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ON THE GEOMETRIC AND TOPOLOGICAL RIGIDITY OF HYPERBOLIC 3-MANIFOLDS

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0. Introduction

The main result of this paper asserts that if N is a closed hyperbolic 3-manifold satisfying a certain geometric/topological *insulator* condition (Conjecture 0.6 says that it always happens), then N is both topologically and geometrically rigid. As a special case we have

Theorem 0.1. *Let N be a closed hyperbolic 3-manifold containing an embedded hyperbolic tube of radius $(\log 3)/2 = .549306\cdots$ about a closed geodesic. Then*

i) *If $f : M \to N$ is a homotopy equivalence where M is an irreducible 3-manifold, then f is homotopic to a homeomorphism.*
ii) *If $f, g : M \to N$ are homotopic homeomorphisms, then f is isotopic to g.*
iii) *The space of hyperbolic metrics on N is path connected.*

Remarks 0.2. i) If M is hyperbolic, then conclusion i) follows from Mostow's rigidity theorem [Mo]. Actually [Mo] implies that f is homotopic to an isometry. If N is instead Haken, then conclusions i)–ii) are due to Waldhausen [W]. If N is Haken and hyperbolic, then conclusion iii) follows by combining [Mo] and [W]. Conclusions i), ii)–iii) can be viewed as a 2-fold generalization of [Mo]. See §5.

ii) There are only six known closed 3-manifolds which are probably hyperbolic and have a shortest geodesic δ with tube radius $(\delta) < \log(3)/2$. The first, known as Vol 3, was found using Jeff Weeks' tube radius/ortholength program [We]. Nathaniel Thurston very recently found 5 other such manifolds. All six of these manifolds appear to satisfy the insulator condition.

iii) An application of the hyperbolic law of cosines shows that if the shortest geodesic δ in N has length ≥ 1.353, then tube radius $(\delta) > \log(3)/2$. See Remark 5.11.

iv) If N has a geodesic δ of length $\leq .0978$, then Meyerhoff's tube radius formula [Me] implies that tube radius $(\delta) > \log(3)/2$. See Remark 5.11. Combined with the work of Jorgensen [Gr], we obtain Corollary 5.12 which asserts that for any $n > 0$, all but finitely many hyperbolic 3-manifolds of volume $< n$ are both topologically and geometrically rigid.

v) Farrell and Jones [FJ] showed that if $f : M \to N$ is a homotopy equivalence between closed manifolds and N is a hyperbolic manifold of dimension ≥ 5, then f is homotopic to a homeomorphism.

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The theme of this paper is to abstract the ideas of the following example to the setting of homotopy hyperbolic 3-manifolds.

Example 0.3. Let δ be a simple closed geodesic in the hyperbolic 3-manifold N. δ lifts to a collection $\Delta = \{\delta_i\}$ of hyperbolic lines in $\mathbf{H}^3$. To each pair δ_i, δ_j, there exists the *midplane* D_{ij}, i.e. the hyperbolic halfplane orthogonal to and cutting the middle of the *orthocurve* (i.e. the shortest line segment) between δ_i and δ_j. Each D_{ij} extends to a circle λ_{ij} on S^2_∞, which separates $\partial\delta_i$ from $\partial\delta_j$. Now fix i. Let $\bar{H}_{ij}$ be the closed $\mathbf{H}^3$-halfspace bounded by D_{ij} containing δ_i. $W_i = \bigcap \bar{H}_{ij} = D^2 \times \mathbf{R}$ is the *Dirichlet torus domain* associated to the geodesic δ_i. $\overset{\circ}{W}_i$ projects to an open solid torus containing δ as its core. In fact $W = W_i/\langle\delta_i\rangle$ is a solid torus with boundary a finite union of totally geodesic polygons.

Definition 0.4. Let G be a group of homeomorphisms of S^2 and $\mathcal{A} = \{A_i\}$ a countable set of pairwise disjoint G-equivariant pairs of points of S^2, i.e. if $g \in G, A_i \in \mathcal{A}$, then $g(A_i) \in \mathcal{A}$. Let $\{\lambda_{ij}\}$ be a collection of smooth simple closed curves in S^2. $\{\lambda_{ij}\}$ is called a $(G, \mathcal{A})$ *insulator family* and each λ_{ij} is an *insulator* if

i) *Separation*: If $i \neq j$, then λ_{ij} separates A_i from A_j.
ii) *Equivariance*: If $g \in G$, then $g(\lambda_{ij})$ is the curve associated to the pair $g(A_i), g(A_j)$. Also $\lambda_{ij} = \lambda_{ji}$.
iii) *Convexity*: To each λ_{ij} there exist round circles respectively containing A_i and A_j and disjoint from λ_{ij}.
iv) *Local Finiteness*: For every $\epsilon > 0$ there exist only finitely many λ_{ij} such that i is fixed and $\operatorname{diam}(\lambda_{ij}) > \epsilon$.

Definition 0.5. A $(G, \mathcal{A})$ insulator family is *noncoalescable* if it satisfies the following *no trilinking* property. For no i, does there exist $\lambda_{ij_1}, \lambda_{ij_2}, \lambda_{ij_3}$ whose union separates the points of A_i. See Figure 0.1. A hyperbolic 3-manifold satisfies the *insulator condition* if there exists a geodesic δ in N and a $(\pi_1(N), \{\partial\delta_i\})$ noncoalescable insulator family. Here $\{\delta_i\}$ is the set of lifts of δ to $\mathbf{H}^3$.

Conjecture 0.6. *Every closed hyperbolic* 3*-manifold satisfies the insulator condition.*

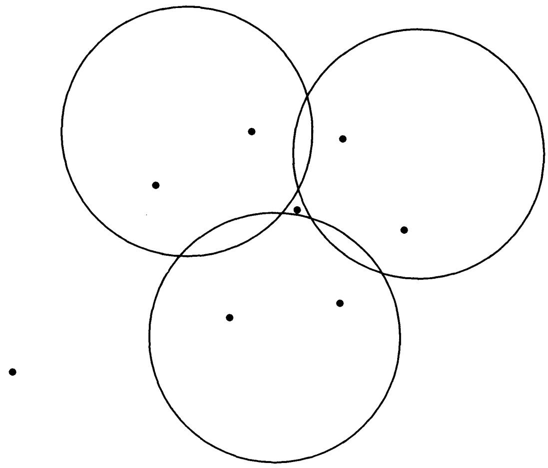
Definition 0.7. If δ is a geodesic in the hyperbolic 3-manifold N, then the insulator family $\{\lambda_{ij}\}$ constructed as in Example 0.3. is called the *Dirichlet insulator family.*

Strong conjecture 0.8. *The Dirichlet insulator family associated to a shortest geodesic in a closed hyperbolic* 3*-manifold, is noncoalescable.*

Lemma 5.9. *If a geodesic δ has tube radius* $> (\log 3)/2$*, then its Dirichlet insulator family is noncoalescable.*

Theorem 0.9. *Let $f : M \to N$ be a homotopy equivalence, where M is a closed, connected, irreducible* 3*-manifold and N is a hyperbolic* 3*-manifold. If N possesses a geodesic δ with a noncoalescable insulator family, then f is homotopic to a homeomorphism.*

Outline of the proof. Let $f : M \to N$ be a homotopy equivalence. M and N are covered by the same hyperbolic manifold X [G3]. In §1 we show that we can assume



Trilinking

FIGURE 0.1

that the homotopy equivalence lifts and extends to a mapping $\tilde{f} : \mathbf{B}^3 \to \mathbf{B}^3$, where $\mathbf{B}^3 = \mathbf{H}^3 \cup S^2_\infty$, so that $\tilde{f} \mid S^2_\infty = \text{id}$. Furthermore the group actions on S^2_∞ defined by $\pi_1(M)$ and $\pi_1(N)$ are identical. In §2 we show that if there exists a simple geodesic δ in N and a simple closed curve γ in M such that the $\mathbf{B}^3$-link Δ is equivalent to the $\mathbf{B}^3$-link Γ, then f is homotopic to a homeomorphism. Here Δ is the preimage of δ in $\mathbf{H}^3$ extended to $\mathbf{B}^3$ and Γ is defined similarly. These links are equivalent means that there exists a homeomorphism $k : (\mathbf{B}^3, \Gamma) \to (\mathbf{B}^3, \Delta)$ so that $k \mid S^2_\infty = \text{id}$. To prove that equivalent links imply topological rigidity we show that f can be homotoped to a map which sends $N(\gamma)$ homeomorphically onto $N(\delta)$ and restricts to a homotopy equivalence between $M - \overset{\circ}{N}(\gamma)$ and $N - \overset{\circ}{N}(\delta)$. By Waldhausen [W], f is homotopic to a homeomorphism.

In §3 we establish technical lemmas about least area planes, discs and laminations in $\mathbf{H}^3$. In particular we show that to each smooth simple closed curve λ_{ij} in S^2_∞, there exists a lamination σ_{ij} by injectively immersed least area (with respect to the metric induced by M) planes in $\mathbf{H}^3$, with limit set λ_{ij} such that σ_{ij} lies in a fixed width hyperbolic regular neighborhood of the hyperbolic convex hull of λ_{ij}. Here $\{\lambda_{ij}\}$ is the $(\pi_1(N), \{\partial\delta_i\})$ and hence $(\pi_1(M), \{\partial\delta_i\})$ noncoalescable insulator family. Fix i. Let H_{ij} be the $\mathbf{H}^3$-complementary region of σ_{ij} containing the ends of δ_i. In §4 we show that $\bigcap H_{ij}$ contains a $\tilde{V}_i = \overset{\circ}{D}{}^2 \times \mathbf{R}$ which projects to an open solid torus in M. (It is this step that requires the noncoalescable insulator hypothesis.) Define γ to be the core of this solid torus and γ_i the lift which lives in $\tilde{V}_i$. We show that the isotopy class of γ is independent of all choices, i.e. the metric on M and the choice of $\{\sigma_{ij}\}$ for a fixed metric. Let τ_0 be the link in X which is the preimage of γ, so $\{\gamma_i\}$ is also the set of lifts of components of τ_0 to $\mathbf{H}^3$. The Riemannian metric μ_0 on X induced from M and the hyperbolic metric μ_1 on X induced from N are connected by a smooth path μ_t of metrics. These metrics lift to $\pi_1(X)$ equivariant metrics $\tilde{\mu}_t$ on $\mathbf{H}^3$, so the above construction applied to the $(\pi_1(X), \{\partial\delta_i\})$ insulator family $\{\lambda_{ij}\}$ with respect to the $\tilde{\mu}_t$ metric yields a link

τ_t in X. Since the isotopy class of τ_t is independent of t, τ_0 is isotopic to τ_1, the preimage of δ in X. We conclude that the $\mathbf{B}^3$-link Γ is equivalent to the $\mathbf{B}^3$-link Δ and so by §2 f is homotopic to a homeomorphism. □

Theorem 0.10. *If N is a closed hyperbolic 3-manifold possessing a geodesic δ with a noncoalescable insulator family, and $f : N \to N$ is a homeomorphism homotopic to* id, *then f is isotopic to* id.

Idea of the proof. Let ρ_0 denote the hyperbolic metric on N. Let ρ_1 be the pull back hyperbolic metric on N induced via f, which we can assume is a diffeomorphism. These metrics are connected by a family ρ_t of Riemannian metrics. As in the proof of Theorem 0.9, to each ρ_t there is associated an oriented simple closed curve γ_t where $\gamma_0 = \delta$ and $\gamma_1 = f^{-1}(\delta)$ and that all of these γ_t's are isotopic. Therefore f is isotopic to a map which fixes δ pointwise. A theorem of Siebenmann [BS] implies that f is isotopic to id. □

Corollary 5.3. *If N satisfies the insulator condition, then* $\mathrm{Homeo}(N)/\mathrm{Homeo}_0(N) = \mathrm{Out}(\pi_1(N)) = \mathrm{Isom}(N)$.

Section 5 contains the proofs of Theorem 0.1, Corollary 5.3 and some concluding results, remarks and conjectures.

1. Preliminary results and notation

From now on we will assume that all 3-manifolds in this paper are connected, irreducible, and orientable. Recall that by Remark 0.2 i), we can assume that M is non-Haken and hence orientable.

Proposition 1.1. *If $f : M \to N$ is a homotopy equivalence, where M is a closed 3-manifold and N is a hyperbolic 3-manifold, then there exists a closed hyperbolic 3-manifold X and regular covering maps $p_1 : X \to M$, $q_1 : X \to N$ such that $f \circ p_1$ is homotopic to q_1. A lift $\tilde{f} : \mathbf{H}^3 \to \mathbf{H}^3$ extends to* $\mathrm{id} : S^2_\infty \to S^2_\infty$. *Furthermore the action of $\pi_1(M)$ on $\mathbf{H}^3$ extends to Möbius action on S^2_∞ which is identical to the action of $\pi_1(N)$ on S^2_∞.*

Proof. By the *Proof of Theorem 1.1* [G3], there exists a hyperbolic 3-manifold X and regular covering projections $p_1 : X \to M$, $q_1 : X \to N$ such that $f_1 : X \to X$, a lift of f to X, is homotopic to a homeomorphism h. By replacing p_1 by $p_1 \circ h^{-1}$ and f_1 by $f_1 \circ h^{-1}$, we can assume that f_1 is homotopic to id. Let $x \in X$. Fixing $m = p_1(x), n = q_1(x)$ and modifying f via a homotopy we can assume that $f(m) = n$ and the homotopy F_1 of id to f_1 is rel x. □

Let $\pi : \mathbf{H}^3 \to X$ be the universal covering projection. Let $p : \mathbf{H}^3 \to M, q : \mathbf{H}^3 \to N$ be the induced coverings. If $g \in \pi_1(M, m)$, then let g' denote $f_\#(g) \in \pi_1(N, n)$. Fix $\bar{e} \in \pi^{-1}(x)$. Let $g \to \bar{g}$ denote the canonical 1-1 correspondence between $\pi_1(M, m)$ and points of $p^{-1}(m)$ such that $e \to \bar{e}$; $g' \to \bar{g}'$ has the similar meaning. The distance between points $\bar{g}, \bar{g}'$ of $\mathbf{H}^3$ differs by a bounded amount, i.e. the diameter of F_1 homotopy tracks. More generally the action of $\pi_1(M)$ on $\bar{g}$ differs from the corresponding action of $\pi_1(N)$ on $\bar{g}'$ by this same amount. Since the action of $\pi_1(M)$ on $\mathbf{H}^3$ is approximated by the action on the points corresponding to $\pi_1(M)$, it follows that the actions of $\pi_1(M), \pi_1(N)$ on S^2_∞ are identical. Here are more details.

In what follows d_ρ (resp. d_μ) will denote distance with respect to the hyperbolic metric ρ (resp. the metric μ induced from M) between points in $\mathbf{H}^3$.

Claim. There exists $k < \infty$ such that if $g \in \pi_1(M, m)$ and $y \in \mathbf{H}^3$, then $d_\rho(g(y), f_\#(g)(y)) \leq k$.

Proof of the Claim. Let $\tilde{F}$ denote the lift of F_1 to $\mathbf{H}^3$ and $\tilde{f}$ the lift of f to $\mathbf{H}^3$ such that $\tilde{F}(z, 0) = z$ and $\tilde{f}(z) = \tilde{F}(z, 1)$.

The compactness of X implies the following facts i)–iii) and the covering space theory implies iv)–v).

i) There exists $b_0 = \sup\{\mathrm{diam}_\rho(\tilde{F}(z \times I)) \mid z \in \mathbf{H}^3\}$. So $z' = \tilde{f}(z)$ implies $d_\rho(z, z') \leq b_0$.

ii) There exists b_1 such that for each $y \in \mathbf{H}^3$ there exists $\bar{r} \in p^{-1}(m)$ such that $d_\mu(\bar{r}, y) < b_1$.

iii) There exists b_2 such that $d_\mu(y_0, y_1) < b_1$ implies $d_\rho(y_0, y_1) < b_2$.

iv) If $g, r \in \pi_1(M)$, then $\tilde{f}(g(\bar{r})) = g'(\bar{r}')$, so $d_\rho(g(\bar{r}), g'(\bar{r}')) \leq b_0$. (Recall that $g(\bar{r}) = \tilde{j}(1)$, where $j : [0,1] \to M$, $[j] = g * r \in \pi_1(M, m)$, and $\tilde{j}(0) = \bar{e}$. Since $[g' * r'] = [f_\#[g * r]]$, it is represented by $f \circ j$, so $g'(\bar{r}') = \tilde{f}(\tilde{j}(1))$. Now apply i).)

v) The covering transformation associated to $g \in \pi_1(M, m)$ (resp. $g' \in \pi_1(N, n)$) is an isometry in the μ (resp. ρ) metric.

If $y \in \mathbf{H}^3$, then $d_\rho(g(y), g'(y)) \leq d_\rho(g(y), g(\bar{r})) + d_\rho(g(\bar{r}), g'(\bar{r}')) + d_\rho(g'(\bar{r}'), g'(y)) \leq b_2 + b_0 + d_\rho(\bar{r}', y) \leq b_2 + b_0 + d_\rho(\bar{r}', \bar{r}) + d_\rho(\bar{r}, y) \leq b_0 + b_2 + b_0 + b_2$. □

Since the actions of $\pi_1(M), \pi_1(N)$ on $\mathbf{H}^3$ differ by a hyperbolically bounded amount, the action of $\pi_1(M, m)$ extends to the same action on S^2_∞ as that of $\pi_1(N, n)$. I.e., for each $g \in \pi_1(M)$ and $y \in S^2_\infty$, $g(y) = g'(y)$. Also $b_0 < \infty$ implies that $\mathrm{id} = \tilde{f} : S^2_\infty \to S^2_\infty$.

Notation 1.2. $M = \mathbf{H}^3/G, N = \mathbf{H}^3/H$, where $\pi_1(M) = G \cong H = \pi_1(N)$, $G \subset \mathrm{Homeo}_+(\mathbf{B}^3)$, $H \subset \mathrm{Isom}_+(\mathbf{H}^3) \subset \mathrm{Homeo}_+(\mathbf{B}^3)$, $G \mid S^2_\infty = H \mid S^2_\infty, G$ and H are naturally identified via the π_1-isomorphism $\phi = f_\#$. The space X and the maps p, q, p_1, q_1, π will be as in Proposition 1.1; μ (resp. ρ) will denote the metric induced on X or $\mathbf{H}^3$ from M (resp. N). In particular ρ will always represent the hyperbolic metric. We abuse notation by letting H and G simultaneously denote the actions on $\mathbf{H}^3, \mathbf{B}^3$ or S^2_∞.

Unless otherwise indicated we will assume that all maps on 3-manifolds without boundary are smooth.

$$\begin{tikzcd}
& \mathbf{B}^3 & \\
& \mathbf{H}^3 \arrow[u] \arrow[dl, "\pi"'] \arrow[dr, "\pi"] \arrow[ddl, bend right=60, "p"'] \arrow[ddr, bend left=60, "q"] & \\
X \arrow[rr, shift left=2, "\mathrm{id}"] \arrow[rr, shift right=2, "f_1"'] \arrow[rr, phantom, "\simeq"] \arrow[d, "p_1"'] & & X \arrow[d, "q_1"] \\
M \arrow[rr, "f"] & & N
\end{tikzcd}$$

$N(E)$ denotes regular neighborhood of E. If $E \subset Y$, then $N(k, E) = \{y \in Y \mid d(y, E) \leq k\}$. Similarly if $x \in Y$, then $B(k, x) = \{y \in Y \mid d(y, x) \leq k\}$. We will use notations such as $N_\rho(k, E)$, $d_\mu(x, y)$ or r-least area when the metric is not clear from the context. $|E|$ denotes the number of components of E, and $\overset{\circ}{E}$ denotes the interior of E.

