Cameron Gordon and knot concordance

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The *knot concordance group* is

$$\mathcal{C} = \frac{\{\text{oriented knots } S^1 \subseteq S^3\}}{\sim}$$

where $K \sim J$ if and only if there is an embedding C making

$$\begin{array}{ccc} S^1 \times \{0,1\} & \xrightarrow{K \cup J} & S^3 \times \{0,1\} \\ \downarrow & & \downarrow \\ S^1 \times I & \xrightarrow{C} & S^3 \times I \end{array}$$

commutes. If $K \sim U$ then K is *slice*.

Here C could be smooth or locally flat – the difference will not matter until slide 35.

The *algebraic concordance group* is

$$\mathcal{AC} = \frac{\{\text{square matrices } V \text{ over } \mathbb{Z} \mid \det(V - V^T) = 1\}}{\sim}$$

where $V \sim W$ if and only if $V \oplus -W$ is metabolic.

J. Levine (1969) defined a group homomorphism

$$\phi \colon \mathcal{C} \to \mathcal{AC}, \quad [K] \mapsto [V_F],$$

where V_F is a Seifert matrix of a Seifert surface F for K.

Levine and Stoltzfus showed:

$$\mathcal{AC} \cong \mathbb{Z}^\infty \oplus (\mathbb{Z}/2)^\infty \oplus (\mathbb{Z}/4)^\infty.$$

J. Levine defined a group homomorphism

$$\phi\colon \mathcal{C} \to \mathcal{AC}, \quad [K] \mapsto [V_F].$$

K is *algebraically slice* if and only if $\phi([K]) = 0$.

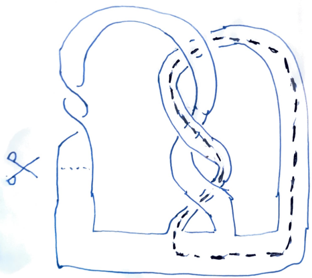
Theorem (Levine 1969)

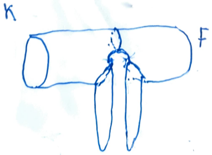
For $(2n-1)$-knots with $n \geq 2$, K is algebraically slice $\Leftrightarrow$ K is slice.

Theorem (Casson-Gordon, $n = 1$, 1978)

There is a non-slice, algebraically slice knot.

This is a slice knot.

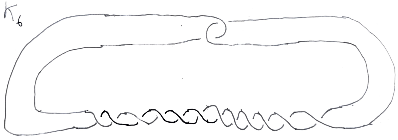




If K has a genus one Seifert surface F with a simple closed curve $\gamma \subseteq F$ representing a metaboliser for the Seifert form, i.e. $\ell\mathsf{k}(\gamma, \gamma^+) = 0$, such that γ is itself a slice knot, then K is slice.

If K is algebraically slice but there is no such γ, can we prove that K is not slice?

Let K_n be the twist knot with n twists.



Seifert form:

$$V_F = \begin{bmatrix} n & 0 \\ 1 & -1 \end{bmatrix}$$

K_n is algebraically slice if and only if $n = t(t-1)$ for some $t \in \mathbb{Z}$.

The classes $[1, t]$ and $[1, 1 - t]$ generate metabolisers, represented by simple closed curves with self linking numbers:

$$\begin{bmatrix} 1 & t \end{bmatrix} \begin{bmatrix} t(t-1) & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ t \end{bmatrix} = 0$$

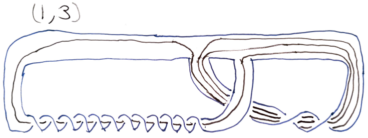
and

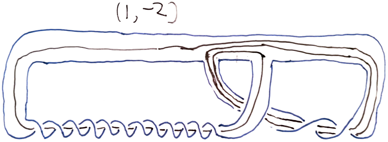
$$\begin{bmatrix} 1 & 1-t \end{bmatrix} \begin{bmatrix} t(t-1) & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1-t \end{bmatrix} = 0.$$

For K_6, the metabolising curves

$$[1, t] = [1, 3] \text{ and } [1, 1 - t] = [1, -2]$$

are trefoils $T_{2,3}$ (non-slice).





