k m}$, $S^3/(D'_{2^k m} \times Z_p)$, or $S^3/(D^*_{4m} \times Z_p)$, where m is odd; and let $\mathscr{R}$ be a rotation parallel to a sphere enclosing one of the factors. Then $\mathscr{R}$ is not isotopic to the identity.*

Proof. Corollary 2.2 follows immediately from Theorem 2.2 and the fact that for N and N' prime, $\mathscr{R}$ is isotopic to the identity for $N \# N'$ iff $\mathscr{R}$ is isotopic to the identity rel. D for either N or N' [4].†

Thus $\mathscr{R}$ is not isotopic but (by Hendriks' Theorem) is homotopic to the identity for the space M.

Remark. An argument analogous to that used in the proof of Theorem 2.2 would extend that theorem and its corollary to the spaces S^3/D^*_{4m}, if Hatcher's conjecture were known to be true (if, in particular, one could show that $\pi_1(\mathscr{H}) \approx \pi_1(\mathscr{I})$ for S^3/D^*_{4m}). Connected sums of these spaces (with m odd) would thus constitute additional examples of compact 3-manifolds for which a rotation parallel to a sphere enclosing a prime factor was homotopic but not isotopic to the identity.

§3. HOMEOTOPY GROUPS OF SPHERICAL SPACES

We now compute $\pi_0(\mathscr{H}_x)$ and $\pi_0(\mathscr{H}_D)$ for spherical spaces of the form S^3/H with $H \subset SU(2)$. The principal results, Theorems 3.1 and 3.2 and their corollaries were summarized in the introduction. For each space we find $\pi_0(\mathscr{H}_x) = \pi_0(\mathscr{I}_x)$, and for all but the lens spaces, $\pi_0(\mathscr{H}_D)$ is the covering group in $SU(2)$ of $\pi_0(\mathscr{H}_x)$. [As before, $\mathscr{H}_x(\mathscr{I}_x)$ and $\mathscr{H}_D$ denote the groups of homeomorphisms (isometries) that fix respectively a point and a disk.]

† The result follows directly from Hatcher's work [4] for N and N' prime and irreducible. For N or N' prime and reducible, $\mathscr{R}$ is isotopic to the identity.

LEMMA 3.1. *Let H be a finite subgroup of $SU(2)$ acting on $S^3 \approx SU(2)$ by left multiplication. The group $\mathscr{I}_x$ of isometries that fix a point of S^3/H is isomorphic to the normalizer in $SU(2)$ of H.*

Proof. Each isometry of S^3/H is induced by an isometry of S^3. The group $SO(4) \approx (SU(2) \times SU(2)/Z_2)$ of isometries of $S^3 \approx SU(2)$ acts on $SU(2)$ by $[g_1, g_2](g) = g_1^{-1} g g_2$, where $[g_1\, g_2] \in (SU(2) \times SU(2)/Z_2)$ and $g \in SU(2)$. Only isometries of the form $[g_1, g_1]$ fix the point $1 \in SU(2)$; and $[g_1, g_1]$ induces an isometry of S^3/H iff $g_1^{-1} H g_1 = H$, that is, iff g_1 is in the normalizer of H. □

LEMMA 3.2. *Let $H \subset SU(2)$ act on $S^3 \approx SU(2)$ by left multiplication. Each distinct automorphism of H of the form $h \to g^{-1}hg$ with $g \in SU(2)$ corresponds to an isotopically distinct homeomorphism in $\mathscr{H}_x(S^3/H)$.*

Proof. An automorphism $h \to g^{-1}hg$ induces an isometry of S^3/H because $g^{-1}Hg = H$ implies $[g^{-1}ug] = [g^{-1}u'g]$ when $[u] = [u']$, $u \in SU(2)$. Each such isometry acts nontrivially on $\pi_1(S^3/H) = H$ because $h \in H$ is mapped to $g^{-1}hg$, $g \notin$ center of H. But any homeomorphism that is isotopic to the identity in $\mathscr{H}_x(S^3/H)$ leaves $\pi_1(S^3/H)$ fixed. Thus distinct automorphisms correspond to isotopically distinct homeomorphisms in $\mathscr{H}_x(S^3/H)$ as claimed. □

In fact (Corollary 1, below), the group of automorphisms of the form $h \to ghg^{-1}$ with $g \in SU(2)$, is isomorphic to the homeotopy group $\pi_0(\mathscr{H}_x)$ for the spaces S^3/H with $H = 0^*$ and D^*_{4m}. Moreover this group $\pi_0(\mathscr{H}_x)$ can be regarded as the group of orientation preserving symmetries of the polyhedron from which the space S^3/H is constructed. Each symmetry of the polyhedron respects the identification of faces and thus induces an isometry of S^3/H. Generators of $\pi_1(S^3/H)$ correspond to curves joining identified faces, and each polyhedral symmetry permutes generators by permuting faces of the polyhedron.