2. A criterion for homeomorphism

Proposition 2.1. *Let $f : M \to N$ be a homotopy equivalence between the closed hyperbolic 3-manifold N and the irreducible 3-manifold M. If there exists a simple closed curve $\gamma \subset M$, a geodesic $\delta \subset N$ and a homeomorphism $k : (\mathbf{B}^3, p^{-1}(\gamma)) \to (\mathbf{B}^3, q^{-1}(\delta))$ such that $k \mid \partial\mathbf{B}^3 = \mathrm{id}$, then f is homotopic to a homeomorphism.*

Definition 2.2. A $\mathbf{B}^3$-link is a collection $\{\alpha_i\}$ of properly embedded arcs in $\mathbf{B}^3$ whose restriction to $\mathbf{H}^3$ is locally finite. Two $\mathbf{B}^3$-links are said to be *equivalent* if there exists a homeomorphism $k : \mathbf{B}^3 \to \mathbf{B}^3$ taking one link to the other such that $k \mid S^2_\infty = \mathrm{id}$.

Remarks. i) Each component of $p^{-1}(\gamma)$ extends to a properly embedded arc γ_i in $\mathbf{B}^3$. We will abuse notation by also referring to γ_i as an arc in $\mathbf{H}^3$. Proposition 2.1 says that if the $\mathbf{B}^3$-link $\{p^{-1}(\gamma)\}$ is equivalent to the link $\{q^{-1}(\delta)\}$, then f is homotopic to a homeomorphism.

ii) The $\mathbf{H}^3$-link determined by δ is the *trivial* link on infinitely many components. Thus Proposition 2.1 crucially depends on the condition $k \mid \partial\mathbf{B}^3 = \mathrm{id}$.

Notation 2.3. Let $\Gamma = p^{-1}(\gamma)$ with components $\{\gamma_i\}, \Delta = q^{-1}(\delta)$ with components $\{\delta_i\}, V = p^{-1}(N(\gamma))$ with components $\{V_i\} = \{N(\gamma_i)\}$, $W = q^{-1}(N(\delta))$ with components $\{W_i\} = \{N(\delta_i)\}$. Indices are chosen so that $i = j$ if and only if $\partial\gamma_i = \partial\delta_i$. Recall that if $g \in G$, then the associated map $g : \mathbf{B}^3 \to \mathbf{B}^3$ has two fixed points, one attracting and one repelling. Thus the correspondence of γ_i to δ_i is determined by ϕ.

Idea of the proof. We show that f is homotopic to a map $h : (M, M - \overset{\circ}{N}(\gamma), N(\gamma)) \to (N, N - \overset{\circ}{N}(\delta), N(\delta))$ which restricts to a homeomorphism on $N(\gamma)$ and a π_1-injective map on $M - \overset{\circ}{N}(\gamma)$. The map f is homotopic to a homeomorphism, since [W] implies that h is homotopic to a homeomorphism rel $N(\gamma)$. Given a handle structure on M which contains $N(\gamma)$ as a 1-handle we use k to define h so that the other handles miss $\overset{\circ}{N}(\delta)$, approximately by considering $q \circ k \circ p^{-1}$ restricted to a handle. There is never a problem with the 1-handles. The difficulty is that the image of the 2-handles might cross δ, e.g. consider the case that δ, γ are distinct knots in $S^3 = M = N$. In our setting this difficulty vanishes if for each $g \in G, \phi(g) \circ k \circ g^{-1} \circ k^{-1} \mid \mathbf{H}^3 - \overset{\circ}{N}(\Delta)$ is homotopic to id rel $\partial N(\Delta)$. We show that k can be chosen so that this holds.

The following result follows by the usual isotopy/uniqueness of regular neighborhood arguments.

Lemma 2.4. *Under the hypothesis of Proposition 2.1, $k : \mathbf{B}^3 \to \mathbf{B}^3$ can be chosen so that for each $g \in G, k \circ g \mid V \cup S^2_\infty = \phi(g) \circ k \mid V \cup S^2_\infty$.*

□

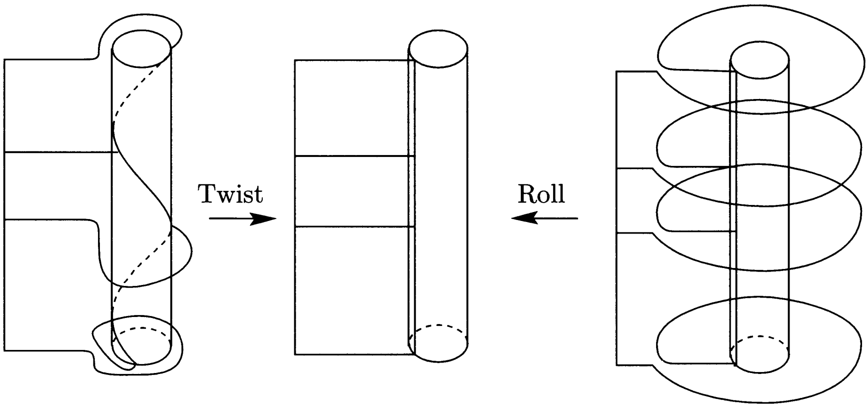


FIGURE 2.1

Remark 2.5. Replace the covering projection $p : \mathbf{H}^3 \to M$ by $p \circ k^{-1} : \mathbf{H}^3 \to M$. The effect on covering transformations is to replace each $g \in G$ by $k \circ g \circ k^{-1}$. We now assume that $V = W, k = \mathrm{id}$ and $g \mid V \cup S^2_\infty = \phi(g) \mid W \cup S^2_\infty$.

Definition 2.6. (Interesting self-maps of $\mathbf{B}^3$ fixing $\delta_0 \cup S^2_\infty$ pointwise.) Parametrize $\mathbf{H}^3$ by $(r, \theta, s), r \geq 0, \theta \in S^1, s \in \mathbf{R}$. Here δ_0 consists of those points with parameters $\{(0,0,s) \mid s \in \mathbf{R}\}$, the s parameter denoting parametrization by arc length. The plane P_t orthogonal to δ_0 at t has $s \equiv t$ and is parametrized by polar coordinates via the (r, θ) coordinates. Finally the parametrization of $\mathbf{H}^3$ is chosen so that the (r, θ) parameters are preserved under pure translation along δ_0. Call $0 \in \delta_0$, the point with s-parameter equal to 0. Assume that $W_0 = \{(r, \theta, s) \mid r \leq c\}$. Let $\alpha, \beta : [0, \infty] \to [0,1]$ be smooth maps such that $\alpha \mid [0,c] = 1$, $\alpha \mid [2c, \infty] = 0$, $\beta \mid [0, 2c] = 0$, and $\beta \mid [3c, \infty] = 1$.

For $a \neq 0$, define $T_a : \mathbf{B}^3 \to \mathbf{B}^3$ by $T_a \mid S^2_\infty = \mathrm{id}$, $T_a(r, \theta, s) = (r, \theta + \alpha(r) 2\pi s/a, s)$. Define $R : \mathbf{B}^3 \to \mathbf{B}^3$ by $R \mid S^2_\infty = \mathrm{id}$, and $R(r, \theta, s) = (r, \theta + \beta(r) 2\pi, s)$.

Define a *twist* to be a map conjugate to T_a, and a *roll* a map conjugate to R. In particular a twist or roll about say W_j, should be viewed as being supported very close to W_j. See Figure 2.1. Figure 2.2 suggests the twist-image of a standardly embedded disc in $\mathbf{B}^3 - \overset{\circ}{W}_0$.

Lemma 2.7. a) *If $h : \mathbf{B}^3 \to \mathbf{B}^3$ is such that $h \mid W_0 \cup S^2_\infty = \mathrm{id}$, then $h \simeq R^n$ rel $W_0 \cup S^2_\infty$ for a unique integer n.*

b) *If $W' = \{W_1, \ldots, W_r\}$ is a finite subset of W, and $h : \mathbf{B}^3 \to \mathbf{B}^3$ a map such that $h \mid S^2_\infty \cup W' = \mathrm{id}$, then h is homotopic rel $S^2_\infty \cup W'$ to a composit of rollings about elements of W'.*

c) *If $h : (\mathbf{B}^3, W_0) \to (\mathbf{B}^3, W_0)$ is a homeomorphism, then $h \circ R^n \simeq R^n \circ h$ rel $W_0 \cup S^2_\infty$.*

d) *If h is a hyperbolic isometry fixing W_0 with real length $a \neq 0$, then $T_a^{-n} \circ h^{-1} \circ T_a^n \circ h \simeq R^{-n}$ rel $W_0 \cup S^2_\infty$.*

Proof. a) The value of n is determined by the class $h(\{(r,0,0) \mid r \in \mathbf{R}\}) \in \pi_1(\mathbf{B}^3, W_0 \cup S^2_\infty)$. For the correct n, the straight line homotopy of $\tilde{R}^{-n} \circ \tilde{h}$ to

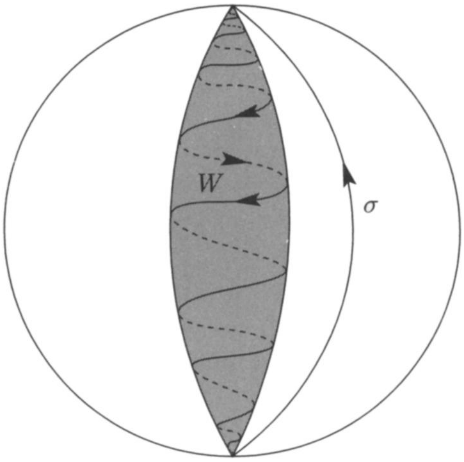


FIGURE 2.2. σ bounds an embedded disc in $\mathbf{B}^3 - \overset{\circ}{W}$

id in the appropriately parametrized universal cover of $\mathbf{B}^3 - \overset{\circ}{W}_0$ projects to a homotopy of $R^{-n} \circ h$ to id.

b) Let $\{A_k\}$ be a set of annuli properly embedded in $\mathbf{B}^3 - \overset{\circ}{W}'$ which cut off the ends of $\{W_i\}$. There are two A_k's for each W_i. If $\mathbf{B}^3$ is the unit ball in $\mathbf{R}^3$, and e is an end of W_i, then the A_k associated to e is the intersection of a very small Euclidean sphere centered at e, with $\mathbf{B}^3 - \overset{\circ}{W}_i$. After a preliminary homotopy we can assume that h fixes each A_k setwise. After precomposing h by rollings the resulting function also called h can be homotoped to fix the A_k's pointwise. Each A_k together with a meridian disc of W' cuts off a halfball B_i containing no other A_j's. Homotop $h \mid B_i$ to id rel $\partial B_i \cup W_i$. As in the first paragrpah, this homotopy can be expressed as the projection of a straight line homotopy assoicated to a lift of h to an appropriately parametrized covering of $\overset{\circ}{B} - W_i$. The manifold $H = \mathbf{B}^3 - (\bigcup_i \overset{\circ}{B}_i \cup \overset{\circ}{W}')$ is a handlebody, for a finite set of hyperbolic geodesics in $\mathbf{B}^3$ are unlinked. Since $h \mid \partial H = \text{id}$, $h \mid H$ is isotopic to id rel ∂H, e.g. by [W].

c) The action of $R^n \circ h \circ R^{-n} \circ h^{-1}$ on $\pi_1(\mathbf{B}^3, W_0 \cup S^2_\infty)$ is trivial.

d)

$$\begin{aligned} T_a^{-n} \circ h^{-1} \circ T_a^n \circ h(r,0,0) &= T_a^{-n} \circ h^{-1} \circ T_a^n(r,b,a) \\ &= T_a^{-n} \circ h^{-1}(r, b + n\alpha(r)2\pi a/a, a) \\ &= T_a^{-n}(r, n\alpha(r)2\pi, 0) = (r, n\alpha(r)2\pi, 0). \end{aligned}$$

Similarly $T_a^{-n} \circ h^{-1} \circ T_a^n \circ h \mid W_0 = \text{id}$. Now apply a). □

Proposition 2.8. *If* $f : (\mathbf{B}^3, \mathbf{B}^3 - \overset{\circ}{W}, W) \to (\mathbf{B}^3, \mathbf{B}^3 - \overset{\circ}{W}, W)$ *is a map such that* $f \mid S^2_\infty \cup W = \text{id}$, *and for each* i, $f : (\mathbf{B}^3, \mathbf{B}^3 - \overset{\circ}{W}_i, W_i) \to (\mathbf{B}^3, \mathbf{B}^3 - \overset{\circ}{W}_i, W_i)$ *is homotopic to* id *rel* $W_i \cup S^2_\infty$, *then* f *is homotopic to* id *rel* $W \cup S^2_\infty$.

Proof. If W' is a finite subset of components of W, then $f : (\mathbf{B}^3, \mathbf{B}^3 - \overset{\circ}{W}', W') \to (\mathbf{B}^3, \mathbf{B}^3 - \overset{\circ}{W}', W')$ is homotopic to id rel W', for by Lemma 2.7 b) f is homotopic to a composite of n_i-rolls supported near components W_i of W'. By hypothesis each n_i=0.

Next we show that if α is a properly embedded closed interval in $\mathbf{H}^3 - \overset{\circ}{W}$, then $f \mid \alpha \simeq \mathrm{id}$ in $\mathbf{B}^3 - \overset{\circ}{W}$, rel $W \cup S^2_\infty$. Let $B \subset \mathbf{H}^3$ be a round ball centered at 0, transverse to Δ containing α and $f(\alpha)$. Let W' be the components of W whose cores hit B. By viewing W as a thin hyperbolic neighborhood of Δ, we can assume that W' is exactly the set of components of W which hit B, and W' is transverse to B with each component hitting ∂B in meridional 2-discs. Now $(\mathbf{H}^3 - \overset{\circ}{B}, (\mathbf{H}^3 - \overset{\circ}{B}) \cap W')$ is homeomorphic to $(S^2 \times [0,1), J)$, where J is a neighborhood of vertical arcs in $S^2 \times [0,1)$, and under this identification $S^2 \times 0 = \partial B$. By the first paragraph $f \mid \alpha \simeq \mathrm{id} \mid \alpha$ in $\mathbf{B}^3 - \overset{\circ}{W'}$ rel $\partial\alpha$. By composing this homotopy with the retraction of $\mathbf{H}^3$ onto B, which retracts each $[0,1)$ fibre to 0, we see that $f \mid \alpha \simeq \mathrm{id} \mid \alpha$ rel $\partial\alpha$ in $\mathbf{B}^3 - \overset{\circ}{W}$.

Now consider any handle decomposition of $\mathbf{H}^3 - \overset{\circ}{W}$ by handles of bounded hyperbolic diameter. The previous paragraph really showed that if h is any 1-handle, then $f \mid h$ is homotopic to id rel ∂h (i.e. the attaching region to W) via a homotopy supported in $B \cap \mathbf{H}^3 - \overset{\circ}{W}$, where B is any closed smooth convex region of $\mathbf{H}^3$ containing both h and $f(h)$. Thus f can be homotoped to f_1, where $f_1 \mid (W \cup \text{1-handles}) = \mathrm{id}$ via a homotopy restricting the trace of each 1-handle h to a neighborhood of the convex hull of $h \cup f(h)$. In a similar way homotope f_1 to id on the 2- and 3-handles. □

Remark. In what follows we will be only using the homotopy described in the first two paragraphs of the proof of Proposition 2.8.

Lemma 2.9. *k can be chosen so that for every $\gamma_i \in \Gamma$ and every $g \in G$,*

$$(*) \qquad k \circ g \simeq \phi(g) \circ k \ \textit{rel}\ V_i \cup S^2_\infty.$$

Proof. Let $g_0 \in G$ be a generator of $\mathrm{Stab}(\gamma_0)$. Suppose that $\mathrm{Re}(\mathrm{length}(\gamma_0))$=$a$. We first show how to adjust k so that $(*)$ holds for $g = g_0$ and $V_i = V_0$. We let g' denote $\phi(g)$. By Lemma 2.7 $g_0 \simeq g'_0 \circ R^n$. Replace k (which is id, by Remark 2.5) by T^n_a near V_0 and conjugates of T^n_a near each V_i (so that Lemma 2.4 still holds). With respect to this new k, $k^{-1} \circ g'^{-1} \circ k \circ g \simeq T_a^{-n} \circ g_0'^{-1} \circ T^n_a \circ g_0 \simeq T_a^{-n} \circ g_0'^{-1} \circ T^n_a \circ g'_0 \circ R^n \simeq R^{-n} \circ R^n = \mathrm{id}$. All homotopies taken rel $V_0 \cup S^2_\infty$.

We show that $(*)$ holds for $g \in \mathrm{Stab}(\gamma_i)$ and V_i. If $g(\gamma_i) = \gamma_i$, then $g = g_i^{-1} \circ g_0^n \circ g_i$ for some g_i taking γ_i to γ_0. For some m, $k \simeq g_i'^{-1} \circ k \circ g_i \circ R^m$ rel $V_i \cup S^2_\infty$, where R^m is an m-roll about V_i. Therefore $g' \circ k \simeq g' \circ g_i'^{-1} \circ k \circ g_i \circ R^m = g_i'^{-1} \circ g_0'^n \circ k \circ g_i \circ R^m \simeq g_i'^{-1} \circ k \circ g_0^n \circ g_i \circ R^m = g_i'^{-1} \circ k \circ g_i \circ g \circ R^m \simeq k \circ R^{-m} \circ g \circ R^m \simeq k \circ g$ rel $V_i \cup S^2_\infty$.

For each i, pick $g_i \in G$ such that $g_i(\gamma_i) = \gamma_0$. Let n_i be such that $g_i'^{-1} \circ k \circ g_i \simeq R^{n_i} \circ k$ rel $V_i \cup S^2_\infty$. Again R^{n_i} denotes an n_i-roll about V_i. The value n_i is a function of i rather than g_i, for if $g(\gamma_i) = \gamma_0$, then $g = g_0^n \circ g_i$ and $g'^{-1} \circ k \circ g = g_i'^{-1} \circ g_0'^{-n} \circ k \circ g_0^n \circ g_i \simeq g_i'^{-1} \circ k \circ g_i \simeq R^{n_i} \circ k$. For each i, replace k near V_i by $R^{n_i} \circ k$. By Lemma 2.7 c), this change does not effect the validity of $(*)$ for g, V_i where $g \in \mathrm{Stab}(V_i)$. However, $(*)$ now holds for g, V_i where $g(V_i) = V_0$ or $g(V_0) = V_i$. This in turn implies that $(*)$ holds in general. □

Lemma 2.10. *Let* $r_g = \phi(g) \circ k \circ g^{-1} \circ k^{-1}$ *(resp.* $l_g = g \circ k^{-1} \circ \phi(g^{-1}) \circ k$*). If* α *is a properly embedded closed interval in* $\mathbf{H}^3 - \overset{\circ}{W}$ *(resp.* $\mathbf{H}^3 - \overset{\circ}{V}$*), then* $r_g \mid \alpha \simeq \mathrm{id}$ *in* $\mathbf{H}^3 - \overset{\circ}{W}$*, rel* ∂W *(resp.* $l_g \mid \alpha \simeq \mathrm{id}$ *in* $\mathbf{H}^3 - \overset{\circ}{V}$*, rel* ∂V*).*

Proof of Lemma 2.10. By Lemma 2.7 b), for each i, r_g is homotopic rel $W_i \cup S^2_\infty$ to an n_i-roll supported near W_i. By Lemma 2.9 n_i=0. Now apply Proposition 2.8. The argument for l_g is similar. □

Proof of Proposition 2.1. We construct maps $f : (M, M - \overset{\circ}{N}(\gamma), N(\gamma)) \to (N, N - \overset{\circ}{N}(\delta), N(\delta))$, $g : (N, N - \overset{\circ}{N}(\delta), N(\delta)) \to (M, M - \overset{\circ}{N}(\gamma), N(\gamma))$ such that $g \circ f \mid M^1 = \mathrm{id}$. Here M has a handlebody structure with a unique 0-handle and M^1 is the union of the 0- and 1-handles. Also $N(\gamma) \subset M$ is the union of the 0-handle and a single 1-handle. It will then follow that degree-$f = 1$, $f \mid N(\gamma)$ is a homeomorphism onto $N(\delta)$ and $(g \circ f)_\# : \pi_1(M - \overset{\circ}{N}(\gamma)) \to \pi_1(M - \overset{\circ}{N}(\gamma)) = \mathrm{id}$ and hence $f : M - \overset{\circ}{N}(\gamma) \to N - \overset{\circ}{N}(\delta)$ is π_1-injective and restricts to a homeomorphism on $\partial N(\gamma)$. Waldhausen [W] implies that $f \mid M - \overset{\circ}{N}(\gamma)$ is homotopic to a homeomorphism rel boundary and hence f is homotopic to a homeomorphism.