Recall K_n is algebraically slice if and only if $n = t(t-1)$ for some $t \in \mathbb{Z}$.

Theorem (Casson-Gordon 1978)

K_n is slice if and only if $n = 0, 2$.

Theorem (Jiang 1981, Se-Goo Kim 2005)

Algebraically slice, non-slice twist knots are linearly independent in $\mathcal{C}$.

I will outline the proof that $K_{t(t-1)}$ is not slice for $t \geq 3$, i.e. for $n = t(t-1) \neq 0, 2$.

Since $K_{t(t-1)}$ is algebraically slice, we need to use *higher-order obstructions*.

This will take the form of the Witt-group valued Casson-Gordon slice obstruction

$$\tau(K, \chi) \in L_0(\mathbb{Q}(\zeta_m)(t)) \otimes \mathbb{Q},$$

depending on an auxiliary χ.

Here $L_0(\mathbb{Q}(\zeta_m)(t))$ is stable isomorphism classes of sesquilinear, Hermitian, nonsingular forms over the field of fractions $\mathbb{Q}(\zeta_m)(t)$.

The zero-framed surgery $S^3_0(K)$ on K is an important 3-manifold in knot concordance.

Theorem (Freedman-Quinn 1982)

A knot K is topologically slice if and only if the zero-framed surgery $S^3_0(K)$ bounds a 4-manifold V such that $H_*(V) \cong H_*(S^1)$ and $\pi_1(V)$ is normally generated by a meridian of K.

For $\ell \in \mathbb{N} \cup \{\infty\}$, let M_ℓ be the ℓ-fold cyclic cover of $S^3_0(K)$.

Note that $TH_1(M_{p^r}) \cong H_1(\Sigma_{p^r}(K))$.

Theorem (Casson-Gordon)

If K is slice, then for any prime p and $r \geq 1$, there is a subgroup P of $TH_1(M_{p^r})$ such that $P = P^\perp$ with respect to the linking form

$$TH_1(M_{p^r}) \times TH_1(M_{p^r}) \to \mathbb{Q}/\mathbb{Z},$$

such that for every $\chi \colon TH_1(M_{p^r}) \to \mathbb{C}^*$ of prime power order $m = q^k$ with $\chi|_P = 0$, we have

$$\tau(K, \chi) = 0.$$

Next I will define $\tau(K,\chi)$ and sketch the proof of this obstruction theorem.

Consider

$$\chi\colon TH_1(M_{p^r}) \to \mathbb{C}^*$$

with image of order m.

Obtain

$$\pi_1(M_{p^r}) \to H_1(M_{p^r}) \xrightarrow{\cong} TH_1(M_{p^r}) \oplus \mathbb{Z} \xrightarrow{(\chi,\mathrm{Id})} \mathbb{Z}/m \oplus \mathbb{Z}.$$

Since $\Omega_3(B(\mathbb{Z}/m \times \mathbb{Z})) \otimes \mathbb{Q} = 0$, there exists $s \in \mathbb{N}$ and a 4-manifold V with covering spaces as shown:

$$s \left(\begin{array}{ccc} \widetilde{M}_\infty & \longrightarrow & \widetilde{M}_{p^r} \\ \downarrow & & \downarrow \\ M_\infty & \longrightarrow & M_{p^r} \end{array} \right) = \partial \left(\begin{array}{ccc} \widetilde{V}_\infty & \longrightarrow & \widetilde{V}_{p^r} \\ \downarrow & & \downarrow \\ V_\infty & \longrightarrow & V_{p^r} \end{array} \right).$$

Here for any X in the bottom row, $\widetilde{X} \to X$ denotes the m-fold χ-cover and X_∞ is the infinite cyclic cover.