Corresponding homeomorphisms that fix a disk are easily obtained from these isometries by twisting a disk at the center of the polyhedron back to its original position. We will prove that one thereby obtains a representative of each class of homeomorphisms in $\pi_0(\mathscr{H}_D)$.

To bound the number of elements of $\pi_0(\mathscr{H}_D)$ we will use Eq. (3) together with the fact that $\operatorname{Im} p_*$ is nonzero.

LEMMA 3.3. *Let H be a finite subgroup of $SU(2)$ with $-1 \in H$. A path $\lambda \mapsto u_\lambda \in SU(2)$ with $u_0 = 1$, $u_1 = -1$ induces a loop $\lambda \mapsto \tilde{u}_\lambda$ of isometries of S^3/H for which $p_*(\tilde{u}_\lambda) = -1$.*

Proof. With H acting on $SU(2) \approx S^3$ by left multiplication, right multiplication by $u_\lambda \in SU(2)$ preserves the equivalence classes of S^3/H, thus providing a family of isometries. Since the orbit of $x \in SU(2)$ is a path xu_λ from x to $-x$, the orbit of $[x] \in S^3/H$ is a loop belonging to $-1 \in \pi_1(S^3/H)$. Thus $p_*(\tilde{u}_\lambda) = -1 \in \pi_1(S^3/H) \approx H$. □

THEOREM 3.1. *The spherical spaces S^3/D^*_{4m}, and $S^3/0^*$ have homeotopy groups $\pi_0[\mathscr{H}_D(S^3/D^*_{4m})] \approx D^*_{8m}$, m even, $m > 2$; $\pi_0[\mathscr{H}_D(S^3/D^*_8)] \approx 0^*$; and $\pi_0[\mathscr{H}_D(S^3/0^*)] \approx 0^*$.*

Proof of Theorem 3.1. (a) S^3/D^*_{4m}. As stated in Section 1, the prism spaces are obtained by identifying opposite faces of a solid $2m$-sided prism, $\mathscr{P}$ (Fig. 4). The normalizer in $SU(2)$ of D^*_{4m} is D^*_{8m} [24], and each element $g \in D^*_{8m}$ induces by conjugation an automorphism of D^*_{4m} and a corresponding isometry of S^3/D^*_{4m}. The resulting group of isometries is D_{4m}, the group of orientation-preserving isometries of the $2m$-sided prism (g and $-g$ induce the same isometry). These isometries are the symmetries of the polyhedron from which S^3/D^*_{4m} is constructed.

An explicit action of D^*_{8m} as a group of homeomorphisms (up to isotopy) in $\mathscr{H}_D(S^3/D^*_{4m})$ is then obtained as follows. Let $r: \mathscr{P} \to [0, 1]$ be continuous with $r = 1$ on $\partial\mathscr{P}$, and for which the $r =$ const. surfaces are non-intersecting spheres concentric to $\partial\mathscr{P}$ and $r = 0$ is a ball (the disk that the homeomorphisms will fix). Regard each element of D^*_{8m} as an element of $SU(2)$ of the form $u(\theta\hat{n})$ as in Eq. (4). Then the path $I \to SO(3)$ given by $r \mapsto R(r\theta\hat{n})$, $0 \le r \le 1$, is a representative of $u(\theta\hat{n}) \in \overline{SO(3)} \approx SU(2)$. Defining a homeomorphism $\psi_u \in \mathscr{H}_D(S^3/D^*_{4m})$ by

$$\psi_u(p) = R[r(p)\theta\hat{n}],$$

we obtain the desired action of D^*_{8m} on S^3/D^*_{4m}.

The induced homomorphism $D^*_{8m} \to \pi_0(\mathscr{H}_D)$ is injective because, apart from $\psi_{(-1)}$, each ψ_u acts nontrivially on $\pi_1(S^3/D^*_{4m})$ by Lemma 3.2, and $\psi_{(-1)}$ is a rotation parallel to a sphere about the fixed disk $r = 0$, which, by Theorem 2.3, is not isotopic to the identity in $\pi_0(\mathscr{H}_D)$. Thus $D^*_{8m} \subset \pi_0(\mathscr{H}_D)$.