The map k induces a homeomorphism $f : N(\gamma) \to N(\delta)$. We extend this map to M as follows. By restricting the size of V, we can assume that $k \circ g \mid N(V) = \phi(g) \circ k \mid N(V)$, where $N(V)$ is an $\mathbf{H}^3$ regular neighborhood of V. Give M a handlebody decomposition with a unique 0-handle A and 1-handles $\{B_0, \ldots, B_r\}$ such that $A \cup B_0 = N(\gamma)$. If $i > 0$, let $\tilde{B}_i$ be a lift of B_i to $\mathbf{H}^3$ and define $f \mid B_i = q \circ k \circ p_i$, where $p_i = p^{-1}$ with the range restricted to $\tilde{B}_i$. By perturbing the handle structure slightly we can assume that $f \mid M^1$ is an embedding. Let C_j be a 2-handle, $\tilde{C}_j$ any lift to $\mathbf{H}^3$. Define $f_j = q \circ k \circ p_j$, where $p_j : C_j \to \tilde{C}_j$ is given by p^{-1}. Let $c_j = C_j \cap M^1$. This annulus is a union of solid squares which alternately lie in 0 and 1-handles. By Lemma 2.4 if s is a square of c_j lying in A or B_0, then $f_j \mid s = f \mid s$, however that conclusion may be false for squares in $B_i, i > 0$. See Figure 2.3. To define the desired extension of f to C_j, i.e. so that $f(C_j) \subset N - \overset{\circ}{N}(\delta)$, it suffices to show that for each square s, $f \mid s \simeq f_j \mid s$ rel $s \cap A$ via a homotopy in $\mathbf{H}^3 - \overset{\circ}{W}$. $s = \alpha \times I$ where α is an embedded path in B_i with endpoints in A. α has two lifts, α_i determined by $\tilde{B}_i$ and α_j determined by $\tilde{C}_j$. For some $g \in G, \alpha_j = g(\alpha_i)$. By Lemma 2.10 it follows that when restricted to α, $f_j = q \circ k \circ p_j \simeq q \circ r(g) \circ k \circ p_j = q \circ \phi(g) \circ k \circ g^{-1} \circ k^{-1} \circ k \circ p_j = q \circ \phi(g) \circ k \circ p_i = q \circ k \circ p_i = f_i$ rel $\partial\alpha$.

The map f extends across the 3-handle of M since $\pi_2(N - \overset{\circ}{N}(\delta)) = 1$. Construct a handle decomposition of N with a single 0-handle E and 1-handles $\{F_0, \ldots, F_s\}$ such that $E = f(A)$ and for $i \leq r$, $F_i = f(B_i)$. Construct $g : (N, N - \overset{\circ}{N}(\delta), N(\delta)) \to (M, M - \overset{\circ}{N}(\gamma), N(\gamma))$ in a manner similar to the construction of f, taking care to use the lifts $k(\tilde{B}_i)$ for $F_i, i \leq r$. By construction $g \circ f \mid M^1 = \mathrm{id}$. □

The methods of this section lead to an elementary proof of the following Proposition 2.11, an unpublished circa 1981 theorem of Siebenmann [BS] communicated to the author by Francis Bonahon. Siebenmann's proof employed Thurston's geometrization theorem, [Th], Mostow's rigidity theorem [Mo], Waldhausen's isotopy

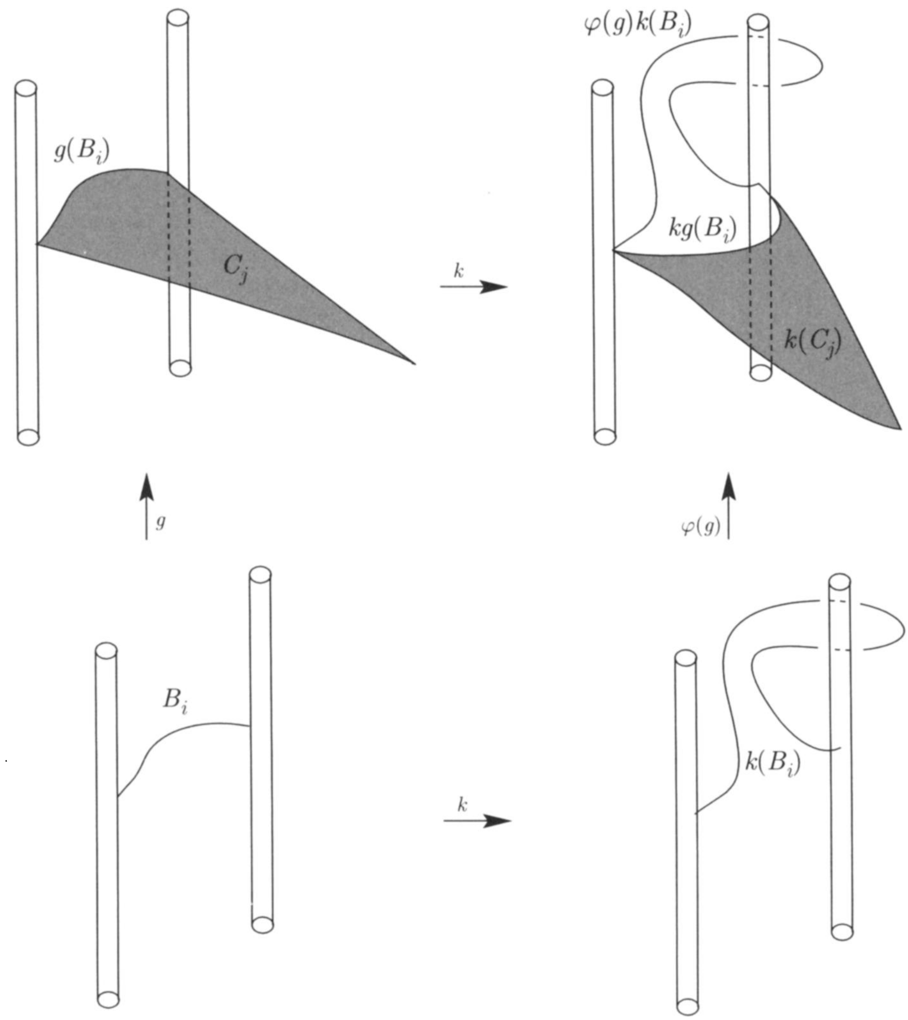


FIGURE 2.3

theorem [W] and the following theorem of Neumann [Ne]: id is the only periodic homeomorphism of $\mathbf{B}^3$ which fixes S^2 pointwise. Here is the idea. $N - \delta$ has a hyperbolic structure [Th], hence $f \mid N - \delta$ is homotopic [Mo], hence isotopic [W] to an isometry, which is necessarily periodic. A lift $\tilde{f}$ extends to a periodic map of $\mathbf{B}^3$ pointwise fixing S^2_∞, so $f = \text{id}$ by [Ne].

Proposition 2.11. *If $f : N \to N$ is a homeomorphism homotopic to* id *and $f \mid \delta = \text{id}$, where N is a closed hyperbolic 3-manifold and δ is a simple closed geodesic in N, then f is isotopic to* id.

Proof. After a preliminary isotopy we can assume that f is smooth by [Mu], [Ki]. We will show that either N is Haken or f is isotopic to a map g such that $g \mid N(\delta) = \text{id}$ and $g \mid N - \mathring{N}(\delta)$ is homotopic to id rel $\partial N(\delta)$. In either case f is isotopic to id by [W].

We can assume that either N is Haken or $f \mid N(\delta) = \mathrm{id}$. In fact, a standard argument shows that $f \mid N(\delta)$ is isotopic to Dehn twists about the meridian of $N(\delta)$. For homological reasons there exists an essential simple closed curve $\alpha \subset \partial N(\delta)$, unique up to isotopy, such that some power is homologically trivial in $N - \mathring{N}(\delta)$. Therefore, $f \mid N(\delta)$ has nontrivial Dehn twisting only if α is a meridian, which implies that $H_2(N) \neq 0$, which implies that N is Haken.

Let $W = \bigcup W_i = q^{-1}(N(\delta))$. Since f fixes $N(\delta)$ pointwise and f is homotopic to id, there exists a lift $\tilde{f}$ such that $\tilde{f} \mid W = \mathrm{id}$. By applying Lemma 2.7 a) and isotoping f to achieve a *rolling* of $N(\delta)$, we can assume that for each i, $\tilde{f}$: $(\mathbf{B}^3, \mathbf{B}^3 - \mathring{W}_i, W_i) \to (\mathbf{B}^3, \mathbf{B}^3 - \mathring{W}_i, W_i)$ is homotopic to id rel $W_i \cup S^2_\infty$. Construct a relative handle decomposition of $N - \mathring{N}(\delta)$. Let h be a 1-handle and $\tilde{h}$ a lift to $\mathbf{H}^3$. By Proposition 2.8 $\tilde{f} \mid \tilde{h}$ is homotopic to id rel $\tilde{h} \cap W$, via a homotopy which does not cross W. Project this homotopy into N to obtain a homotopy of $f \mid h$ to id$\mid h$ which does not cross $N(\delta)$. Construct a similar homotopy on each 1-handle. The vanishing of π_2, π_3 on $N - \mathring{N}(\delta)$ allows us to extend the homotopy over the 2- and 3-handles. Therefore $f \mid N - \mathring{N}(\delta) \simeq \mathrm{id}$ rel $\partial N(\delta)$. By [W] f is isotopic to id rel $N(\delta)$. □

3. Minimal surface lemmas

The main results of this section, Propositions 3.9 and 3.10, assert that given a Riemannian metric r on $\mathbf{H}^3$ arising from the closed hyperbolic 3-manifold X and a simple closed curve λ in S^2_∞, there exists a lamination σ by r-least area planes in $\mathbf{H}^3$ which spans λ. A sequence of such laminations, with underlying metrics $s_t \to s$, converges to a spanning lamination by s-least area planes.

In the setting r is the hyperbolic metric, Anderson [A] showed "λ bounds a least area properly embedded plane". Our σ will be the limit of a sequence of compact r-least area discs in $\mathbf{H}^3$ whose boundaries approach λ. Unlike [A] the restriction of our sequence to a fixed compact region of $\mathbf{H}^3$ may not have uniformly bounded area. As a consequence the leaves of our laminations may not be properly embedded. See Conjecture 3.12.

All the background needed to read this section is contained in pp. 89–99 of Joel Hass' and Peter Scott's sharp and to the point paper [HS]. Our arguments rely on the foundational results of [Mor], [MSY] and [S].

Definition 3.1. Let $\rho, p, q, M, X, \mathbf{B}^3, S^2_\infty$, etc. be as in Notation 1.2. If $E \subset \mathbf{B}^3$, then $C(E)$ denotes its hyperbolic convex hull. We abuse notation by letting a Riemannian metric on M or X also denote the induced metric on X or $\mathbf{H}^3$. An immersed disc with boundary γ is a *least area disc* if it is least area among all immersed discs with boundary γ. An injectively immersed plane is a *least area plane* if each compact subdisc is a least area disc.

A codimension-k lamination σ in the n-manifold Y is a codimension-k foliated closed subset of Y, i.e. Y is covered by charts of the form $\mathbf{R}^{n-k} \times \mathbf{R}^k$ and $\sigma \mid \mathbf{R}^{n-k} \times \mathbf{R}^k$ is the product lamination on $\mathbf{R}^{n-k} \times C$, where C a closed subset of $\mathbf{R}^k$. Here and later we abuse notation by letting σ also denote the *underlying space* of its lamination, i.e. the points of Y which lie in leaves of σ. Laminations in this paper will be codimension-1 in manifolds of dimension 2 or 3.

A *complementary region* J is a component of $Y-\sigma$. Given a Riemannian metric on Y, J has an induced path metric, the distance between two points being the infimum of lengths of paths in J connecting them. A *closed complementary region* is the metric completion of a complementary region with the induced path metric. As a manifold with boundary, a closed complementary region is independent of metric.

Definition 3.2. The sequence $\{S_i\}$ of embedded surfaces or laminations in a Riemannian manifold Y converges to the lamination σ if

ia) $\sigma = \{x = \mathrm{Lim}_{i\to\infty} x_i \mid x_i \in S_i$ and $\{x_i\}$ a convergent sequence in $Y\}$;

ib) $\sigma = \{x = \mathrm{Lim}_{n_i\to\infty} x_{n_i} \mid \{n_i\}$ an increasing sequence in $\mathbb{N}, x_{n_i} \in S_{n_i}$ and $\{x_{n_i}\}$ a convergent sequence in $Y\} \stackrel{\mathrm{def}}{=} \mathrm{Lim}\{S_i\}$.

ii) Given $x, \{x_i\}$ as above, there exist embeddings $f_i : D^2 \to L_{x_i}$ which converge in the C^∞-topology to a smooth embedding $f : D^2 \to L_x$, where $x_i \in f_i(\overset{\circ}{D}{}^2)$, L_{x_i} is the leaf of S_i through x_i, and L_x is the leaf of σ through x, and $x \in f(\overset{\circ}{D}{}^2)$.

The following result is more or less well known to experts.

Lemma 3.3 (convergence of least area discs). i) *Let r be a Riemanninan metric on $\mathbf{H}^3$ which is the lift of a metric on a closed hyperbolic manifold X. If $\{S_i\}$ is a sequence of embedded least area discs in $\mathbf{H}^3$ with the r-metric, where $\partial S_i \to \infty$, then after passing to a subsequence $\{S_i\}$ converges to a (possibly empty) lamination by r-least area planes.*

ii) *Let r_t be a $[0,1]$-parameter family of Riemannian metrics on $\mathbf{H}^3$ obtained by lifting a $[0,1]$-parameter family on a closed hyperbolic manifold X. If S_i is a sequence of embedded least area discs in $\mathbf{H}^3$ with the r_{t_i}-metric, where $\partial S_i \to \infty$ and* $\mathrm{Lim}\, t_i = t$, *then after passing to a subsequence $\{S_i\}$ converges to a (possibly empty) lamination by r_t-least area planes.*

Proof. We first give the proof of i).

Step 1. After passing to a subseqence $\mathrm{Lim}\{S_i\} = \{x = \mathrm{Lim}_{i\to\infty} x_i \mid x_i \in S_i$ and $\{x_i\}$ a convergent sequence in $\mathbf{H}^3\} = Z$, a closed subset of $\mathbf{H}^3$.

Proof of Step 1. For each j subdivide $\mathbf{H}^3$ into a finite number of closed regions, such that the $j+1$'st subdivision subdivides the j'th one and such that for each closed ball B in $\mathbf{H}^3$, the mesh of these subdivisions restricted to B converges to 0. Choose a subsequence of $\{S_i\}$ so that if $i \geq j$ and S_i hits a region of the j'th subdivision, then so does S_k, if $k > i$. □

Step 2. Let $\{z_j\}$ be a countable dense subset of Z. There exists $\epsilon > 0$ such that after passing to a subsequence of $\{S_j\}$ the following holds. For each i there exists a sequence of embedded discs $D_{i_j} \subset S_j$ which converges to a smoothly embedded least area disc D_i such that $z_i \in D_i$ and $\partial D_i \cap B_r(\epsilon, z_i) = \emptyset$.

Proof of Step 2. The compactness of X and Theorem 3 of [S] imply that there exist $n, \epsilon > 0$ such that if $x \in \mathbf{H}^3$ and if S is an embedded r-least area compact disc such that $\partial S \cap B_r(n\epsilon, x) = \emptyset$, then after deleting isolated points $S \cap B_r(2\epsilon, x)$ is a union of properly embedded discs of bounded second fundamental form. (Informally, Schoen's local theorem asserts that a least area surface restricted to a sufficiently small ball B does not bend very much provided that the boundary is sufficiently far from B. The bound on bending depends only on the local curvature tensor. Since the r-metric on $\mathbf{H}^3$ is induced from a closed manifold, we can make the above global statement.) By reducing the size of ϵ, if necessary, we can assume that all closed balls of r-radius α, $\alpha < n\epsilon$, are B^3's with strictly convex boundary. Fix i. For j sufficiently large let D_{i_j} be a component of $S_j \cap B_r(2\epsilon, z_i)$ such that $d(z_i, D_{i_j}) \to 0$. Since the D_{i_j}'s are r-least area, they have area bounded above by Sup $1/2\{$Area $\partial B_r(2\epsilon, x) \mid x \in \mathbf{H}^3\}$. By Lemma 3.3 [HS] after passing to a subsequence and restricting to $B_r(\epsilon, z_i)$ the D_{i_j}'s converge (in the sense of Definition 3.2) to the desired D_i. Since this is true for each i, the usual diagonal subsequence argument completes the proof of Step 2. $\square$

Step 3. There exists a lamination σ with underlying space Z, such that each D_i is contained in a leaf. Furthermore $\{S_i\}$ converges to σ.

Proof of Step 3. By Step 1, i) of Definition 3.2 holds. By Step 2, for each i, $D_i \subset Z$. If $x \in \mathring{D}_i \cap \mathring{D}_j$, then D_i and D_j coincide in a neighborhood of x. Otherwise being minimal surfaces, D_i and D_j would cross transversely at some point close to x (e.g. Lemma 3.6 [HS]), which would imply that S_k was not embedded for k sufficiently large. If $z \in Z$, then the argument of Step 2 shows that there exists a convergent sequence $\{D_{z_i}\} \to D_z$, where D_{z_i} is a subdisc of some D_j, $z \in D_z$ and $\partial D_z \cap B_r(\epsilon, z) = \emptyset$. Again since the D_i's pairwise either locally coincide or are disjoint, D_z is uniquely determined in an ϵ-neigborhood of z. Thus $Z = \bigcup_{z \in Z} D_z$. Using the D_z's to define a topology on Z, it follows that connected components are leaves of a lamination σ with underlying space Z. The uniquenss of D_z in $B_r(\epsilon, z)$ implies that near z leaves of σ are graphs of functions over D_z and that $\{S_i\}$ converges to σ. $\square$

Since $\{S_i\}$ converges to σ, we obtain

Step 4. If $g : D \to L$ is an immersion of a disc into a leaf L of σ, then for all i sufficiently large there exists an immersion $g_i : D \to S_i$ such that $g_i \to g$ in the C^∞ topology. $\square$

Step 5. Each leaf L of σ is a least area plane.

Proof of Step 5. Let τ be an essential simple closed curve in L and $A \subset L$ a thin (e.g. $< .5\epsilon$) regular neighborhood of τ. Let $B \subset \mathbf{H}^3$ be a 3-ball transverse to $\bigcup S_i$ such that $A \subset \mathring{B}$. Let $g : D \to L$ be an isometric immersion of a disc such that $g(D) = A$ and Area$(D) >$Area(∂B). (Think of D as being a long thin rectangle.) By Step 4, for i sufficiently large, g is closely approximated by an isometric immersion of a 2-disc, i.e. $g_i : D_i \to S_i$ and Area$(D_i) >$Area(∂B). For i sufficiently large $g_i(D_i)$ is an annulus which closely approximates A. Otherwise $g_i(D_i)$ is an embedded disc which spirals around and closely approximates A. This contradicts the fact that if B is a ball and $\partial S_i \cap B = \emptyset$, then Area$_r(P) \leq 1/2$Area$_r(\partial B)$, where P is a component of $S_i \cap B$. Thus for each sufficiently large i, there exists an embedded simple closed curve $\tau_i \subset S_i$ such that $\{\tau_i\}$ converges to τ. Each τ_i bounds a disc $E_i \subset S_i$ of uniformly bounded area. The sequence of discs $\{E_i\}$ converges to a disc in L bounded by τ via arguments similar to those of the proof of Step 3. Thus L is simply connected. L is not a sphere else for i sufficiently large each S_i would be a sphere. Since each embedded subdisc of L is the limit of least area discs by Step 4, each embedded subdisc of L is least area and hence L is a least area plane. $\square$

Proof of ii). The proof of ii) follows exactly as the proof of i). Perhaps only Step 2 requires some clarification. Again by [S], there exists an $\epsilon > 0$ independent of x, such that if j is sufficiently large (so that t_j is very close to t and $\partial S_j \cap B_{r_t}(n\epsilon, x) = \emptyset$), then each component of $S_j \cap B_{r_t}(2\epsilon, x)$ is a properly embedded disc of uniformly bounded second fundamental form. Similarly given $\delta > 0$, then for j sufficiently large $\text{Area}_{r_{t_j}}(D_{i_j})$ is bounded above by $1/2\text{Area}_{r_t}(\partial(B_{r_t}(2\epsilon, z_i))) + \delta$ and hence the D_{i_j}'s can be parametrized to have uniformly bounded energy with respect to the r_t-metric. These are the facts needed to invoke the proof of Lemma 3.3 [HS]. $\square$

Remark. The lemma could have been stated in more generality by allowing each S_i to be a finite union of pairwise disjoint least area discs such that $\partial S_i \to \infty$.