Want to consider a twisted intersection form of V_{p^r}. Consider

$$H_*(\widetilde{V}_\infty; \mathbb{Z}) \otimes_{\mathbb{Z}[\mathbb{Z}/m \times \mathbb{Z}]} \mathbb{Q}(\zeta_m)(t) \cong H_*(V_{p^r}; \mathbb{Q}(\zeta_m)(t)).$$

On H_2 we have the equivariant intersection form

$$\lambda^\chi \colon H_2(V_{p^r}; \mathbb{Q}(\zeta_m)(t)) \times H_2(V_{p^r}; \mathbb{Q}(\zeta_m)(t)) \to \mathbb{Q}(\zeta_m)(t).$$

We also have

$$\lambda^{\mathbb{Q}} \colon H_2(V_{p^r}; \mathbb{Q}) \times H_2(V_{p^r}; \mathbb{Q}) \to \mathbb{Q}.$$

Let $\iota \colon \mathbb{Q} \to \mathbb{Q}(\zeta_m)(t)$. Define the Witt-class defect:

$$\tau(K, \chi) = \big([\lambda^\chi] - \iota_*[\lambda^{\mathbb{Q}}]\big) \otimes \frac{1}{s} \in L_0(\mathbb{Q}(\zeta_m)(t)) \otimes \mathbb{Q}.$$

Casson-Gordon Lemma 4

Suppose $m = q^k$ is a prime power and

$$H_*(Y; \mathbb{Z}/q) \cong H_*(S^1; \mathbb{Z}/q).$$

Consider covers

$$\widetilde{X} \xrightarrow{m=q^k} X \xrightarrow{\infty} Y$$

with the degree shown. Then

$$H_*(\widetilde{X}; \mathbb{Q}) \otimes \mathbb{Q}(\zeta_m)(t) = 0.$$

Proof of obstruction theorem

Let $\Delta \subseteq D^4$ be a slice disc and set

$$Y = V := D^4 \setminus \nu\Delta.$$

Let

$$P := \ker\big(TH_1(M_{p^r}) \to TH_1(V_{p^r})\big)$$

and choose χ that vanishes on P.

Ignoring some important slice vs. ribbon issues, χ extends to

$$\overline{\chi}\colon\; TH_1(V_{p^r}) \to \mathbb{Z}/m.$$

Consider corresponding covers:

$$\widetilde{V}_\infty \to V_\infty \to V.$$

By Casson-Gordon Lemma 4,

$$H_2(V_{p^r}; \mathbb{Q}(\zeta_m)(t)) \cong H_2(\widetilde{V}_\infty; \mathbb{Q}) \otimes \mathbb{Q}(\zeta_m)(t) = 0.$$

Hence $[\lambda^\chi] = 0$ and then it is easy to show

$$\tau(K, \chi) = 0.$$

$\square$

For every $n = t(t-1)$, $t \geq 3$, and for every metaboliser P of the linking form, Casson-Gordon chose a χ that vanishes on P and proved that $\tau(K_n, \chi) \neq 0$, using signatures.

For $t = 3$, this is related to $\sigma(T_{2,3}) \neq 0$.

Applications of Casson-Gordon invariants.

Theorem (Livingston 1983)

There exists a knot not concordant to its reverse.

Applications of Casson-Gordon invariants.

Theorem (Livingston 2001, Taehee Kim 2005)

Fix a Seifert form V with $\Delta_V(t) = \det(tV - V^T) \neq \pm t^k$. There exists an infinite family $\{K_i\}$ of knots with Seifert form V that are linearly independent in $\mathcal{C}$.

Applications of Casson-Gordon invariants.

Theorem (Gilmer 1982)

There exist algebraically slice knots with arbitrarily large 4-genus.

A. N. Miller 2022: these can be chosen to be amphichiral and hence order 2 in $\mathcal{C}$.

Applications of Casson-Gordon invariants.

- Ruberman 1983, 1988: Casson-Gordon style double slice obstructions, including for high dimensional knots.
- Livingston 1991, Gilmer-Livingston 1992: Alternative proof using Casson-Gordon invariants of Cochran-Orr's 1990 result that there are links with vanishing Milnor's invariants not concordant to boundary links, including for high dimensional links.
- Livingston-Naik 2001: used Casson-Gordon invariants to show that many knots of order 4 in $\mathcal{AC}$ have infinite order in $\mathcal{C}$.