By Eq. (3) and Lemma 3.3, $o[\pi_0(\mathscr{H}_D)] \le 2o[\pi_0(\mathscr{H}_D)o(D^*_{4m})$. But Rubinstein [5] has shown that $\pi_0(\mathscr{H}) = Z_2$, whence $o[\pi_0(\mathscr{H}_D)] \le 8m = o(D^*_{8m})$. Thus $\pi_0(\mathscr{H}_D) \approx D^*_{8m}$ as claimed.

(b) S^3/O^*, S^3/D^*_8. Both these spaces are constructed from polyhedra (a truncated cube and a cube, respectively) whose group of orientation preserving symmetries is D. These symmetries are isometries of S^3/O^* (S^3/D^*_8) induced by the action on $SU(2)$ of O^*. (O^* is the normalizer in $SU(2)$ of both O^* itself and D^*_8). An action of O^* as a group of homeomorphisms (up to isotopy) in $\mathscr{H}_D(S^3/O^*)$ (or $\mathscr{H}_D(S^3/D^*_8)$) is obtained as in (a). Hendrik's Theorem and Lemma 3.2 again imply that the induced homomorphisms $O^* \to \pi_0(\mathscr{H}_D)$ are injective. Rubinstein and Birman [7] and Rubinstein [5], respectively, have shown that $\pi_0(\mathscr{H}(S^3/O^*)) = 1$ and $\pi_0(\mathscr{H}(S^3/D^*_8)) \approx P_3$, the permutation group on three objects. Equation (3) and Lemma 4 then imply $o(\pi_0(\mathscr{H}_D)) \le o(O^*)$ for each space. Thus, in both cases, $\pi_0(\mathscr{H}_D) \approx O^*$. □

$\mathscr{R}$, the rotation parallel to a sphere is always isotopic to the identity rel. x. Consequently, $\pi_0(\mathscr{H}_x) = \pi_0(\mathscr{H}_D)/Z_2$ for the above spaces, where "Z_2" denotes the two element subgroup $\{1, [\mathscr{R}]\}$ of $\pi_0(\mathscr{H}_D)$.

COROLLARY 3.1. *The spherical spaces S^3/O^* and S^3/D^*_8 have homeotopy group $\pi_0(\mathscr{H}_x) \approx 0$ (the octahedral group). For S^3/D^*_{4m}, with $m > 2$, $\pi_0(\mathscr{H}_x) \approx D_{4m}$.*

Remark. As noted in the remark after Corollary 2.2, if one knew that $\pi_1(\mathscr{H}) \approx \pi_1(\mathscr{I})$ for S^3/D^*_{4m}, m odd, it would follow that $\mathscr{R}$ was not isotopic rel. D to the identity; and Theorem 3.1 could then be extended to include odd m. Corollary 3.1 *is* valid for odd m because $\mathscr{R}$ is isotopic rel. x to the identity in any event.

THEOREM 3.2. *For the lens spaces $L(p, q)$, $\pi_0(\mathscr{H}_D) \approx \pi_0(\mathscr{H}_x) \approx \pi_0(\mathscr{H})$.*

Again we will bound the order of $\pi_0(\mathscr{H}_D)$ and $\pi_0(\mathscr{H}_x)$ by finding Im $\hat{p}_*$ and Im p_* in the exact sequences (1) and ($\hat{1}$).

LEMMA 3.4. *The map p_*: $\pi_1(\mathscr{H}) \to Z_p$ is surjective.*

Proof. Regard $L(p, q)$ as the space obtained from a disk by identifying the upper and lower boundary hemispheres after a relative $(2\pi q/p)$ rotation. In cylindrical coordinates ρ, z, ϕ, the identification is

$$z \to -z \qquad \phi \to \phi + 2\pi\frac{q}{p}, \tag{6}$$

and a path in the ball joining the identified points is the generator of Z_p. Define a path ψ_λ of homeomorphisms of $L(p, q)$ by

$$\psi_\lambda(\rho, z, \phi) = \begin{cases} (\rho, z - \lambda z_s, \phi + \lambda \frac{q}{p}\pi), & z > (\lambda - 1)z_s \\ (\rho, z + (2-\lambda)z, \phi + (\lambda - 2)\frac{q}{p}\pi), & z < (\lambda - 1)z_s, \end{cases}$$

where $z_s = \sqrt{(1-\rho^2)}$ is the value of z at the surface for fixed ρ. Note that ψ_λ respects the identifications and so is well defined. Because the orbit of a point $L(p, q)$ is a path joining identified points, i.e. a generator of Z_p, $p_*(\psi_\lambda) = 1 \in Z_p$. Thus $\operatorname{Im} p_* \approx Z_p$. □