Definition 3.4. A lamination σ which is a limit as in Lemma 3.3 of a sequence of embedded least area discs $\{S_j\}$ (or more generally a lamination by finite unions of pairwise disjoint least area discs) such that $\partial S_i \to \infty$ will be called a *D^2-limit lamination.* The D^2-limit lamination σ *spans* the simple closed curve $\tau \subset S^2_\infty$, if there exists $e > 0$ such that $\sigma \subset N_\rho(e, C(\tau))$ and the components of $S^2_\infty - \tau$ lie in different components of $\mathbf{B}^3 - \sigma$. Recall that ρ denotes the hyperbolic metric.

The following standard result records all the other elementary least area surface facts needed in this section. Most of these observations were made either implicitly or explicitly in the proof of Lemma 3.3. For convenience we record several D^2-limit laminations facts too.

Lemma 3.5. *Let X be a closed hyperbolic 3-manifold. Let $r_t, t \in [0,1]$ be a family of metrics on $\mathbf{H}^3$ induced from a 1-parameter family of Riemannian metrics on X.*

i) *For each $t \in [0,1]$, the r_t-area differs infinitesimally from the ρ-area by bounded factors $1/c_2, c_2$, where $c_2 \geq 1$ and c_2 is independent of t.*

ii) *There exist constants c_0, c_1 such that if P is a least area disc or plane in $\mathbf{H}^3$ with the r_t-metric, $y \in P$ and $B = B_\rho(c_0, y) \subset \mathbf{H}^3$ is such that $B \cap \partial P = \emptyset$, then $\text{Area}_\rho(P \cap B) > c_1$.*

iii) *If P is an r_t-least area disc, $y \in P$ and $d_\rho(y, \partial P) > 3c_0$, then $d_\rho(y, \partial P) < 3c_0 \,\text{Area}_\rho(P)/c_1$.*

iv) *If σ is a D^2-limit lamination, then σ has no holonomy.*

v) *Let W be a smooth compact codimension-0 submanifold of $\mathbf{H}^3$ transverse to the D^2-limit lamination σ. Then $\sigma \mid W$ is a finite union of product laminations. I.e. there exist finitely many pairwise disjoint submanifolds W_i of W of the form $F_i \times I$, where F_i is a compact surface, $(F_i \times I) \cap \partial W = (\partial F_i) \times I$ and $\sigma \mid F_i \times I$ is the product lamination $F_i \times C_i$, where $C_i \subset \mathring{I}$ is compact.*

In particular the leaves of $\sigma \mid W$ have uniformly bounded area and the leaves of $\sigma \mid \partial W$ are simple closed curves of uniformly bounded length.

vi) *Each leaf $\tilde{L}$ of a D^2-limit lamination σ has an exhaustion by compact discs P_i, such that $\partial P_i \to \infty$. Furthermore if $\sigma = \text{Lim}\{S_j\}$, then for each i, there exist least area discs $\{E_{i_j}\}$ converging to P_i, such that $E_{i_j} \subset S_j$.*

vii) *If, for $i = 1, 2$, L_i is a leaf of the D^2-limit lamination σ_i, then no component of $L_1 \cap L_2$ contains a simple closed curve.*

viii) *If, for $i = 1, 2$, σ_i is a D^2-limit lamination spanning $\lambda_i \subset S^2_\infty$ and $\sigma_1 \cap \sigma_2 \neq \emptyset$, then $\lambda_1 \cap \lambda_2 \neq \emptyset$.*

ix) *If σ is a codimension-1 lamination in the 3-manifold Y, then there exists a possibly empty, nowhere dense sublamination κ such that each closed complementary region of σ is a closed complementary region of κ.*

x) *If κ is a nowhere dense lamination in the 3-manifold Y and W is a compact codimension-0 submanifold of Y, then W can be isotoped slightly to be transverse to κ.*

Proof. i) The metrics on $\mathbf{H}^3$ arise from a $[0,1]$-family of metrics on a closed manifold.

ii) follows from i) and the monotonicity formula (e.g. Lemma 2.3 [HS]).

iii) Apply ii).

iv) Since each leaf is simply connected, this follows from the Reeb stability theorem applied to laminations. E.g. see [GO].

v) We first show that each leaf of $\sigma \mid W$ is compact. Let B be a ball such that $W \subset \mathring{B}$ and B is transverse to σ except possibly at finitely many points. At these points the tangencies should be Morse like. If a leaf P of $\sigma \mid B$ was noncompact, then P would pass through a lamination chart in B infinitely often and so P would have infinite area. By Step 4 of the proof of Lemma 3.3, compact regions of P are closely approximated by compact regions of S_j for j sufficiently large. This contradicts the fact that the r_t-area of components of $S_j \cap \mathring{B}$ is bounded above by $c_2\text{Area}_\rho(\partial B)$. Since each leaf of $\sigma \mid W$ is contained in a leaf of $\sigma \mid B$, the leaves of $\sigma \mid W$ are compact and of uniformly bounded area.

By the Reeb stability theorem and iv) each leaf F of $\sigma \mid W$ has a neighborhood $W_i \subset W$ such that $\sigma \mid W_i$ is a product lamination. Conclusion v) now follows from the compactness of $\sigma \mid W$. The condition $C_i \subset \mathring{I}$ follows if one uses maximal product laminations. In reality, by ix)–x), our laminations will never be locally dense, so the condition $C_i \subset \mathring{I}$ is essentially automatic.

vi) Fix $x \in \tilde{L}$. Using the proof of v) construct $P_j, j \in \mathbf{N}$ so that for each $i, P_i \subset P_{i+1}$ and $\partial P_i \subset \tilde{L} \cap B_\rho(i,x)$. The second part follows from Step 4 of the proof of Lemma 3.3.

vii) If such a curve exists, then apply the Meeks-Yau exchange roundoff trick to show that one of L_1, L_2 is not least area.

viii) Proof by contradiction. If $x \in \sigma_1 \cap \sigma_2$, then let B be a ball such that $x \in B$ and $\partial B \cap N_\rho(e_1, C(\lambda_1)) \cap N_\rho(e_2, C(\lambda_2)) = \emptyset$. The $e_i > 0$ are chosen to have the property that, for $i = 1,2$, $\sigma_i \subset N_\rho(e_i, C(\lambda_i))$. If for $i = 1,2$ there exists a leaf L_i of $\sigma_i \mid B$ such that $x \in L_i$, then $L_1 \cap L_2$ contains a circle of intersection, for each L_i is compact. (Recall 2.6 [HS].) This contradicts vii).

ix) Take κ to be the closure of all the boundary leaves of σ. This lemma allows us to avoid some technicalities in the very unlikely event that a lamination arising from Propositions 3.9 and 3.10 is somewhere locally dense. I.e. we can treat the lamination more like a properly embedded surface than like a foliation.

x) Use general position. □

Definition 3.6. Let α be an unknotted simple closed curve in $\mathbf{H}^3$ with the r-metric. Change the r-metric of $U = \mathbf{H}^3 - \mathring{N}(\alpha)$ by one which coincides with r away from a very small neighborhood of ∂U and which gives U a strictly convex boundary. It follows by [MSY] that an essential simple closed curve on $\partial N(\alpha)$, also called α, bounds a properly embedded disc $D \subset U$, least area among all immersed discs $E \subset U$ with $\partial E \subset \partial U$ and ∂E essential in ∂U. Call a disc that arises from this construction a *relatively least area* disc in $\mathbf{H}^3$.

Lemma 3.7. *Let r_t be a $[0,1]$-parameter family of Riemannian metrics on* $\mathbf{H}^3$ *obtained by lifting a $[0,1]$-parameter family on a closed hyperbolic manifold X. There exists $e > 0$ such that if S is a relatively least area disc in* $\mathbf{H}^3$ *with the r_i-metric, then $S \subset N_\rho(e, C(\partial S))$.*

The proof we give is a technically simpler version of the following more concisely stated outline. Either Lemma 3.7 holds or by applying Lemma 3.3 to a sequence of discs we obtain an embedded least area plane lying in a horoball based at a point in S^2_∞. Such a plane can be chosen to be disjoint from all its translates under G. The projection to X is a leaf of an essential lamination κ by least area planes. By Imanishi (see [G2]) only the 3-torus has an essential lamination by planes.

Proof of Lemma 3.7. Step 1. There exists an r-least area plane $\tilde{L}$ which is a leaf of a D^2-limit lamination and which lies in a horoball of $\mathbf{H}^3$.

Proof of Step 1. Suppose that for each i, there exists a relatively r_i-least area disc D_i' such that $D_i' \not\subset N_\rho(i, C(\partial D_i'))$. Let $z_i \in D_i'$ be a point farthest from $C(\partial D_i')$. A covering transformation of $q : \mathbf{H}^3 \to X$ is an isometry in both the r_i and hyperbolic metrics. Therefore by replacing each D_i' by a covering translate and passing to a subsequence, we can assume that the z_i converge to a fixed $z \in \mathbf{H}^3$. After reparametrizing $\mathbf{H}^3$ and using the unit disc model, we can assume that $z = 0$. By passing to another subsequence we can assume that $\operatorname{Lim}\{C(\partial(D_i'))\} = w \subset S^2_\infty$. Conclude that $\operatorname{Lim}\{D_i'\} \subset H$, the horoball which contains both 0 and w. [Note that if $y \in \mathbf{H}^3 - H$ and $t \in \mathbf{H}^3 \subset \mathbf{B}^3$ is sufficiently Euclidian close to w, then $d_\rho(0,t) < d_\rho(y,t)$.]

Cut down the size of the D_i' and pass to a subsequence to obtain a new sequence as above, called $\{D_i\}$, such that for each i, D_i is an r_i-least area disc (i.e. rather than just relatively least area). To prove this use the following observations. For $N \in \mathbf{N}$ let $B(N)$ denote $B_\rho(N,0)$ perturbed slightly to be transverse to $\bigcup D_i'$. If τ is a component of $D_i' \cap \partial B(N)$ and i is sufficiently large, then the subdisc E of D_i' bounded by τ is an r_i-least area disc. Otherwise, since $\operatorname{Area}_{r_i}(E) < 1/2\operatorname{Area}_{r_i}(\partial B(N))$, any least area disc F with $\partial F = \partial E$ (which exists by [Mor] (see [HS])) must be somewhat close to 0 by Lemma 3.5 iii) and hence be disjoint from $\partial D_i'$ for i sufficiently large. Since D_i' is relatively least area, F and E have the same r_i-area and hence E is r_i-least area. Finally observe that

$$\operatorname{Lim} d_\rho(0, C(\partial B(N) \cap H)) \to \infty.$$

Apply Lemma 3.3 ii) to pass to a subsequence of the D_i and obtain the D^2-limit lamination σ, each of whose leaves is an r-least area plane. Let $\tilde{L}$ be the leaf which contains 0. By Lemma 3.5 vi) $\tilde{L}$ is a union of embedded nested least area discs whose boundaries go to infinity. Replace the old sequence of discs by this sequence, also denoted $\{D_i\}$. □

Step 2. Let G_X denote the group of covering translations of $\mathbf{H}^3$ associated to X. There exists an r-least area plane $\tilde{Q}$ such that for each $g \in G_X$, either $g(\tilde{Q}) = \tilde{Q}$ or $g(\tilde{Q}) \cap \tilde{Q} = \emptyset$. Furthermore either $g(\tilde{Q}) \cap \tilde{L} = \emptyset$ or $g(\tilde{Q}) = \tilde{L}$.

Proof of Step 2. If w is not the fixed point of any element of G_X, then $\tilde{L}$ is the desired $\tilde{Q}$, otherwise there exists $g \in G_X$ such that $g \neq \mathrm{id}$ and $g(\tilde{L}) \cap \tilde{L} \notin \{\emptyset, \tilde{L}\}$. Since $g(w) \neq w$, there exists some i such that $g(D_i) \cap D_i \neq \emptyset$ but $g(\partial D_i) \cap (\partial D_i) = \emptyset$. This leads to a contradiction by the exchange roundoff trick.

The other possibility is that w is the fixed point of some primitive element f of G_X. We find Q as follows. Let A_f denote the hyperbolic axis of f. There does not exist $N > 0$ such that $\tilde{L} \subset N_\rho(N, A_f)$. Otherwise, for any $t \in A_f$, each component of $H_t \cap \tilde{L}$ would have area bounded by $c_2\cdot$(area of the hyperbolic disc of radius N). Here H_t is the $\mathbf{B}^3$ halfspace disjoint from w and bounded by the hyperbolic plane orthogonal to A_f at t. This contradicts Lemma 3.5 iii), for t close to w.

Let $\{y_i\}$ be a sequence of points in $\tilde{L}$ such that $d(y_i, A_f) > i$. Let $g_i \in G_X$ be such that $g_i(y_i) = v_i$ lies in a fixed X-fundamental domain V in $\mathbf{H}^3$. By passing to a subsequence we can assume that $v_i \to v \in \mathbf{H}^3$ and $g_i(w) \to w'$. By passing to another subsequence we can assume that $i \neq j$ implies that $w_i \overset{\text{def}}{=} g_i(w) \neq g_j(w) \overset{\text{def}}{=} w_j$. Suppose on the contrary that for all $i, j, g_i(w) = g_j(w)$. Then $g_i(w) = g_j(w) \implies g_j^{-1} \circ g_i(w) = w \implies g_j^{-1} \circ g_i = f^{n_i} \implies g_i = g_j \circ f^{n_i}$. Now $g_i(y_i) \subset V \implies y_i \in g_i^{-1}(V) = f^{-n_i} \circ g_j^{-1}(V) \implies d(y_i, A_f) \leq \max\{d(g_j^{-1}(z), A_f) \mid z \in V\}$. The finiteness of the latter contradicts the choice of y_i, for i large.

Let $\tilde{Q}$ be a least area plane passing through v, obtained by applying Lemma 3.3 to the sequence $g_i(\tilde{L}) = \tilde{L}_i$, or more precisely to $\{g_i(D_{n_i})\}$, where $\{n_i\}$ is a sufficiently fast growing sequence. There exists no $h \in G_X$ such that $h(\tilde{Q}) \cap \tilde{Q} \notin \{\emptyset, \tilde{Q}\}$; else for sufficiently large i, j, $h(\tilde{L}_j) \cap \tilde{L}_i \neq \emptyset$. Therefore there exists i, j such that $h(\tilde{L}_j) \cap \tilde{L}_i \neq \emptyset$ and $w_i \neq h(w_j)$. This implies that $g_i^{-1} \circ h \circ g_j(\tilde{L}) \cap \tilde{L} \neq \emptyset$ and $g_i^{-1} \circ h \circ g_j(w) \neq w$, which is a contradiction. A similar argument shows that $h(\tilde{L}) \cap \tilde{Q} \in \{\emptyset, \tilde{Q}\}$ ☐

Step 3. There exists an r-least area properly embedded plane $\tilde{P}$ contained in a horoball in $\mathbf{H}^3$ such that for each $g \in G_X, g(\tilde{P}) = \tilde{P}$ or $g(\tilde{P}) \cap \tilde{P} = \emptyset$. If $\pi : \mathbf{H}^3 \to X$ is the covering projection, then $\pi(\tilde{P})$ projects to a leaf P of an essential lamination κ in X. Finally the leaves of κ lift to r-least area planes in $\mathbf{H}^3$ and each leaf of κ is dense in κ.

Proof of Step 3. Let λ be the lamination in X obtained by taking the closure of the injectively immersed surface Q which is the projection of $\tilde{Q}$. We show that λ is essential by showing that each leaf is incompressible and end incompressible [GO]. Each leaf Q_α of λ lifts to a surface $\tilde{Q}_\alpha$ in $\mathbf{H}^3$ which is a limit of translates of subdiscs of $\tilde{Q}$, hence $\tilde{Q}_\alpha$ is a leaf of a D^2-limit lamination and hence is a least area plane, so Q_α is incompressible. An end compression of Q_α would imply the existence of a monogon in $\mathbf{H}^3$ connecting two very close together subdiscs of $\tilde{Q}$ of very much larger area, contradicting the fact that $\tilde{Q}_\alpha$ is least area as in Figure 4 of [HS].

Let κ be a nontrivial sublamination of λ such that each leaf of κ is dense in κ.

The lift $\tilde{\kappa}$ of κ to $\mathbf{H}^3$ is a sublamination of the lamination which is the closure of all the G_X-translates of $\tilde{Q}$. Since $\tilde{L}$ is either disjoint from $\tilde{\kappa}$ or a leaf of $\tilde{\kappa}$, it follows that $L = \pi(\tilde{L})$ is either a leaf of κ or disjoint from κ. By construction $\kappa \subset \bar{L}$ since $\tilde{Q}$ is in the closure of $G_X(\tilde{L})$.

If $\tilde{L}$ is a leaf of $\tilde{\kappa}$, then Step 3 holds with $\tilde{P} = \tilde{L}$. In that case since $\tilde{L}$ is the lift of a leaf of an essential lamination, it follows by [GO] that $\tilde{L}$ is properly embedded in $\mathbf{H}^3$.

Next consider the case that $L \subset J$ is a closed complementary region of κ. The proof of Step 3 follows from the following

Claim. $J = \mathring{D}^2 \times I$ and L is homotopic to $\mathring{D} \times 1/2$ via a homotopy in J in which points of L are moved by homotopy tracks of uniformly bounded length.

Proof of the Claim. As in [GO] J is of the form $A \cup Z$, where each component of A is an I-bundle over a noncompact surface, Z is a connected compact 3-manifold and $A \cap Z$ is a union of annuli. Since X is of finite volume, by taking Z to be sufficiently big (by reducing the size of A) we can assume that the I-fibres are very short ρ-geodesic arcs nearly orthogonal to ∂J. Recall that by [S] $\tilde{L}$ and hence L have bounded second fundamental form. This implies that if the I-fibres are sufficiently short, then they must be transverse to L. Thus we can assume that L is transverse to the I-fibres of A.

Assume $A \neq \emptyset$. If E is a vertical annulus in A, i.e. a union of I-fibres, then either E spans a $D^2 \times I \subset J$ or $E \cap L = \emptyset$. Otherwise E lifts to an $I \times \mathbf{R}$ whose core α is properly homotopic (by the previous paragraph) to a curve lying in $\tilde{L}$, contradicting Step 1, for α has distinct endpoints in S^2_∞. Since $\kappa \subset \bar{L}$, it follows that some component A_1 of A and hence each component of $A_1 \cap Z$ nontrivially intersect L and hence $A_1 = A$ and J is obtained by attaching 2-handles to A along $A \cap Z$. Since each vertical annulus in A bounds a $D^2 \times I$, it follows that $J = \mathring{D}^2 \times I$. Since J is simply connected, it lifts to $\mathbf{H}^3$ and hence L is embedded in J since $\tilde{L}$ is embedded in $\mathbf{H}^3$. Therefore if $E \subset A$ is a vertical annulus, then $E \cap L$ is a union of embedded circles. Each such circle bounds a disc in L which is isotopic rel boundary to a horizontal disc in the associated $D^2 \times I$. If P is a component of ∂J, then vertical projection of $L \cap A$ to $P \cap A$ extends to an immersion of L to P. P being simply connected implies that this is in fact a diffeomorphism. Again as in [GO] each lift of P is properly embedded.