Applications of Casson-Gordon invariants.

- Friedl 2003: Interpreted Casson-Gordon invariants as η-invariants.
- Friedl-Powell 2012: Casson-Gordon style obstructions for a 2-component link to be concordant to the Hopf link.
- A. N. Miller 2017: computed topological slice status of odd 3-strand pretzel knots.

Applications of Casson-Gordon invariants.

- A. N. Miller 2023: used Casson-Gordon obstructions to rule out many satellite operators from being homomorphisms. (See also later work of Cahn–Kjuchukova 2023, Miller–Lidman–Pinzón-Caicedo 2024, Cha-Taehee Kim 2025, and Johanningsmeier–Kim–Miller 2025.)
- Livingston 2025: there exist rank-expanding satellite operators P, i.e. for some K, $\{P(nK)\}_{n\in\mathbb{Z}}$ generates an infinite rank subgroup of $\mathcal{C}$.

Computing Casson-Gordon invariants.

- Twisted Alexander polynomials: Lin 1990, Wada 1994, Kirk-Livingston 1999, Herald-Kirk-Livingston 2010. Distinguished more knots from their reverses and mutant knots, in $\mathcal{C}$. Resolved slice status of most knots up to 12 crossings.

- Twisted Blanchfield pairings: Leidy 2006, A. N. Miller-Powell 2018, Powell 2016, Friedl-Leidy-Nagel-Powell 2017, Borodzik-Conway-Politarczyk 2022-25.
  Boundary of a Witt class in $L_0(\mathbb{Q}(\zeta_m)(t))$ is a Witt class of linking forms.
  These capture most of the Casson-Gordon invariants and refine twisted Alexander polynomials.

Computing Casson-Gordon invariants.

- Hedden-Kirk-Livingston 2012 and Conway-Min Hoon Kim-Politarczyk 2023: Used Casson-Gordon invariants and twisted Blanchfield pairings respectively to show many algebraic knots (which are all iterated torus knots) are linearly independent in $\mathcal{C}$: evidence for Rudolph's conjecture.

Extension of Casson-Gordon invariants.

Theorem (Cooper 1982, Cochran-Orr-Teichner 2004)

Let K be a slice genus one knot. For every genus one Seifert surface F, there exists a simple closed curve $\gamma \subseteq F$ with $\mathrm{lk}(\gamma, \gamma^+) = 0$ and

$$\int_{S^1 \subseteq \mathbb{C}} \sigma_\gamma(z) \, \mathrm{d}z = 0.$$

Here let V be a Seifert form for γ and $z \in S^1 \subseteq \mathbb{C}$. Define

$$\sigma_\gamma(z) = \mathrm{sign}\big((1 - z)V + (1 - \overline{z})V^T\big)$$

See also Gilmer-Livingston 2013 for a related statement on sums of signatures at certain prime power roots of unity.

Theorem (Cochran-Davis 2013)

There exists a genus one slice knot K such that for every genus one Seifert surface F and every simple closed curve γ on F with $\mathrm{lk}(\gamma, \gamma^+) = 0$, $\sigma_\gamma(-1) \neq 0$ and $\mathrm{Arf}(\gamma) \neq 0$.

Kaufmann had conjectured otherwise.

Extension of Casson-Gordon invariants.

Theorem (Cochran-Orr-Teichner 2003)

There exist (topologically) non-slice knots with vanishing Casson-Gordon invariants.

Extension of Casson-Gordon invariants.

Cochran-Orr-Teichner filtration 2003:

$$\mathcal{C} \supseteq \mathcal{F}^{(0)} \supseteq \mathcal{F}^{(0.5)} \supseteq \mathcal{F}^{(1)} \supseteq \mathcal{F}^{(1.5)} \supseteq \mathcal{F}^{(2)} \supseteq \cdots$$

- $\mathcal{C}/\mathcal{F}^{(0.5)} \cong \mathcal{AC}$;
- $K \in \mathcal{F}^{(1.5)} \Rightarrow \tau(K, \chi) = 0$ for all χ.