LEMMA 3.5. *$\mathscr{R}$ is isotopic to the identity in $L(p, q)$.*

Proof. Choose for $\mathscr{R}$ the homeomorphism used in the proof of Lemma 2.1,

$$\mathscr{R}(x) = R(\hat{z}\, 2\pi f)x,$$

with $L(q, p)$ as in the previous proof. Let $f_\lambda(|x|)$ be a continuous sequence of functions with $f_0 = f$ and $f_1(|x|) = 1$ and $f_\lambda(|x|) = 1, |x| < \varepsilon$. Because rotations of the boundary sphere about the $\hat{z}$-axis respect the identification (6), the family of homeomorphisms

$$\mathscr{R}_\lambda(x) = R(\hat{z}\, 2\pi f_\lambda)x$$

is well defined, and $\mathscr{R}_1$ is the identity. Consequently $\mathscr{R}_\lambda$ is an isotopy of $\mathscr{R}$ to the identity rel. D.

Proof of Theorem 3.2. In the exact sequence (1), ker $\partial_* = 0$, since p_* is surjective by Lemma 3.4. The sequence then implies $\pi_0(\mathscr{H}_x) = \pi_0(\mathscr{H})$. By Lemmas 2.1 and 3.5 $\hat{p}_*$ is also surjective in the exact sequence ($\hat{1}$). Thus $\pi_0(\mathscr{H}_D) = \pi_0(\mathscr{H})$ as well. □

Remark. The remaining spherical spaces S^3/H with $H \subset SU(2)$ are S^3/T^* and S^3/I^*. If, as one expects, $\pi_0(\mathscr{H}) = \pi_0(\mathscr{I})$ [viz. $\pi_0(\mathscr{H}) \approx Z_2$ for S^3/T^* and $\pi_0(\mathscr{H}) = 1$ for S^3/I^*] then $\pi_0(\mathscr{H}_D)$ and $\pi_0(\mathscr{H}_x)$ can be found easily by the technique of Theorem 3.1. In each case $\pi_0(\mathscr{H}_x)$ is isomorphic to the symmetry group of the polyhedron from which the spaces are constructed, i.e., $\pi_0(\mathscr{H}_x) \approx \pi_0(\mathscr{I}_x)$; and $\pi_0(\mathscr{H}_D)$ is isomorphic to the double covering in $SU(2)$ of $\pi_0(\mathscr{H}_x)$:

$$\pi_0[\mathscr{H}_D(S^3/T^*)] \approx 0^* \quad \text{and} \quad \pi_0[\mathscr{H}_D(S^3/I^*)] \approx I^*.$$

At present, however, our result is weaker:

THEOREM 3.3. *$O^* \subset \pi_0[\mathscr{H}_D(S^3/T^*)]$ and $I^* \subset \pi_0[\mathscr{H}_D(S^3/I^*)]$. Similarly $O \subset \pi_0[\mathscr{H}_x(S^3/T^*)]$ and $I \subset \pi_0[\mathscr{H}_x(S^3/I^*)]$, and the groups $\pi_0(\mathscr{H}_x)$ are extensions of $\pi_0(\mathscr{H})$ by T and I, respectively.*

Proof. The proofs in each case are similar to those for the spaces of Theorem 3.1. Here S^3/T^* and S^3/I^* are constructed from polyhedra (an octahedron and a dodecahedron) whose symmetry groups are O and I respectively. An action of O^* (or I^*) as a group of homeomorphisms in $\mathscr{H}_D(S^3/T^*)$ or $\mathscr{H}_D(S^3/I^*)$) is obtained as in (a) of the proof of Theorem 3.1. Hendrik's Theorem and Lemma 3.2 imply that the induced homomorphisms $O^* \to \pi_0(\mathscr{H}_D)$ and $I^* \to \pi_0(\mathscr{H}_D)$ are injective. Thus $O^* \subset \pi_0[\mathscr{H}_D(S^3/T^*)]$, $I^* \subset \pi_0[\mathscr{H}_D(S^3/I^*)]$, $O \subset \pi_0[\mathscr{H}_x(S^3/T^*)]$ and $I \subset \pi_0[\mathscr{H}_D(S^3/I^*)]$. From Lemma 3.3 and the fact [12] that in the exact sequence (1) $\operatorname{Im} p_* \subset \text{center}(H)$, we have $\operatorname{Im} p_* \approx Z_2$. Then (1) implies

$$1 \to Inn\, H \to \pi_0(\mathscr{H}_x) \to \pi_0(\mathscr{H}) \to 1,$$

with $Inn\, H = T, I$ for $H = T^*$ and I^*, respectively. □

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