If $A = \emptyset$, derive a contradiction as follows. In this case κ is a closed π_1-injective surface S_0. Consider an incompressible surface S_1 in X split open along S_0 which nontrivially intersects S_0 and consider $L \cap S_1$ to argue that the limit set of $\tilde{L}$ consists of more than a point.

Since each leaf of κ is dense in κ the above argument shows that κ has no closed leaves. □

Step 4. Proof of Lemma 3.7.

Proof of Step 4. Note that $\tilde{P}$ could have been chosen so that $w \in S^2_\infty$ is the basepoint of the horoball containing $\tilde{P}$. If B is the region in $\mathbf{B}^3$ bounded by $\tilde{P}$ such that $B \cap S^2_\infty = w$, then $G_B = \{g \in G_X \mid g(\mathring{B}) \cap \mathring{B} \neq \emptyset\}$ is a subgroup of the stabilizer G_w of w. Since G_w is generated by f, G_B is generated by f^n for some $n \in \mathbf{Z}$. First suppose that $G_B \neq \text{id}$. We can assume that $f^n(B - w) \subset \mathring{B}$. Since $\tilde{P}$ is proper, each $z \in \tilde{P}$ has a neighborhood $W \subset \mathbf{H}^3$ such that $W \cap (f^n(\tilde{P}) \cup f^{-n}(\tilde{P})) = \emptyset$ and hence $\{g \in G_X \mid g(\tilde{P}) \cap W \neq \emptyset\} = \text{id}$. This implies that P is isolated, contradicting the fact that each leaf of κ is dense and κ has no closed leaves. Finally consider the case $G_B = \text{id}$. In this case $\mathring{B} \cap \tilde{\kappa} = \emptyset$, otherwise P is dense in κ implies that some covering translate of $\tilde{P}$ lies in $\mathring{B}$. Let I be an I-fibre of A and let $\tilde{I}$ be the lift which intersects $\tilde{P}$. Since P is nonisolated, $\tilde{I} \subset B$, with one endpoint $i \in \mathring{B}$. We obtain the contradiction $\tilde{\kappa} \cap \mathring{B} \neq \emptyset$. □

Lemma 3.8 (Convex hull facts). *Let τ be a smooth simple closed curve in S^2_∞ and $k > 0$. Then*

i) *$N_\rho(k, C(\tau))$ is a convex, smooth, properly embedded $\mathring{D}^2 \times I$.*

ii) *The product structure can be chosen so that for every $\epsilon > 0, \{x \in D^2 \mid \text{length}_\rho(x \times I) > 2k + \epsilon\}$ is bounded.*

iii) *If $\tau_\epsilon \subset \mathbf{B}^3$ is a Euclidean ϵ-neighborhood of τ, then $N_\rho(k, C(\tau)) = \bigcap_{\epsilon>0} N_\rho(k, C(\tau_\epsilon))$.*

Proof. A proof of smoothness of $\partial N(k, C(\tau))$ due to Bowditch can be found in [EM, p. 119]. The remainder is an elementary exercise in hyperbolic geometry. Use the fact that if $x \in C(\tau) \cap \mathbf{H}^3$ and x is Euclidean very close to τ, then in the visual sphere of x, τ approximates a great circle. $\square$

Proposition 3.9 (Least area spanning laminations exist). *Let τ be a smooth simple closed curve in S^2_∞ and r a Riemannian metric on $\mathbf{H}^3$ induced from a metric on a closed hyperbolic 3-manifold. Then there exists a D^2-limit lamination $\sigma \subset \mathbf{H}^3$ by r-least area planes spanning τ. Furthermore there exists $e > 0$, which depends only on r (and hence independent of τ), such that if σ is any spanning lamination by r-least area planes, then $\sigma \subset N_\rho(e, C(\tau))$.*

Proof. Let $e > 0$ be as in Lemma 3.7. Let ω be a properly embedded path in $\mathbf{B}^3$ connecting points in distinct components of $S^2_\infty - \tau$.

Step 1. There exists a sequence of relatively least area discs $\{E_i\}$ such that for each $i, E_i \subset N_\rho(2e, C(\tau))$, $\partial E_i \to \infty$, and $|\langle E_i, \omega\rangle| \neq 0$. Here $\langle , \rangle$ denotes oriented intersection number.

Proof of Step 1. By Lemma 3.8 $N_\rho(e, C(\tau))$ has an exhaustion by regions $R_0 \subset R_1 \subset \cdots$, where $R_i = D^2 \times I$ and $\delta R_i = (\partial D^2) \times I$. Here δR_i denotes the relative boundary of R_i. Choose R_0 such that $\omega \cap N_\rho(e, C(\tau)) \subset R_0$. The disc E_i is obtained by applying the procedure of Definition 3.6 to an essential circle in δR_i. By Lemma 3.7 $E_i \subset N_\rho(2e, C(\tau))$. $\square$

Step 2. There exists a sequence $\{D_i\}$ of least area discs such that for each $i, D_i \subset N_\rho(2e, C(\tau)), \partial D_i \to \infty$, and $|\langle D_i, \omega\rangle| \neq 0$.

Proof of Step 2. Apply a procedure, similar to the one of the second paragraph of the proof of Lemma 3.7, to obtain the sequence $\{D_i\}$ from the sequence $\{E_i\}$. $\square$

Step 3. After passing to a subsequence, $\{D_i\}$ converges to a lamination σ by r-least area planes which spans τ.

Proof of Step 3. Let σ be a D^2-limit lamination obtained by applying Lemma 3.3 to $\{D_i\}$. We need to show that each component of $S^2_\infty - \tau$ lies in a different complementary region of σ, the other conditions being self evident. If $\omega_1 \subset \mathbf{B}^3 - \sigma$ is a properly embedded path connecting these two components, then since $\omega_1 \cap N_\rho(2e, C(\tau))$ is compact and disjoint from σ, it follows that for i sufficiently large $D_i \cap \omega_1 = \emptyset$. This contradicts the fact that for i sufficiently large, $|\langle \omega, D_i\rangle| = |\langle \omega_1, D_i\rangle|$. $\square$

Step 4. If σ spans τ, then $\sigma \subset N_\rho(e, C(\tau))$.

Proof of Step 4. If L is a leaf of σ, then by Lemma 3.5 vi) L has an exhaustion by compact discs P_i such that $\partial P_i \to \infty$. Hence for each $\epsilon > 0$, there exists N_ϵ such that if $i > N_\epsilon$, then $\partial P_i \subset N_E(\epsilon, \tau) \subset \mathbf{B}^3$, where E denotes Euclidean metric. Therefore by Lemmas 3.7 and 3.8 iv), $L \subset N_\rho(e, C(\tau))$. □

Proposition 3.10 (Convergence of spanning laminations). *Let $r_t, t \in [0,1]$ be a smooth family of Riemannian metrics on $\mathbf{H}^3$ induced from Riemannian metrics on a closed hyperbolic 3-manifold and let $\{t_i\}$ be a sequence in $[0,1]$ such that* $\operatorname{Lim} t_i = t$. *Let r_i (resp. r) denote the r_{t_i} (resp. r_t) metric. Let τ be a smooth simple closed curve in S^2_∞. If $\{\sigma_i\}$ is a sequence of D^2-limit laminations by r_i-least area planes spanning τ, then after passing to a subsequence $\{\sigma_i\}$ converges to a D^2-limit lamination σ by r-least area planes which spans τ.*

Proof. Let $e > 0$ be as in Lemma 3.7 for the metrics $r_s, s \in [0,1]$. The proof of Lemma 3.3 works equally well for sequences of D^2-limit laminations as it does for sequences of least area discs. In fact suppose that for each i, D_i is chosen to be a finite union of discs embedded in leaves of σ_i which are 2^{-i} dense in $\sigma \cap B_\rho(i,0)$ (i.e. $\sigma_i \cap B_\rho(i,0) \subset N_\rho(2^{-i}, D_i \cap B_\rho(i,0)))$ and $\partial D_i \to \infty$. Then the subsequence $\{D_{i_\alpha}\}$ of $\{D_i\}$ converges to the lamination σ if and only if the subsequence $\{\sigma_{i_\alpha}\}$ of $\{\sigma_i\}$ converges to the lamination σ.

Now suppose that $\{\sigma_i\}$ converges to the D^2-limit lamination σ. Again, we need to show that each component of $S^2_\infty - \tau$ lies in a different complementary region of σ, the other conditions being self evident. Let R_i be as in the proof of Step 1 of Proposition 3.9. After a small perturbation assume further that R_i is transverse to $\bigcup \sigma_i$. Since σ_i separates the components of $S^2_\infty - \tau$, some leaf ζ_i of $\sigma_i \mid \delta R_i$ is essential in δR_i. ζ_i bounds a disc $F_i \subset \sigma_i$. Thus $|\langle F_i, \omega\rangle| = 1$. As in the proof of the previous lemma, a subsequence of $\{F_i\}$ limits on a lamination σ' which separates the components of $S^2_\infty - \tau$. Since σ' is a sublamination of σ, σ separates the components of $S^2_\infty - \tau$. □

Corollary 3.11. *A limit of D^2-limit laminations is a D^2-limit lamination.*

Conjecture 3.12. *Let r be a Riemannian metric on $\mathbf{H}^3$ induced from a Riemannian metric on a closed hyperbolic 3-manifold. If λ is a smooth simple closed curve in S^2_∞, then λ spans a properly embedded r-least area plane in $\mathbf{H}^3$.*

Remarks 3.13. i) Freedman and He [FH] have constructed an example of a non-properly-embedded plane which is least area with respect to the hyperbolic metric on $\mathbf{H}^3$.

ii) A result similar to Proposition 3.9 can be found in [L2]. The lamination that arises there is properly embedded but not necessarily by planes. On the other hand, it is a theorem about all dimensions and codimensions and requires only that the Riemannian metric induce a topological metric Lipshitz equivalent to the hyperbolic metric. I suspect that under this weaker hypothesis on the Riemannian metric, the planes that span $\lambda \subset S^2_\infty$ need not be properly embedded. One can deduce an independent proof of Lemma 3.7 from the proof of Theorem 2 of [L1].

Remark 3.14. In a natural way the results of this section generalize to 3-manifolds with negatively curved fundamental group.

4. Constructing the tubes

Lemma 4.1. *Let μ be a Riemannian metric on $\mathbf{H}^3$ induced from a closed hyperbolic 3-manifold. Let $\{\sigma_i\}$ be a locally finite set of D^2-limit laminations of $\mathbf{H}^3$ by μ-least area planes. If J is a component of $\mathbf{H}^3 - \bigcup \sigma_i$, then*

i) $\pi_1(J) = 1$.

ii) *If J^* is the metric completion of J with the induced path metric, then J^* is a manifold with boundary.*

iii) *The natural map $\Pi : J^* \to \mathbf{H}^3$ is an injective immersion.*

Proof of i). Let τ be a closed curve in J and $f : D^2 \to \mathbf{H}^3$ be transverse to each σ_i such that $f \mid \partial D^2 = \tau$. Then $f(D^2) \cap \sigma_i = \emptyset$ for all but finitely many i, say $i = 1, \ldots, n$. Assume that f is chosen to minimize n. If $n = 0$, then τ is homotopically trivial in J. If $n > 0$ we obtain a contradiction as follows. By Lemma 3.5 σ has no holonomy, so $f^{-1}(\sigma_n)$ is a lamination by circles so there is a finite number of outermost circles $\tau_1, \ldots, \tau_m$ of $f^{-1}(\sigma_n)$ bounding discs in D whose union contains $f^{-1}(\sigma_n)$. Each τ_r maps to an immersed curve α_r in a leaf L_i of σ_n. L_i is a least area plane and α_r is disjoint from σ_k for $k > n$ imply that α_r is homotopically trivial in L_i via a homotopy disjoint from $\sigma_k, k > n$. In fact, the outermost component of $\partial N(\alpha_r) \subset L_i$ bounds an embedded disc E whose boundary is disjoint from $\sigma_k, k > n$, hence E is disjoint from $\sigma_k, k > n$. Being outermost, $f(\tau_r)$ lies on the boundary of a closed complementary region of σ_n. By replacing the image of the subdisc of D bounded by each τ_s with a disc close to but disjoint from σ_n, we obtain a new immersed disc spanning τ intersecting at most $\sigma_i, 1 \le i \le n-1$. □

Proof of ii). By Lemma 3.5 ix) we can assume that each σ_i is nowhere dense. Given $x \in \bigcup \sigma_i$, there exists a short geodesic arc α passing through x and transverse to $\bigcup \sigma_i$ with $\partial\alpha \cap (\bigcup \sigma_i) = \emptyset$. Let $\hat{\sigma}_i$ denote $\sigma_i \mid D^2 \times I$, the $D^2 \times I$ being a regular neighborhood of α where $0 \times 1/2 = x$. If the D^2 factor is sufficiently small, then each leaf of any $\hat{\sigma}_i$ is the graph of a function $g : D^2 \to I$ and $\hat{\sigma}_i$ is a product lamination. Also the projection of the intersection of any two leaves $L_i \subset \hat{\sigma}_i, L_j \subset \hat{\sigma}_j$ into the D^2 factor is either empty or a smooth properly embedded arc or n properly embedded arcs which intersect at a single point in $\mathring{D}^2$, the arcs having distinct slopes at the common point. The latter occurs if L_i and L_j are tangent and uses the normal form theorem for tangencies between least area surfaces, e.g. Lemma 2.6 [HS]. By local finiteness and reindexing we assume that $D^2 \times I$ intersects only $\sigma_i, 1 \le i \le q$.

Let K be a component of $D^2 \times I - \bigcup \sigma_i$. To prove ii) it suffices to show that $\Pi \mid K^*$ is injective and $\Pi(K^*)$ is a manifold with boundary. For each $i \le q$ there exist leaves A_i, B_i of $\hat{\sigma}_i$ such that K lies in the complementary region of $\hat{\sigma}_i$ defined by points lying above A_i and below B_i, though possibly one of $A_i, B_i = \emptyset$. The region above the leaves $A_1, \ldots, A_q$ (resp. below $B_1, \ldots, B_q$) is the region above (resp. below) the graph of a function $A : D^2 \to I$ (resp. $B : D^2 \to I$). So A (resp. B) is the maximum (resp. minimum) of a finite set of smooth functions. Therefore each component K of $D^2 \times I - \bigcup \sigma_i$ either lies between the graph of two functions defined over an open subset U of D^2, or lies either above or below the graph of a function on D^2. The proof of ii) in the latter case is clear. To show that $\Pi \mid K^*$ is injective and $\Pi(K^*)$ is a manifold with boundary it suffices to show that $\bar{U}$ is a manifold with boundary. The projection c_{ij} of $A_i \cap B_j$ into $\mathring{D}^2$ has a natural normal orientation, i.e. the normal points into the side where the region below B_j and above A_i is nontrivial. $\bar{U}$ is the closure of a connected region defined by the c_{ij}, with all normals pointing in. Since $L_i \cap L_j$ contains no embedded circles, the region U is a disc. It is routine to check that $\bar{U}$ is topologically a closed disc. □

Proof of iii). Proving that Π is an injective immersion reduces to showing that if the discs U_1 and U_2 in $\mathring{D}^2$ are defined by the same leaves $B_1, \ldots, B_q$ and $A_1, \ldots, A_q$, and $\bar{U}_1 \cap \bar{U}_2 \neq \emptyset$, then the regions K_1 and K_2 associated to them lie in distinct path components of $\mathbf{H}^3 - \bigcup \sigma_i$. If not, then one could pass to a minimal example of the saddle or spike type which are described and dispatched in the following paragraphs.

Saddle example. $\bigcup \sigma_i = \sigma_1 \cup \sigma_2$ and $\bar{U}_1 \cap \bar{U}_2$ correspond to a saddle tangency between leaves $A_1 \subset \sigma_1$ and $B_2 \subset \sigma_2$, where $B_1 = A_2 = \emptyset$. See Figure 4.3(b). A *down* isotopy of σ_1 near the saddle creates a non-simply-connected component of $\mathbf{H}^3 - \sigma_1' \cup \sigma_2$, where σ_1' denotes the isotoped σ_1. The intersection of a leaf A of σ_1' and a leaf B of σ_2 must contain a simple closed curve; else one could argue as in the proof of i) to conclude that each component of $\mathbf{H}^3 - (\sigma_1' \cup \sigma_2)$ was simply connected. Since σ_1 (resp. σ_2) is isolated above (resp. below) A_1 (resp. B_2), the only possibility is that A is the isotoped A_1 and $B = B_2$. This implies that $A_1 \cap B_2$ contains a simple closed curve, again contradicting Lemma 3.5 vii).

Spike example. Here $\bigcup \sigma_i = \sigma_1 \cup \sigma_2 \cup \sigma_3$; $A_1, B_2, B_3 \neq \emptyset$ and the lines $A_1 \cap B_2, A_1 \cap B_3$ intersect tangentially at a single point $x \in \bar{U}_1 \cap \bar{U}_2$. See Figure 4.1. A small down isotopy of σ_1 creates a non-simply-connected component of $\mathbf{H}^3 - \sigma_1' \cup \sigma_2 \cup \sigma_3$ where σ_1' denotes the isotoped σ_1. On the other hand, since σ_1 is transverse to $\sigma_2 \cup \sigma_3$ near x, $\sigma_1' \cap \sigma_j$ contains a simple closed curve if and only if $\sigma_1 \cap \sigma_j$ contains a simple closed curve, for $j = \{2, 3\}$. Thus one obtains a contradiction as in the saddle example.

□

Remark 4.2. The following observations follow from the proof of Lemma 4.1.

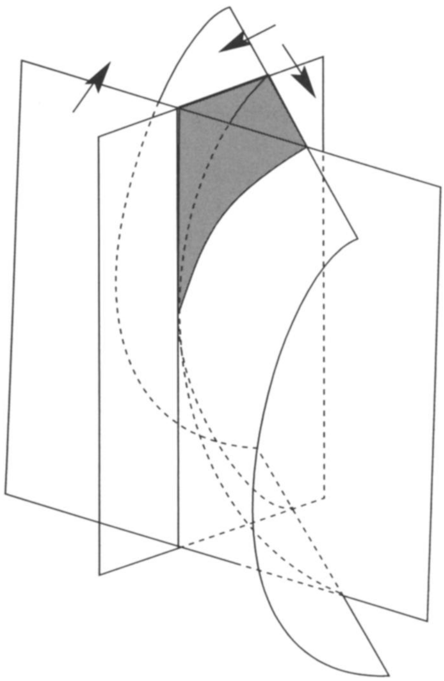
i) If K_1 and K_2 are distinct connected complementary regions of $D^2 \times I - \bigcup \sigma_i$ which limit on x, then one of the following three situations occur, up to reversing the parametrization on I.

a) There exists j such that K_1 and K_2 lie in distinct complementary regions of $\sigma_j \mid D^2 \times I$. In this case $\sigma_j \mid D^2 \times I$ has an isolated leaf separating K_1 and K_2. Therefore K_1, K_2 lie in distinct complementary regions of $\mathbf{H}^3 - \sigma_j$.

b) There exist j, k, leaves $A_j \subset \sigma_j \mid D^2 \times I, B_k \subset \sigma_k \mid D^2 \times I$, such that A_j and B_k have a saddle tangency at x, K_1, K_2 lie above A_j and below B_k, and K_1, K_2 lie in distinct components of $D^2 \times I - (\sigma_j \cup \sigma_k)$.

c) There exist j, k, l leaves $A_j \subset \sigma_j \mid D^2 \times I$, $B_k \subset \sigma_k \mid D^2 \times I$, $B_l \subset \sigma_l \mid D^2 \times I$ such that K_1, K_2 lie above A_j and below B_k, B_l. Finally K_1, K_2 lie in different components of $\mathbf{H}^3 - \sigma_j \cup \sigma_k \cup \sigma_l$.

ii) If $J_1, \ldots, J_n$ are finitely many components of $\mathbf{H}^3 - (\sigma_j \cup \sigma_k \cup \sigma_l)$ such that for each $r \neq s$ each point of $\Pi(J_r^*) \cap \Pi(J_s^*)$ occurs at a saddle or spike as in i), then $\bigcup \Pi(J_i^*)$ is simply connected. Otherwise one obtains a contradiction as in the proof of Lemma 4.1.