Theorem (Cochran-Teichner 2007, Cochran-Taehee Kim 2008, Cochran-Harvey-Leidy 2008–10, Taehee Kim 2016, Franklin 2013)

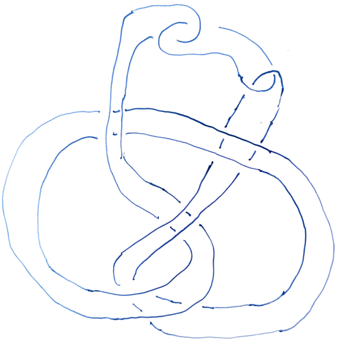
For every $n \in \mathbb{N}_0$, there are subgroups

$$\mathbb{Z}^\infty \oplus (\mathbb{Z}/2)^\infty \subseteq \mathcal{F}^{(n)}/\mathcal{F}^{(n.5)}.$$

Extension of Casson-Gordon invariants.

Theorem (Cha 2010, Cha-Friedl 2013)

The Bing double of the figure eight knot is not slice.

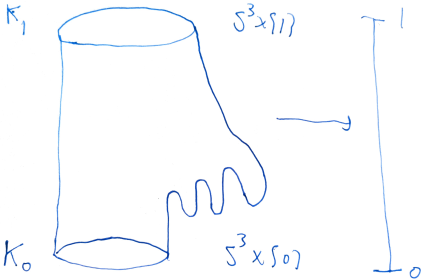


Ribbon concordance (Gordon, 1981).

We say that K_1 is *ribbon concordant* to K_0, and write $K_1 \geq K_0$, if there is a smooth concordance C from K_1 to K_0 such that

$$C \to S^3 \times I \to I$$

is a Morse function with no maxima.



Theorem (Gilmer 1984)

If $K_1 \geq K_0$, then $\Delta_{K_0} \mid \Delta_{K_1}$.

Conjecture (Gordon 1981)

The relation $\geq$ is a partial order on the set of isotopy classes of knots.

Theorem (Gordon 1981)

If $K_1 \geq K_0$ and $K_0 \geq K_1$ and K_1 is transfinitely nilpotent, that is if the lower central subgroup of the commutator subgroup

$$\left[\pi_1(S^3 \setminus K_1), \pi_1(S^3 \setminus K_1)\right]_\alpha$$

is trivial for some ordinal α, then $K_0 = K_1$.

Holds for K_1 constructed by connected sums and cables from 2-bridge knots and fibred knots.

Theorem (Zemke, Levine-Zemke 2019)

If $K_1 \geq K_0$ and $K_0 \geq K_1$, then $\widehat{HFK}(K_0) \cong \widehat{HFK}(K_1)$ and $Kh(K_0) \cong Kh(K_1)$.

Theorem (Agol 2022)

If $K_1 \geq K_0$ and $K_0 \geq K_1$, then $K_0 = K_1$.

Question

Does every concordance class contain a unique minimal representative with respect to $\geq$?

There is another very influential Casson-Gordon article. I expect Zupan's talk to discuss this in detail, so I'll just briefly mention it here.

Homotopically ribbon knots.

Definition (Casson-Gordon)

A knot K in a homology 3-sphere M is *homotopically ribbon* if it bounds a smoothly embedded disc D in a homology 4-ball V such that

$$\pi_1(M \setminus K) \twoheadrightarrow \pi_1(V \setminus D)$$

is surjective.

If $M = S^3$, V is homeomorphic to D^4 so K is *exotically slice*.

Theorem (Casson-Gordon 1983)

A fibred knot in a homology 3-sphere is homotopically ribbon if and only if its closed monodromy extends over a handlebody.

Conjecture (Homotopically ribbon implies slice)

A knot K in S^3 is topologically slice if and only if K is homotopically ribbon.

If true this would have far reaching consequences for our understanding of the (topological) knot concordance group.