A spike

FIGURE 4.1

Lemma 4.3. *Let $x, y \in S^2_\infty$, r a Riemannian metric on $\mathbf{H}^3$ induced from the closed 3-manifold M or X, and $\lambda_1, \ldots, \lambda_m$ smooth simple closed curves in $S^2_\infty - \{x, y\}$ such that no λ_i separates x from y. For each i, let σ_i be a lamination by r-least area planes which spans λ_i and let H_i be the complementary region of $\mathbf{B}^3 - \sigma_i$ which contains x, y. Then either*

i) *x, y lie in the same component H of $\bigcap_{i=1}^m H_i$, or*

ii) *there exist $\lambda_i, \lambda_j, \lambda_k$ such that $\lambda_i \cup \lambda_j \cup \lambda_k$ separate x from y in S^2_∞.*

Proof. If $m \leq 3$ and ii) does not hold, then x, y lie in the same component of $S^2_\infty - \bigcup \lambda_i$, so i) holds. Assuming inductively that the lemma is true for $n < m$, we will establish it for cardinality m. Therefore either ii) holds or

$$(*) \qquad \text{for every } j \leq m,\ x \text{ and } y \text{ lie in the same component of } \bigcap_{i \neq j} H_i.$$

We show that if $(*)$ holds, then either i) holds or for each j, k, $\lambda_j \cap \lambda_k \neq \emptyset$. Let $\tau_j \subset \bigcap_{i \neq j} H_i$ (resp. $\tau_k \subset \bigcap_{i \neq k} H_i$) be a path from x to y transverse to σ_j (resp. σ_k). By Lemma 4.1 i), there exists $h : I \times I \to \bigcap_{i \notin \{j,k\}} H_i$, a homotopy from τ_j to τ_k, which is transverse to both σ_j and σ_k. Either $\sigma_k \cap \sigma_j \neq \emptyset$ and hence $\tau_k \cap \tau_j \neq \emptyset$ by Lemma 3.5 viii) or $h^{-1}(\sigma_k \cup \sigma_j)$ is a lamination by circles and arcs. Each arc has both endpoints on one of $I \times 0$ or $I \times 1$. Thus $\tau_k \cap \tau_j = \emptyset$ implies that there exists an embedded path from $0 \times I$ to $1 \times I$ disjoint from $h^{-1}(\sigma_j \cup \sigma_k)$ and hence conclusion i) holds.

Either i) holds or there exists a minimal $s > 0$ and a reordering of the λ_i so that $\lambda_1 \cup \cdots \cup \lambda_s$ separate x from y in S^2_∞. Minimality implies that some component τ of $\lambda_s - (\lambda_1 \cup \cdots \cup \lambda_{s-1})$ has the property that $\tau \cup \lambda_1 \cup \cdots \cup \lambda_{s-1}$ separate x, y and there exists a path $\alpha \subset S^2_\infty$ from x to y such that α intersects $\tau \cup \lambda_1 \cup \cdots \cup \lambda_{s-1}$ transversely exactly once in $\overset{\circ}{\tau}$. The closure of τ has endpoints on λ_i and λ_j, where $i, j \leq s-1$. Since $\lambda_i \cap \lambda_j \neq \emptyset$, there exists a simple closed curve $\beta \subset \lambda_i \cup \lambda_j \cup \tau$ such that $\tau \subset \beta$. Therefore $\lambda_i \cup \lambda_j \cup \lambda_s$ separate x from y. □

Definition 4.4. If $Y = D^2 \times S^1$ or $\overset{\circ}{D}{}^2 \times S^1$, then a *core* of Y is a curve of the form $z \times S^1$, where $z \in \overset{\circ}{D}{}^2$.

Lemma 4.5. i) *Cores of solid tori are unique up to isotopy.*

ii) *If $h : Y_2 \to Y_1$ is a covering map between solid tori, then c is a core of Y_1 if and only if $h^{-1}(c)$ is a core of Y_2.*

iii) *If Y_1 and Y_2 are $\overset{\circ}{D}{}^2 \times S^1$'s, such that $c_1 \subset Y_2 \subset Y_1$, where c_1 is a core of Y_1, then c_1 is a core of Y_2.*

iv) *Let $\{D_i\}$ be a locally finite collection of pairwise disjoint properly embedded planes in $Y_2 = \overset{\circ}{D}{}^2 \times S^1$. If Y_1 is a $\overset{\circ}{D}{}^2 \times S^1$ component of $Y_2 - \bigcup D_i$, then any core of Y_1 is a core of Y_2.*

v) *Let Y_2, Y_3 be $D^2 \times S^1$'s embedded in the 3-manifold Y and let c_1 be a core of Y_3. If $c_1 \subset \overset{\circ}{Y}_2$ and there exists an embedded 2-disc $E \subset \partial Y_3$ such that $Y_2 \cap \partial Y_3 \subset E$, then c_1 is a core of Y_2.*

Proof. We first establish versions of i)–iii) in the case of closed solid tori.

ic) Here is a hint to this well known result. Let c_1 and c_2 be cores of Y, the c_2 being an S^1 fibre of $Y = D^2 \times S^1$. First isotope c_1 to be transverse to this D^2-fibration. Then isotope c_1 to c_2. The first isotopy follows from the existence of an embedded annulus connecting c to a simple closed curve d in $\partial D^2 \times S^1$ and the isotopy classification of simple closed curves on the torus.

iic) The product structure of Y_1 lifts to a product structure on Y_2, so a core of Y_1 lifts to a core of Y_2. Conversely suppose that $h^{-1}(c) = d$ is a core of Y_2. Let R be a connected embedded orientable surface of maximal Euler characteristic with two boundary components, one boundary component c and one boundary component on ∂Y_1. The curve c is necessarily a generator of $\pi_1(Y_1)$ so R exists for homological reasons. The surface R lifts up to $\tilde{R}$ in Y_2. Each primitive element of $H_2(Y_2 - \overset{\circ}{N}(d), \partial Y_2 \cup \partial N(d))$ and in particular $[\tilde{R}]$ is represented by an annulus. By [G1, Corollary 6.13] $\deg(h)\chi(R) = \chi(\tilde{R}) = 0$, therefore R is an annulus and c is a core of Y_1.

iiic) Here we are assuming that Y_1, Y_2 are $D^2 \times S^1$'s and $c_1 \subset \overset{\circ}{Y}_2 \subset Y_2 \subset Y_1$ and c_1 is a core of Y_1. ∂Y_2 is incompressible in $N(Y_1) - \overset{\circ}{N}(c) = T^2 \times I$, so by [W], ∂Y_2 is isotopic in $N(Y_1) - \overset{\circ}{N}(c)$ to $T^2 \times 1/2$. Therefore Y_2 has a product structure which restricts to a product structure on $N(c)$.

We now prove i)–iv) for open solid tori.

iii) For $i = 1, 2$, let c_i be a core of Y_i. Let $N_2 \subset Y_2$ be a large regular neighborhood of c_2 such that $c_1 \subset N_2$. Let $N_1 \subset Y_1$ be a large regular neigborhood of c_1 such that $N_2 \subset N_1$. Then iiic) applied to the inclusions $c_1 \subset N_2 \subset N_1$ yields that c_1 is a core of N_2. By ic) c_1 is isotopic to c_2 and hence c_1 is a core of Y_2.

i) Apply iii).

ii) Suppose $c \subset Y_1$ lifts to a core $d \subset Y_2$ and let N_1 be a regular neighborhood of a core of Y_1 such that $d \subset \text{int}(\tilde{N}_1)$ where $\tilde{N}_1$ is the lift of N_1 to Y_2. By iii) d is a core of $\tilde{N}_1$ and so c is a core of N_1 and hence Y_1 by iic).

iv) Let N_2 be a large standardly embedded $D^2 \times S^1 \subset Y_2$ such that $c_1 \subset N_2$ and N_2 is transverse to $\bigcup D_j$. Since c_1 is homotopically nontrivial in Y_2, any disc component of $D_j \cap N_2$ must separate off a ball in N_2 disjoint from c_1. On the other hand, a subdisc D of D_j with $\partial D \subset N_2, \mathring{D} \cap N_2 = \emptyset$ together with a subdisc of ∂N_2 bounds a ball in $Y_2 - \mathring{N}_2$. Therefore the usual innermost disc (in D_j) and an isotopy argument allows us to assume, after isotopy of N_2, that $c_1 \subset N_2 \subset Y_1$. Now apply iii).

v) By doing a finite sequence of compressions and 2-handle attachments to Y_2 in a small neighborhood of E, we obtain a new manifold W such that $c_1 \subset W \subset Y_3$. Topologically W must be a $D^2 \times S^1$, possibly with some balls removed, otherwise one obtains a π_1 contradiction. The contradiction depends on whether $|\pi_1(W)| < \infty$ or $W = S^2 \times S^1$−3-balls. Therefore every compression or 2-handle attachment was trivial. Since c_1 is a core of W if and only if c_1 is a core of Y_2, v) follows from iii). $\square$

If $\delta \subset \mathbf{H}^3$ is a geodesic, then $S^2_\infty - \partial\delta$ is naturally parametrized by $S^1 \times \mathbf{R}$, where each $x \times \mathbf{R}$ lies in the ideal boundary of a hyperbolic half-plane bounded by δ, and the $\mathbf{R}$ parameter is given by the hyperbolic nearest point projection of $\mathbf{B}^3 - \partial\delta$ to δ.

Definition 4.6. If $R \subset S^2_\infty - \partial\delta$, then define *$\delta$-visual angle*$(R) = \inf\{\theta_2 - \theta_1 \mod 2\pi \mid R \subset [\theta_1, \theta_2] \times \mathbf{R}\} \in [0, 2\pi]$. The possible choice of 0 or 2π is made in the obvious manner.

Lemma 4.7. *If $P \subset \mathbf{H}^3$ is a hyperbolic plane with ideal boundary λ and $P \cap \delta = \emptyset$, then δ-visual angle$(\lambda) = 2\sin^{-1}(1/\cosh(d))$, where d is the hyperbolic distance between δ and P.*

Proof. Let τ be the orthogonal geodesic segment between δ and $P, x = \delta \cap \tau$ and $y = P \cap \tau$. Let Q be the hyperbolic plane orthogonal to δ containing τ and $\sigma = Q \cap P$. The $\mathbf{H}^2$ visual angle of σ viewed from $x \in Q$ is equal to the δ-visual angle of λ. Now apply the formula (e.g. [F], p. 92) $\sin(\alpha)\cosh(d) = 1$ associated to the right angle triangle xyz, where z is an endpoint of $\sigma, d = d_\rho(x, y)$ and α is the angle zxy. $\square$

Corollary 4.8. *If $P \subset \mathbf{H}^3$ is a geodesic plane with ideal boundary λ and $d_\rho(\delta, P) = (\log(3))/2 = .549306\cdots$, then δ-visual angle$(\lambda) = 2\pi/3$.*

Mark Culler told me that $.549306\cdots = (\log(3))/2$.

Proofs of Theorems 0.9 and 0.10. By hypothesis there exists a $(\pi_1(N), \{\partial\delta_j\})$ non-coalescable insulator family $\{\lambda'_{jk}\}$, where δ is a closed geodesic in N and $\{\delta_j\} = q^{-1}(\delta)$. To prove Theorem 0.9 it suffices by Proposition 2.1 to find a simple closed curve γ in M, such that the $\mathbf{B}^3$-link $p^{-1}(\gamma) = \Gamma$ is isotopic rel S^2_∞ to the $\mathbf{B}^3$-link $q^{-1}(\delta) = \Delta$. In the context of Theorem 0.10 $M = N$ and f is a homeomorphism homotopic to id. To prove Theorem 0.10 it suffices by Proposition 2.11 to show that $f^{-1}(\delta)$ is isotopic to δ.

Our terminology will follow that of Notation 1.2. In particular G denotes the action of $\pi_1(N), \pi_1(M)$ on S^2_∞ as well as the action of $\pi_1(M)$ on $\mathbf{B}^3$.

Step 1. We can assume that, with only finitely many G-orbits of exceptions, each λ'_{ij} is the ideal boundary of the midplane (see Example 0.3) D_{ij} between δ_i and δ_j. If E_{ij} denotes the component of $S^2_\infty - \lambda'_{ij}$ which does not contain $\partial\delta_i$, then $S^2_\infty - \partial\delta_i = \bigcup_j E_{ij}$.

Proof of Step 1. By convexity and local finiteness there exists $\beta < \pi$ such that for each j, δ_i-visual angle$(\lambda'_{ij}) < \beta$. Let $\alpha = \min\{2\pi - 2\beta, 2\pi/3\}$ and let $d = \cosh^{-1}(1/\sin(\alpha/2))$. Define a new $(\pi_1(N), \{\partial\delta_j\})$ insulator family by the rule

$$\lambda_{ij} = \begin{cases} \lambda'_{ij}, & d_\rho(\delta_i, \delta_j) \leq d, \\ \partial D_{ij}, & \text{otherwise.} \end{cases}$$

Using Lemma 4.7 and the choice of d it follows that this family satisfies the no-trilinking condition of Definition 0.3 hence is noncoalescable. The first part of Step 1 is established by replacing $\{\lambda'_{ij}\}$ by $\{\lambda_{ij}\}$.

Let $x \in S^2_\infty - \partial\delta_i$. Let τ be a geodesic from x to δ_i, which is orthogonal to δ_i. An extended hyperbolic plane $P \subset \mathbf{B}^3$ disjoint from $\delta_i \cup x$ separates δ_i from x if and only if $P \cap \tau \neq \emptyset$. If $C = 2(\operatorname{diam}_\rho(X))$, then there exists a δ_j such that $d_\rho(\delta_j, \tau) < C$, but $d_\rho(\delta_j, \delta_i) > \max\,(10C, d)$. The midplane D_{ij} between δ_j and δ_i crosses τ and hence ∂D_{ij} separates x from $\partial\delta_i$. By definition $\partial D_{ij} = \lambda_{ij}$. □

From now on i will denote a fixed integer and $g \in G$ will denote a fixed generator of $\operatorname{Stab}(\partial\delta_i) = \langle g\rangle$. By the equivariance and local finiteness properties of insulator families, there exists an integer $n > 0$ such that $g^n \in \pi_1(X)$ and for all j and all $r \neq 0$, $g^{rn}(\lambda_{ij}) \cap \lambda_{ij} = \emptyset$. By Step 1, the compactness of $(S^2_\infty - \partial\delta_i)/\langle g^n\rangle$ and the G-equivariance of insulator families, there exists only finitely many outermost $\langle g^n\rangle$-orbits of $\{\lambda_{i1}, \lambda_{i2}, \dots\}$. λ_{ij} is outermost means that there exists no E_{ik} such that $E_{ij} \subset \mathring{E}_{ik}$. From now on n will be the integer determined as above.

Fix a Riemannian metric μ on M and let μ denote the induced metric on $\mathbf{H}^3$. For each j, k, let σ_{jk} be a lamination spanning λ_{jk} by μ-least area planes. The σ_{jk} should be chosen G-equivariantly, i.e. $h(\lambda_{jk}) = \lambda_{rs}$ implies $h(\sigma_{jk}) = \sigma_{rs}$ and $\sigma_{kj} = \sigma_{jk}$. Let H_{jk} denote the $\mathbf{H}^3$-complementary region of σ_{jk} which contains the ends of δ_j. Let $H_j = \bigcap_k H_{jk}$.

Reorder the δ_j's so that $\{\lambda_{i1}, \dots, \lambda_{im}\}$ denote representatives of the outermost $\langle g^n\rangle$-orbits of λ_{ij}'s. By Lemma 3.5 viii) it follows that $H_i = \bigcap_{j=1}^m \bigcap_{r\in\mathbf{Z}} g^{rn}(H_{ij})$. By equivariance, $h \in G, h(\delta_i) = \delta_j$ implies that $h(H_i) = H_j$.

Step 2. Establish the following properties of the H_j.

i) There exists $a > 0$ such that $H_i \subset N_\rho(a, \delta_i)$.

ii) $H_i \cap H_j \neq \emptyset$ if and only if $\delta_i = \delta_j$.

iii) If $h \in G$, then $h(H_i) \cap H_i \neq \emptyset$ if and only if $h = g^k$ for some k.

iv) If J is a component of H_i, then for each $x \in \bar{J}$, there exists a *standard* $D^2 \times I$ neighborhood of x as in the proof of Lemma 4.1 with the following additional properties. $J \cap D^2 \times I$ is the region which lies above the leaves $A_1, \dots, A_q$ and below the leaves $B_1, \dots, B_q$, notation as in the proof of Lemma 4.1 ii). Both $\bar{J} \cap D^2 \times I$ and $\bar{J} \cap (\partial D^2) \times I$ are connected. Finally if $C \neq D$ where $C, D \in \{A_1, \dots, A_q, B_1, \dots, B_q\}$ then ∂C is transverse to ∂D in $\partial D^2 \times I$.

v) If J is a component of H_i and J^* is the metric completion of J with the induced path metric, and $\Pi : J^* \to \mathbf{H}^3$ is the natural map, then Π is an embedding (rather than injective immersion) of J^* onto $\bar{J} \subset \mathbf{H}^3$.

vi) If J is a component of H_i and $h \in \langle g \rangle$, then $h(\bar{J}) \cap \bar{J} \in \{\bar{J}, \emptyset\}$.

vii) There exists N_1 such that every $x \in \mathbf{H}^3$ lies in the closure of at most one component of H_i having ρ-diameter $\geq N_1$. Each bounded component of H_i has ρ-diameter $\leq N_1$.

Proof of Step 2. i) Consider a fundamental domain F of $(\mathbf{B}^3 - \partial\delta_i)/\langle g^n \rangle$. $F \cap S^2_\infty$ is covered by a finite number of E_{ij}, hence a $\mathbf{B}^3$-neighborhood N of $F \cap S^2_\infty$ is covered by a finite number of $\hat{H}_{ij}$, where $\hat{H}_{ij}$ is the component of $\mathbf{B}^3 - N_\rho(e, C(\lambda_{ij}))$ which does not contain $\partial\delta_i$. By Proposition 3.9 $\hat{H}_{ij} \cap \sigma_{ij} = \emptyset$, so $\hat{H}_{ij} \cap H_i = \emptyset$. Choose a to be sufficiently large so that $F - N \subset N_\rho(a, \delta_i)$. Using the equivariance of insulator families and the fact that g^n is an isometry in both the ρ and μ metrics, i) follows.

ii) If $i \neq j$, then λ_{ij} separates $\partial\delta_i$ from $\partial\delta_j$. By definition of spanning lamination, the complementary regions of σ_{ij} which contain $\partial\delta_i$ and $\partial\delta_j$ are distinct and hence $H_i \cap H_j \subset H_{ij} \cap H_{ji} = \emptyset$.

iii) The σ_{ij} were chosen to be G-equivariant, hence $h(\partial\delta_i) = \partial\delta_j$ implies that $h(H_i) = H_j$. Conversely if $h \notin \langle g \rangle$, then $h(\partial\delta_i) \neq \partial\delta_i$, so by ii) $h(H_i) \cap H_i = \emptyset$.

iv) By Lemma 3.5 ix) we can assume that each σ_{ij} is nowhere dense. Let $x \in \mathbf{H}^3$. Here we are identifying, via Lemma 4.1, J^* with the region $\Pi(J^*) \subset \mathbf{H}^3$. Let $x \in \sigma_{ij} \cap \partial J^*$. Let $B \subset \mathbf{H}^3$ be a large ball such that $x \in B$, B is transverse to $\bigcup_j \sigma_{ij}$, and $N_\rho(a, \delta_i) \cap \partial B \cap N_\rho(e, C(\lambda_{ij})) = \emptyset$. Let L be the leaf of $\sigma_{ij} \mid B$ which contains x. By Lemma 3.5 v) L is compact and has a neighborhood of the form $L \times I$ such that $\sigma_{ij} \mid L \times I = L \times C$, with the product lamination, where C is compact. (It is this technical point which allows us to treat σ_{ij} as though it is a proper plane.) Thus if $y, z \subset H_i \cap L \times I$ and y, z lie in distinct complementary regions of $\sigma_{ij} \mid L \times I$, then y and z lie in distinct components of H_i for $(\partial L) \times I \cap H_i = \emptyset$. Therefore any sufficiently small $D^2 \times I$ neighborhood of x satisfies the second sentence of iv).

By making the $D^2 \times I$ sufficiently small, both $\bar{J} \cap D^2 \times I$ and $\bar{J} \cap \partial D^2 \times I$ are connected. See Remark 4.2. By making ∂D^2 transverse to the various $C \cap D$ arcs of intersection as well as choosing ∂D^2 to avoid the finitely many tangencies among the leaves $\{A_1, \dots, A_q, B_1, \dots, B_q\}$, $\partial D^2 \times I$ has the desired transversality property.

v) This follows by iv) and Lemma 4.1.

vi) Let $D^2 \times I$ be a small neighborhood of $x \in h(\bar{J}) \cap \bar{J}$ as in iv), i.e. such that each of $h(\bar{J}), \bar{J}$ intersects $D^2 \times I$ in a single component. By the proof of iv) $h(J) \cap D^2 \times I$ is not separated from $J \cap D^2 \times I$ by a leaf of some $\sigma_{ij} \mid D^2 \times I$.

Therefore, by Remark 4.2 i), if $x \in h(\bar{J}) \cap \bar{J} \neq \bar{J}$, then $\bar{J}, h(\bar{J})$ meet at a spike or saddle point created by the laminations $\sigma_{ij_s}, 1 \leq s \leq k \in \{2, 3\}$. If J_r is the component of $\mathbf{H}^3 - (\bigcup_{s=1}^k \sigma_{ij_s})$ containing $h^r(J)$, then for some finite t, $\bigcup_{r=-t}^t \Pi(J_r^*)$ is connected but not simply connected. A nontrivial cycle passing through x is obtained as follows. First chain together a path, through x and all the $\Pi(J_r^*)$, from $\Pi(J_{-t}^*)$ to $\Pi(J_t^*)$. This path (which has endpoints near $\partial\delta_i$) together with a path "near" S^2_∞ and disjoint from $\bigcup_{s=1}^k \lambda_{ij_s}$ yields the desired cycle, contradicting Remark 4.2 ii). This uses the fact that $\bigcup_{s=1}^k \lambda_{ij_s}$ does not separate $\partial\delta_i$ in S^2_∞.

vii) By equivariance, local finiteness and the last conclusion of Proposition 3.9, there exists an $N_0 > 0$ such that the image under orthogonal projection of any σ_{ij} into δ_i has ρ-diameter bounded above by N_0. As in vi), if there exist two components J_1, J_2 of H_i which limit on the same point x and are sufficiently large, e.g. ρ-diameter $> 2(a + N_0) = N_4$, one can find a cycle contradicting Remark 4.2 ii). In fact say J_1, J_2 are locally separated near x by $\bigcup_{s=1}^k \sigma_{ij_s}$ where $k \leq 3$. If x projects to $y \in \delta_i$, via orthogonal projection, then $\bigcup_{s=1}^k \sigma_{ij_s}$ projects into $B_\rho(N_0, y) \cap \delta_i$, for each of these k laminations have the point x in common. If for $r = 1, 2$, $\mathrm{diam}(J_r) > N_4$, then since $J_r \subset N_\rho(a, \delta_i)$ the orthogonal projection of J_r is not contained in $B_\rho(N_0, y) \cap \delta_i$. Therefore J_1, J_2 lie in the same component of $\mathbf{H}^3 - \bigcup_{s=1}^k \sigma_{ij_s}$ and one constructs the desired cycle.

By iv), any $x \in \mathbf{H}^3$ has a standard $D^2 \times I$ neighborhood whose boundary intersects at most finitely many components of H_i. This together with Step 2 i) implies that given $\epsilon > 0$, then modulo the action of $\langle g^n \rangle$ which acts isometrically in both the μ and ρ metrics, there are only finitely many components of H_i with ρ-diameter $> \epsilon$. Therefore there exists an N_2 such that the ρ-diameters of bounded H_i components are uniformly bounded by N_2. Finally take $N_1 = \max(N_2, N_4)$. □

Step 3. Let $P' : \mathbf{B}^3 - \partial\delta_i \to D^2 \times S^1$ be the quotient map under the action of $\langle g \rangle$. Then $P'(H_i)$ is the union of open balls and exactly one open solid torus, whose closures are respectively closed balls and one solid torus T. Therefore $H_i \subset \mathbf{H}^3$ is a union of uniformly bounded open balls and exactly one component $\tilde{V}_i$ whose $\mathbf{H}^3$-closure is a $D^2 \times \mathbf{R}$ whose ends limit on $\partial\delta_i$. Finally $p(\tilde{V}_i) = V$ is a $\overset{\circ}{D}{}^2 \times S^1$, where $p : \mathbf{H}^3 \to M$ is the universal covering map.

Proof of Step 3. Each component Z of $P'(H_i)$ has $\pi_1(Z) \in \{1, \mathbf{Z}\}$ since it is covered by a simply connected component of H_i, by Lemma 4.1, with covering translations contained in $\langle g \rangle$. $\bar{Z}$ is a compact manifold with boundary by Lemma 4.1 and v)–vii) of Step 2. Z is irreducible since it is covered by an irreducible manifold [MSY]. Therefore $\bar{Z}$ is a closed ball or solid torus.

We show that there exists some $\overset{\circ}{D}{}^2 \times S^1$ component of $P'(H_i)$. Parametrize $\mathbf{B}^3 - \partial\delta_i$ by $D^2 \times \mathbf{R}$ so that g acts by $(x, t) \to (x, t+1)$. If $P'(H_i)$ contains no $\overset{\circ}{D}{}^2 \times S^1$ component, then by Step 2 vii) the components of H_i have uniformly bounded ρ-diameter. Hence there exists an integer $N_3 > 0$ such that if $\tilde{Z}$ is a component of H_i and $\tilde{Z} \cap D^2 \times 0 \neq \emptyset$, then $\tilde{Z} \subset D^2 \times (-N_3, N_3)$. By Step 1 and Lemma 3.5 viii), $H_i \cap (D^2 \times [-N_3, N_3]) = Z_\alpha \cap (D^2 \times [-N_3, N_3])$ where $Z_\alpha = \bigcap_{j_k \in \alpha} H_{ij_k}$ for some finite set $\alpha = \{j_1, \ldots, j_r\}$. By Lemma 4.3 some component Z_β of Z_α contains an embedded path τ connecting the points of $\partial\delta_i$. This implies that some component of H_i nontrivially intersects $D^2 \times (-N_3)$ and $D^2 \times N_3$ and hence $D^2 \times 0$, which is a contradiction.

Suppose that $P'(H_i)$ had two solid tori T_1 and T_2. Their cores c_1, c_2 would lift to paths $\tilde{c}_1, \tilde{c}_2$ between the elements of $\partial\delta_i$ such that for some component S of $Z_\alpha \cap D^2 \times 0$, $\langle S, \tilde{c}_1 \rangle = 1$ and $S \cap \tilde{c}_2 = \emptyset$. Again $\langle , \rangle$ denotes the algebraic intersection number. Since the ends of $\tilde{c}_1$ and $\tilde{c}_2$ lie in a neighborhood of $\partial\delta_i$ contained in Z_α, the ends can be truncated and fused to create a closed curve τ in Z_α such that $\langle \tau, S \rangle = 1$, contradicting the simple connectivity of Z_α established in Lemma 4.1. □

Definition. The solid torus V constructed above is said to arise from the *insulator construction.*

Let $P : \mathbf{B}^3 - \partial\delta_i \to Y = D^2 \times S^1$ be the quotient map given by the action of $\langle g^n \rangle$. Let γ be a core of $V = p(\tilde{V}_i)$. (Recall Notation 1.2.) For each j let γ_j denote the lift of γ to $\tilde{V}_j$ extended to be a properly embedded arc in $\mathbf{B}^3$.

Step 4. The isotopy class of γ and hence the $\mathbf{B}^3$-link $\{\gamma_j\} = \Gamma$ is independent of the choice of $\{\sigma_{ij}\}$.

Proof of Step 4. Let $\{\sigma'_{ij}\}$ be another collection of laminations spanning $\{\lambda_{ij}\}$, with associated regions $\{H'_{ij}\}$, $\{H'_i\}$. Let $H''_{ij} = H_{ij} \cap H'_{ij}$ and $H''_i = \bigcap_j H''_{ij}$. The arguments of Steps 1–3 show that H_i, H'_i, H''_i each contain a unique unbounded $\mathring{D}^2 \times \mathbf{R}$ respectively called $\tilde{V}_i, \tilde{V}'_i, \tilde{V}''_i$, which project respectively to open solid tori V, V', V'' in M such that $V'' \subset V$ and $V'' \subset V'$.

To prove Step 4 we will pass to the n-fold cyclic covering space $Y_0 \subset Y$ of V and there find $Y_m \subset Y_{m-1} \subset \cdots \subset Y_0$ which are respectively open solid tori with a common core. Y_m will be the n-fold cyclic covering space of V''. Since a core c of V'' lifts to a core of Y_m, which is a core of Y_0, this curve c when viewed in V is therefore a core by Lemma 4.5 ii). A similar argument shows that any core of V'' is a core of V'. Therefore V and V' have common cores. Since cores are unique up to isotopy (Lemma 4.5 i)), Step 4 is established.

Recall that $\lambda_{i1}, \ldots, \lambda_{im}$ are representatives of the distinct outermost $\langle g^n \rangle$-orbits of $\{\lambda_{ij}\}$ and that n was chosen so that $r \neq 0$ implies that for all k, $g^{rn}(\lambda_{ik}) \cap \lambda_{ik} = \emptyset$. For $1 \leq j \leq m$ let $\kappa_j = \bigcup_{r=-\infty}^{\infty} g^{rn}(\sigma'_{ij})$ and $H_i^{\kappa_j} = \bigcap_{r=-\infty}^{\infty} g^{rn}(H'_{ij})$. Again by Lemma 3.5 viii) $H'_i = \bigcap_{j=1}^{m} H_i^{\kappa_j}$. For $1 \leq t \leq m$, let $W_t = \bigcap_{j=1}^{t} H_i^{\kappa_j} \cap H_i$, so $W_m = H''_i$. Define $W_0 = H_i$. As in the proof of Steps 1–3, each W_j contains a unique $\mathring{D}^2 \times \mathbf{R}$ component $\tilde{V}_i^j$ which projects via P to a $\mathring{D}^2 \times S^1$ called Y_j. Step 4 will follow from the following claim, for when combined with Lemma 4.5 iii) it asserts that for all j, any core of Y_{j+1} is a core of Y_j, and hence any core of Y_m is a core of Y_0.

Claim. Each leaf L_α of $P(\kappa_{j+1}) \mid Y_j$ is a properly embedded separating disc, exactly one complementary component being a $\mathring{B}^3$. The collection $\{L_\beta\}$ of outermost such discs is locally finite in Y_j. (Outermost means, maximal in the partial order defined by inclusion of complementary $\mathring{B}^3$'s.) Finally $Y_{j+1} = \bigcap C_\beta$, where C_β is the non-simply-connected component of $Y_j - D_\beta$ and D_β is an outermost disc.

Proof of the Claim. By Lemma 3.5 vi) and Step 2 i) each leaf L of $\hat{\kappa}_{j+1} = \kappa_{j+1} \mid \tilde{V}_i^j$ is a properly embedded planar surface. By Lemma 3.5 vii) L must be a disc. It follows from the choice of n that for each $j, P \mid \lambda_{ij} \to P(\lambda_{ij})$ and hence $P \mid \sigma_{ij} \to P(\sigma_{ij})$ were embeddings. Arguing as in the proofs of Steps 1)–3), it follows that the $\mathbf{H}^3$-closures of the components of $\tilde{V}_i^j - L$ consist of a B^3 called B_L and a properly embedded $D^2 \times R$. Finally by the choice of n, the complementary regions of $P(L) \subset Y_j$ consist of an $\mathring{D}^2 \times S^1$ and a $\mathring{B}^3 = P(\mathring{B}_L) = P(\bigcup_{r=-\infty}^{\infty}(g^{rn}(\mathring{B}_L)))$.

By Lemma 3.5 v) $\tilde{V}_i^j$ is covered by a locally finite $\langle g^n \rangle$-equivariant collection of lamination charts $\{U_\alpha\}$ such that each leaf L of $\hat{\kappa}_{j+1}$ passes through U_α at most once. To see this, apply Lemma 3.5 v) where W is a large ball transverse to κ_{j+1} such that $\partial W \cap \kappa_{j+1} \cap \tilde{V}_i^j = \emptyset$ and argue as in the proof of Step 2 iv). Since at most 2 outermost leaves can pass through a given U_α , local finiteness of outermost leaves is established.

Again by the choice of n, if L and $\hat{L}$ are distinct leaves of $\hat{\kappa}_{j+1}$, then they bound balls $B_{\hat{L}}, B_L \subset \tilde{V}_i^j$ which are either disjoint or nested. Thus $\tilde{V}_i^{j+1} = \bigcap(\tilde{V}_i^j - B_L)$, the intersection taken with respect to outermost leaves of $\hat{\kappa}_{j+1}$. By projecting into $P(\mathbf{B}^3 - \partial\delta_i)$ the last assertion follows. □

Remark. An isotopy of γ to a core γ' of V' can be expressed as the composition of two isotopies, the first (resp. second) of which is supported in V (resp. V').

Step 5. The isotopy classes of $\gamma \subset M$ and the $\mathbf{B}^3$-link Γ are independent of the choice of metric on M.

Proof. Since the space of Riemannian metrics is path connected, it suffices to show that if μ_s is a $[0,1]$-family of smooth metrics and γ_s is a core curve arising from the insulator construction with respect to μ_s, then the isotopy class of $\gamma_s \subset M$ is locally constant as a function of s. (For fixed s the isotopy class of γ_s is well defined by Step 4.) Suppose that there exists a sequence $t_k \to t$ such that for each k, γ_{t_k} is not isotopic to γ_t. Let σ_{ij}^k be a lamination spanning λ_{ij} by μ_{t_k}-least area planes. By Proposition 3.10 after passing to a subsequence we can assume that for $j = 1, \dots, m$, $\sigma_{ij}^k \to \sigma_{ij}$, where σ_{ij}^k (resp. σ_{ij}) is a lamination spanning λ_{ij} by μ_{t_k}-least area (resp. μ_t-least area) planes. Using $\{\sigma_{ij}^k\}$ (resp. $\{\sigma_{ij}\}$) the insulator construction associates to each t_k (resp. t) a $\tilde{V}_i^k = \mathring{D}^2 \times \mathbf{R}$ (resp. $\tilde{V}_i$) which covers $V^k \subset M$ (resp. V), where V^k, V are $\mathring{D}^2 \times S^1$'s. Let T^k (resp. T) denote the $D^2 \times S^1$ whose interior is $P(\tilde{V}_i^k)$ (resp. $P(\tilde{V}_i)$).

Let $\hat{\gamma}$ be the lift of γ_t to Y. To complete the proof of Step 5 it suffices to show that for k sufficiently large, $\hat{\gamma}$ is a core of T^k. For by Lemma 4.5 ii) for k sufficiently large γ_t would be a core of V^k. By Lemma 4.5 i) cores are unique up to isotopy, so we conclude that γ_{t_k} is isotopic to γ_t, a contradiction.

By Lemma 4.5 v) it suffices to find a regular neighborhood $N(T)$ of T, and a disc $E \subset \partial N(T)$ such that for k sufficiently large $\hat{\gamma} \subset T^k$ and $\partial N(T) \cap T^k \subset E$. Reconcile our notation here with that of Lemma 4.5 by letting $c_1 = \hat{\gamma}, Y_2 = T^k, Y_3 = N(T), Y = Y$ and $E = E$.

Let γ_i be the lift of $\hat{\gamma}$ to $\tilde{V}_i$. For fixed j and for k sufficiently large $\gamma_i \cap \sigma_{ij}^k = \emptyset$; else by Definition 3.2 $\gamma_i \cap \sigma_{ij} \neq \emptyset$, a contradiction. The finiteness of m (recall Step 2) and the $\langle g^n \rangle$-invariance of γ_i imply that for k sufficiently large $\gamma_i \subset \tilde{V}_i^k$ and hence $\hat{\gamma} \subset \mathring{T}^k$.

We now find the desired $N(T)$ and E. Let $N_1(T) \subset Y$ be a regular neighborhood of T transverse to $P(\bigcup_j \sigma_{ij})$.

Claim 1. i) There exist properly embedded compact separating surfaces $S_1, \dots, S_m \subset N_1(T)$ such that for $p \leq m$ there exists a closed complementary region $J_p \subset N_1(T)$ of S_p such that $T \subset \bigcap_{p=1}^m J_p = J$. Each S_p is a (possibly disconnected) surface lying in leaves of $P(\sigma_{ip})$ and $\mathring{T} \cap P(\sigma_{ip}) = \emptyset$.

ii) Fix $\epsilon > 0$. For k sufficiently large, there exist properly embedded compact separating surfaces $S_1^k, \dots, S_m^k \subset N_1(T)$ such that for $p \leq m$ there exists a closed complementary region $J_p^k \subset N_1(T)$ of S_p^k such that $T^k \subset \bigcap_{p=1}^m J_p^k = J^k$. Each S_p^k is a (possibly disconnected) surface lying in leaves of $P(\sigma_{ip}^k)$. Finally for every j, S_j^k

is ϵ-close to S_j in the C^2 topology and J^k is ϵ-close to J with respect to Hausdorff distance.

Proof of Claim 1. i) Fix $j \in \{1,\dots,m\}$. Let $\tilde{N}_1(T)$ be the lift of $N_1(T)$ to $\mathbf{H}^3$ which contains $\tilde{V}_i$. Using Lemma 3.5 v), the closure of the component of $H_{ij} \cap \tilde{N}_1(T)$ which contains $\tilde{V}_i$ intersects $\sigma_{ij} \cap \tilde{N}_1(T)$ in a compact surface L_j lying in leaves of σ_{ij}. Part i) follows by taking $S_j = P(L_j)$ and J_j the closed complementary region containing $\hat{\gamma}$.

ii) Since $\sigma_{ij}^k \to \sigma_{ij}$ and $\tilde{N}_1(T) \cap \sigma_{ij}$ is compact, it follows that for k sufficiently large $\sigma_{ij}^k \mid \tilde{N}_1(T)$ very closely approximates $\sigma_{ij} \mid \tilde{N}_1(T)$. In particular $\sigma_{ij} \mid \tilde{N}_1(T)$ consists of finitely many families of parallel compact surfaces, so $\sigma_{ij}^k \mid \tilde{N}_1(T)$ consists of parallel families which approximate in a bijective fashion the σ_{ij} families. In fact by Definition 2.2 for k sufficiently large, each $x \in \sigma_{ij}^k \cap \tilde{N}_1(T)$ must lie very close to a σ_{ij} family. Conversely a σ_{ij} family is a product lamination of the form $F \times C \subset F \times \mathring{I}$, C compact, and the proof of Lemma 3.3 shows that for k sufficiently large $\sigma_{ij}^k \cap F \times I \subset F \times \mathring{I}$ and each connected leaf is a compact surface transverse to the I-fibres. Therefore $\sigma_{ij}^k \mid F \times I$ is isotopic to a product lamination of the form $F \times C', C' \neq \emptyset$. Hence for k sufficiently large, the component of $H_{ij}^k \cap \tilde{N}_1(T)$ which contains $\tilde{V}_i^k$, intersects $\sigma_{ij}^k \cap \tilde{N}_1(T)$ in a compact surface L_j^k which is ϵ-close to L_j. Part ii) follows by taking $S_j^k = P(L_j^k)$ and J_j^k the closed complementary region containing T^k. By construction for k sufficiently large, then for every j, J_j^k is ϵ-close to J_j and J^k is ϵ-close to J. □

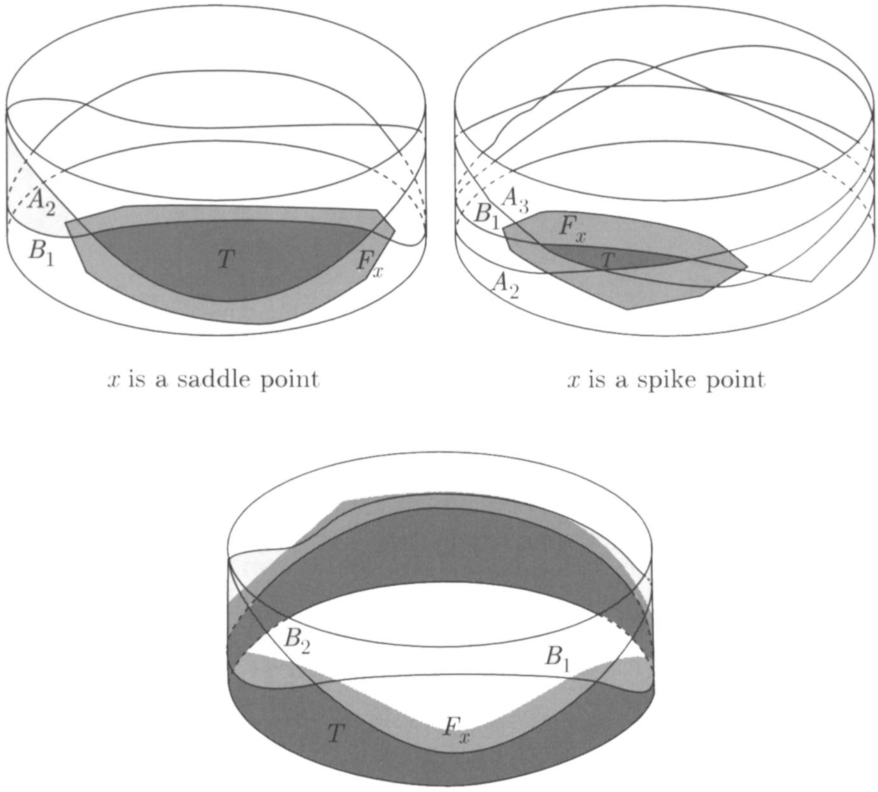
Claim 2. There exist an $\epsilon > 0$, a neighborhood $N(T)$ of T and an embedded disc $E \subset \partial N(T)$ such that if $x \in \partial N(T) - E$, then there exists j such that $d_\rho(x, J_j) > \epsilon$.

Proof of Claim 2. Let $S = \bigcup_{j=1}^m S_j$. Each $x \in \partial T$ has a $D^2 \times I \subset N_1(T)$ neighborhood satisfying the conclusion of Step 2 iv). In our context $S_j \cap D^2 \times I = A_j \cup B_j$, where at most one of A_j, B_j is nonempty. The last conclusion of Step 2 iv) implies that associated to $D^2 \times I \cap \partial T$ is a small annulus $F_x \subset (\partial D^2) \times I$ such that if $F_x = S^1 \times I$, then $F_x \cap \partial T = S^1 \times 0$, and if $y \in S^1 \times t, t > 0$, then there exists j such that $d_\rho(y, J_j) > 0$. See Figure 4.2. From a finite collection of curves of the form $((\partial D^2) \times I) \cap \partial T$, we obtain a 1-complex $C \subset \partial T$ whose complement is a union of $\mathring{D}^2$'s. By the usual transversality arguments there exists a regular neighborhood $N(T) \subset N_1(T)$, a regular neighborhood $N(C) \subset \partial T$ and an $\epsilon > 0$ such that if $y \in N(C)$, then there exists j such that $d_\rho(y \times 1, J_j) > \epsilon$. Here $N(T) - \mathring{T}$ (resp. ∂T) is identified with $\partial T \times I$ (resp. $\partial T \times 0$). Finally let $E \subset \partial N(T)$ be a 2-disc such that $\partial N(T) - (\mathring{N(C)} \times 1) \subset E$. □

To complete the proof of Step 5, let $N(T)$ and E be as in Claim 2. By Claim 1, if k sufficiently large, then for each $y \in \partial N(T) - \mathring{E}$ there exists j such that $d_\rho(y, J_j^k) > \epsilon/2$. Therefore $(\partial N(T) - \mathring{E}) \cap T^k \subset (\partial N(T) - \mathring{E}) \cap J^k = \emptyset$. □

Remark. i) Near saddle tangencies T^k may spill way out of T. In Figure 4.3 $\tilde{V}_i^k$ (resp. $\tilde{V}_i$) is locally defined by two leaves, whose $\tilde{V}_i^k$ (resp $\tilde{V}_i$) sides are indicated by arrows. T^k may also spill out near spikes of T. Compare Figure 4.1.

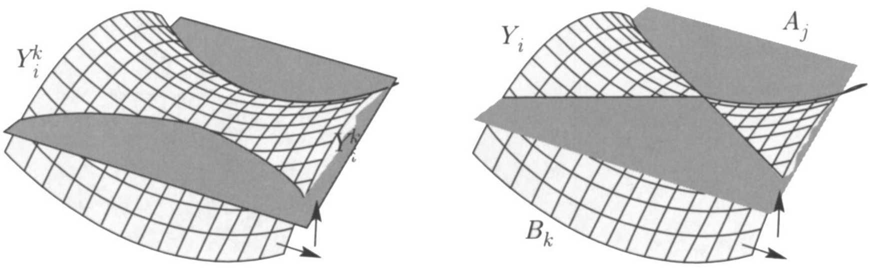
ii) The isotopy from γ_t to γ_{t_k} was supported in V^k.



x is a saddle point x is a spike point

x is a generic point

FIGURE 4.2



(a) (b)

FIGURE 4.3

iii) One could alternatively show that $\hat{\gamma}$ is a core of T^k by first showing that $\hat{\gamma}$ is a core of Y and then invoking Lemma 4.5 iii) with respect to the inclusion $\hat{\gamma} \subset T^k \subset Y$.

Step 6. Steps 1–5 applied to the $(\pi_1(X), \{\partial\delta_i\})$ insulator family $\{\lambda_{ij}\}$ yields the isotopy class of the link $\mathbf{\Gamma}_1 = p_1^{-1}(\gamma) \subset X$ and the isotopy class of the $\mathbf{B}^3$-link

Γ. These classes are independent of the metric on X and the choice of spanning laminations. □

Remark. We leave it to the reader to check that the analogues of Steps 1–5 work in the context of the $(\pi_1(X), \{\partial\delta_i\})$ insulator family $\{\lambda_{ij}\}$. In particular Steps 1–3 show that a metric μ on X and a collection $\{\sigma_{ij}^{\mu_\alpha}\}$ of μ-least area laminations spanning $\{\lambda_{ij}\}$ give rise to a finite set $\mathcal{V}_{\mu_\alpha}$ of pairwise disjoint $\mathring{D}^2 \times S^1$'s in X. Each isotopy required in the proofs of Steps 4–5 can be expressed as the composition of isotopies each of which is supported in some $\mathcal{V}_{\mu_\alpha}$. Each component of $\mathcal{V}_{\mu_\alpha}$ supports the isotopy restricted to exactly one component of the link. See the remarks following Steps 4–5.

Step 7. (Proof of Theorem 0.9.) Using the hyperbolic metric on $\mathbf{H}^3$, the $(\pi_1(X), \{\partial\delta_i\})$ insulator family $\{\lambda_{ij}\}$ yields the $\mathbf{B}^3$-link Δ. The $\mathbf{B}^3$-link Γ is isotopic to the $\mathbf{B}^3$-link Δ.

Proof of Step 7. By the convexity property of Definition 0.4 there exists, for each j, a hyperbolic half-plane P_{ij} separating $\partial\delta_i$ from λ_{ij}. Thus P_{ij} separates δ_i from any ρ-least area lamination σ_{ij} which spans λ_{ij}. Applying the insulator construction to the $(\pi_1(X), \{\partial\delta_i\})$ insulator family $\{\lambda_{ij}\}$ using the hyperbolic metric on $\mathbf{H}^3$, we get for each j: $\delta_j \subset \tilde{V}_j$ and $\Delta_1 = \pi(\Delta) = q_1^{-1}(\delta)$ are cores of the tori $\pi(\{\tilde{V}_i\})$. To see that $\pi(\delta_j)$ is a core of $\pi(\tilde{V}_j)$, consider $P^j : (\mathbf{H}^3 - \partial\delta_j) \to (\mathbf{H}^3 - \partial\delta_j)/\langle g_j\rangle = Y^j = \mathring{D}^2 \times S^1$, where g_j generates $\mathrm{Stab}_{\pi_1(X)}(\delta_j)$. Apply Lemma 4.5 iii) to $P(\delta_j) \subset P^j(\tilde{V}_j) \subset Y^j$. By Step 6, Δ_1 is isotopic to $\Gamma_1 = p_1^{-1}(\gamma)$. Lift this isotopy to $\mathbf{H}^3$ to complete the proof of Theorem 0.9. □

Step 8. (Proof of Theorem 0.10.) The curve $f^{-1}(\delta)$ is isotopic to δ.

Proof of Step 8. In the context of Theorem 0.10 $M = N$ and f is a diffeomorphism (after a preliminary isotopy) homotopic to id. M has two Riemannian metrics, the given hyperbolic metric ρ and the pullback metric $f^*(\rho)$. The construction of

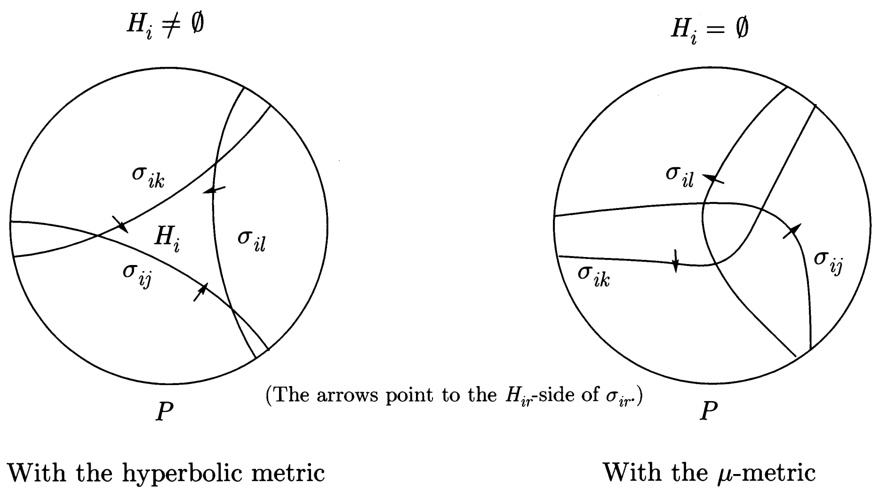


FIGURE 4.4

Steps 1–5 using the metric ρ yields the geodesic δ and using the metric $f^*(\rho)$ yields the curve $f^{-1}(\delta)$. Since the isotopy class is independent of metric, it follows that $f^{-1}(\delta)$ is isotopic to δ. $\square$

Remark 4.9. (Why a coalescable insulator is bad.) It is possible that the H_i resulting from the construction applied to a coalescable insulator family would contain no unbounded $\mathring{D}^2 \times \mathbf{R}$ component, i.e. Step 3 fails. Perhaps even $H_i = \emptyset$. It is conceivable that, as in [GS], using the wrong metric some ρ-totally geodesic plane P transverse to δ_i may be disjoint from H_i. See Figure 4.4.

5. Applications and concluding remarks

Mostow's Rigidity Theorem [Mo]. *If $f : M \to N$ is a homotopy equivalence between closed hyperbolic manifolds of dimension > 2, then f is homotopic to an isometry.*

Remark 5.1. (What Mostow does not say.) If ρ_0 is a hyperbolic metric on N, then associated to a nontrivial element α of $\pi_1(N)$, there exists a unique geodesic $\delta \subset N$ which is freely homotopic to α. Mostow does not rule out the possibility that with respect to a different hyperbolic metric ρ_1, the geodesic δ' associated to α would lie in a different isotopy class than δ. What Mostow does assert is that there exists a diffeomorphism $f : N \to N$, *homotopic* to id, such that $f(\delta) = \delta'$.

If N satisfies the insulator condition, then this diffeomorphism is *isotopic* to id, by Theorem 0.10, so we obtain

Theorem 5.2. *Let N be a closed hyperbolic 3-manifold satisfying the insulator condition. If ρ_0 and ρ_1 are hyperbolic metrics on N, then there exists a diffeomorphism $f : N \to N$ isotopic to* id *such that $f^*(\rho_1) = \rho_0$. In particular the space of hyperbolic metrics on N is path connected.* $\square$

Remark. So for manifolds satisfying the insulator condition, hyperbolic structures are unique up to isotopy.

Corollary 5.3. *Let N be a closed orientable 3-manifold satisfying the insulator condition. Then* $\mathrm{Homeo}(N)/\mathrm{Homeo}_0(N) = \mathrm{Out}(\pi_1(N)) = \mathrm{Isom}(N)$. ($\mathrm{Homeo}_0(N)$ *is the group of homeomorphisms isotopic to* id.)

Proof. The second equality folows from Mostow. Let $\mathcal{H} : \mathrm{Homeo}(M)/\mathrm{Homeo}_0(N) \to \mathrm{Out}\ \pi_1(N)$ be the map induced by the action of $\pi_1(N)$. Since N is a $k(\pi, 1)$, it follows that $\mathcal{H}([h]) = \mathcal{H}([g])$ if and only if h is homotopic to g. By Mostow $\mathcal{H}$ is surjective and by Theorem 0.10 $\mathcal{H}$ is injective. $\mathrm{Homeo}(N)/\mathrm{Homeo}_0(N)$ is often called the *mapping class group* of N. $\square$

Definition 5.4. i) Call a finite set of pairwise disjoint simple closed curves in N a *homotopy essential link* if each component is homotopically nontrivial in N and no two components lie in the same $\mathbf{Z}$ subgroup of $\pi_1(N)$.

ii) If $q : \mathbf{H}^3 \to N$ is the universal covering projection of a hyperbolic 3-manifold N, then q^{-1} induces the map Q : {isotopy classes of homotopy essential links in N} $\to${isotopy classes of $\mathbf{B}^3$-links}. If $q_X : X \to N$ is a finite covering map, then q_X^{-1} induces the map Q_X : {isotopy classes of homotopy essential links in N} $\to${isotopy classes of homotopy essential links in X}.

Recall that isotopies of $\mathbf{B}^3$-links are required to fix S^2_∞ pointwise.

Conjecture 5.5A. *Q is injective.*

Conjecture 5.5B. *Q_X is injective.*

Corollary 5.6. *Let δ be a simple closed geodesic and δ' a simple closed curve in N such that $Q(\delta) = Q(\delta')$. Then δ is isotopic to δ' if N satisfies the insulator condition.*

Proof. Apply Proposition 2.1 to find a homeomorphism $f : N \to N$ such that $f(\delta) = \delta'$. By construction $\tilde{f}$ fixes S^2_∞ pointwise, so f is homotopic to id. Now apply Theorem 0.10. □

Remark 5.7. By Corollary 5.6 if N satisfies the insulator condition, then Conjecture 5.5A is true for geodesics. Also if X satisfies the insulator condition, then Conjecture 5.5B implies Conjecture 5.5A for geodesics.

Definition 5.8. If δ is a closed geodesic in the hyperbolic 3-manifold N, then the *tube radius* of δ = Sup {radii of embedded hyperbolic tubes about δ} = $1/2 \min\{d(\delta_i, \delta_j) \mid \delta_i, \delta_j$ are distinct preimages of δ in $\mathbf{H}^3\}$.

Lemma 5.9. *If the hyperbolic manifold N has a geodesic δ with tube radius $> (\log 3)/2$, then the Dirichlet insulator family associated to δ is noncoalescable. In particular N satisfies the insulator condition.*

Proof. By Corollary 4.8, if tube radius$(\delta) > (\log 3)/2$ and δ_i is a lift of δ, then δ_i-visual angle$(\lambda_{ij}) < 2\pi/3$ where λ_{ij} is the ideal boundary of the Dirichlet midplane between δ_i and δ_j. Therefore for no j, k, l does $\lambda_{ij} \cup \lambda_{ik} \cup \lambda_{il}$ separate $\partial\delta_i$. □

Corollary 5.10. *Each hyperbolic 3-manifold N is covered by a hyperbolic 3-manifold X satisfying the insulator condition.*

Proof. By [G3] for every $\epsilon > 0$, there exists a finite regular cover X of N such that tube radius $(\delta) > \epsilon$, where δ is a shortest geodesic in X. □

Proof of Theorem 0.1. Combine Theorems 0.9, 0.10, and 5.2 with Lemma 5.9. □

Remark 5.11. i) Via a technique called *fudging*, the inequality given in Lemma 5.9 can be improved at least to an equality. The idea is that a small perturbation of a *just barely* coalescable Dirichlet insulator would create a noncoalescable insulator family.

ii) An application of the hyperbolic law of cosines shows that if the shortest geodesic δ in N has length $L \geq 1.353$, then tube radius$(\delta) > (\log 3)/2$. One finds a right triangle as in Figure 5.1.

iii) Applying the Meyerhoff tube radius formula, i.e. Corollary of [Me, §3] with $n \leq 8$, we conclude that if N has a geodesic δ of length $\leq .0978$, then tube radius$(\delta) > (\log 3)/2$. Combined with the work of Jorgenson [Gr], this shows that for $n > 0$ there are at most finitely many hyperbolic 3-manifolds of volume $< n$ which can fail to satisfy the insulator condition.

Corollary 5.12. *For any $n > 0$, all but at most finitely many closed hyperbolic 3-manifolds of volume $< n$ are both geometrically and topologically rigid.* □

Remark 5.13. A 3-manifold N is *topologically rigid* means that any homotopy equivalence between N and an irreducible 3-manifold is homotopic to a homeomorphism. The hyperbolic 3-manifold N is *geometrically rigid* means that its hyperbolic metric is unique up to isotopy.

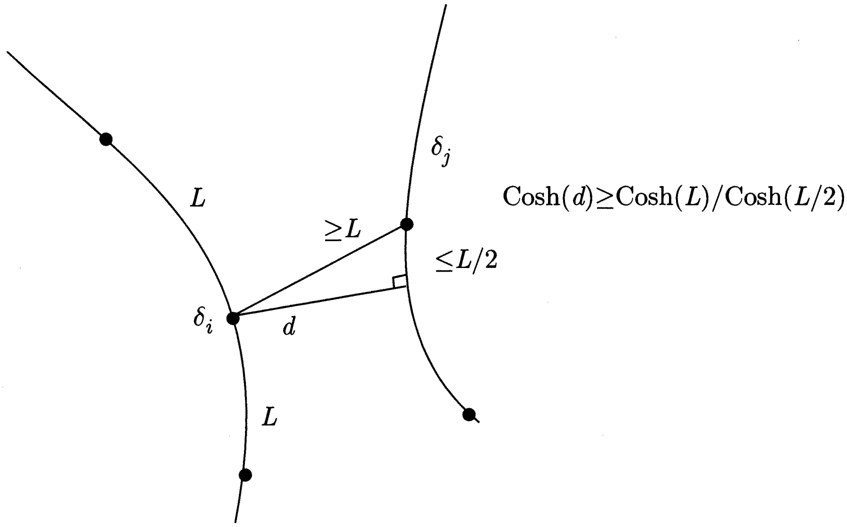


FIGURE 5.1

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