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ON THE DIFFEOMORPHISM GROUP OF $S^1 \times S^2$

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ABSTRACT. A disjunction technique for families of 2-spheres in 3-manifolds is applied to determine the homotopy type of the diffeomorphism group of $S^1 \times S^2$.

We operate in the C^∞ category throughout. Let M^3 be a connected 3-manifold having exactly two isotopy classes of submanifolds diffeomorphic to S^2. It is an exercise in 3-manifold topology to see that a closed 3-manifold satisfying this condition is one of the two S^2-bundles over S^1, or the connected sum of two irreducible 3-manifolds. Let $\mathcal{E}$ denote the space of embeddings $f\colon S^2 \to M^3$ whose image does not bound a ball in M^3. For a bicollar neighborhood $[-1, 1] \times S^2 \subset M^3$ of the image of a fixed element of $\mathcal{E}$, let $\mathcal{E}' \subset \mathcal{E}$ be the subspace of embeddings $f\colon S^2 \to M^3$ whose image is disjoint from $\{x\} \times S^2$ for some $x \in [-1, 1]$ (depending on f).

THEOREM. *The inclusion map $\mathcal{E}' \to \mathcal{E}$ is a homotopy equivalence.*

If one assumes the Smale Conjecture, $\mathrm{Diff}(S^3) \simeq O(4)$ [3], then the Theorem easily implies

COROLLARY. $\mathrm{Diff}(S^1 \times S^2)$ *has the homotopy type of the product* $O(2) \times O(3) \times \Omega O(3)$.

The calculation of $\pi_0 \, \mathrm{Diff}(S^1 \times S^2)$ has been known for a long time, using Cerf's "$\Gamma_4 = 0$" theorem. In his thesis [4], B. Jahren showed $\pi_1(\mathcal{E}, \mathcal{E}') = 0$, which reduces the calculation of $\pi_1 \, \mathrm{Diff}(S^1 \times S^2)$ to $\pi_1 \, \mathrm{Diff}(S^3)$.

Results equivalent to the Theorem and Corollary were announced in [1], based on the belief that the disjunction technique of [2] for surfaces of higher genus extended in straightforward manner to the case of 2-spheres. However, as Laudenbach has pointed out, this extension is not straightforward, and the purpose of the present paper is to recast the technique of [2] so it does apply to 2-spheres. (Laudenbach also has a method for handling 2-spheres in $S^1 \times S^2$, not as general as the one in this paper.)

PROOF OF THE THEOREM. Let $f_t \in \mathcal{E}$, $t \in D^k$, be a smooth family representing an element of $\pi_k(\mathcal{E}, \mathcal{E}')$, so that $f_t \in \mathcal{E}'$ for $t \in \partial D^k$. Choose a basepoint $* \in S^2$ and let $p_t = f_t(*)$ and $M_t = f_t(S^2)$. By Sard's theorem and the compactness of D^k, we may choose a finite number of slices $N_i = \{x_i\} \times S^2 \subset [-1, 1] \times S^2 \subset M^3$ and

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closed k-balls $B_i \subset D^k$ such that:

(1) M_t is transverse to N_i for $t \in B_i$.

(2) $\cup_i \operatorname{int}(B_i) = D^k$.

(3) $N_i \neq N_j$ for $i \neq j$.

(4) $p_t \notin N_i$ for $t \in B_i$.

Let C_t^i be the collection of circles of $M_t \cap N_i$ for $t \in B_i$, and let $C_t = \cup_i C_t^i$, the union over all i such that $t \in B_i$. Each circle $c_t \in C_t$ bounds a unique disk $D_M(c_t) \subset M_t - \{p_t\}$. Proceeding inductively over the multiple intersections $B_{i_1} \cap \cdots \cap B_{i_n}$, from larger to smaller values of n, we may choose a smooth family of functions $\varphi_t: C_t \to (0, 1)$ satisfying

$$\varphi_t(c_t) < \varphi_t(c_t') \quad \text{whenever } D_M(c_t) \subset D_M(c_t'). \tag{5}$$

With a bit more effort, we may also achieve

$$\varphi_t(c_t) \neq \varphi_t(c_t') \quad \text{for all pairs } c_t \neq c_t' \text{ in } C_t^i. \tag{6}$$

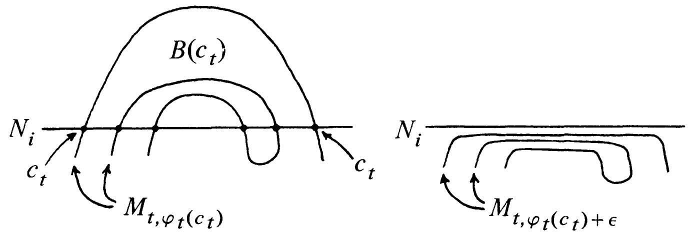
To obtain this, first replace each N_i by $k+1$ nearby slices $N_{ij} = \{x_{ij}\} \times S^2$, $j = 1, \ldots, k+1$, for which (1), (3), and (4) still hold. Let C_t^{ij} be the set of circles of $M_t \cap N_{ij}$ and let C_t be the union of the C_t^{ij}'s, as before. Choose $\varphi_t: C_t \to (0, 1)$ again satisfying (5). For each N_{ij} there is a subset K_{ij} of B_i where φ_t is not injective on C_t^{ij}. If φ_t is in "general position", each K_{ij} will be a finite union of codimension-one submanifolds of B_i and $\bigcap_{j=1}^{k+1} K_{ij}$ will be empty, for each i. Thus

$$\bigcup_{i,j} \operatorname{int}\big(B_i - N(K_{ij})\big) = D^k$$

for small enough neighborhoods $N(K_{ij})$ of K_{ij}. By construction, φ_t is injective on C_t^{ij} for $t \in B_i - N(K_{ij})$. Now choose finitely many balls B_{ijl} in $B_i - N(K_{ij})$ and corresponding slices N_{ijl} near N_{ij} so that (1)–(4) hold for these. Each circle $c_t \subset M_t \cap N_{ij}$ then determines a nearby circle $c_t^l \subset M_t \cap N_{ijl}$, and we choose for $\varphi_t(c_t^l)$ a value near $\varphi_t(c_t)$ such that (5) holds for the circles $c_t^1, c_t^2, \ldots$. With $\{B_{ijl}\}$ for $\{B_i\}$ and $\{N_{ijl}\}$ for $\{N_i\}$ we have now achieved (1)–(6), as one can easily check.

For fixed t, we now construct an isotopy M_{tu} of $M_t = M_{t0}$ which eliminates all the circles of C_t. Choose first an $\varepsilon > 0$, independent of t, so that the inequalities in (5) and (6) take the forms $\varphi_t(c_t) < \varphi_t(c_t') - \varepsilon$ and $|\varphi_t(c_t) - \varphi_t(c_t')| < \varepsilon$, respectively. Next, suppose inductively that for some $c_t \in C_t^i$, M_{tu} has been constructed for $u \leqslant \varphi_t(c_t)$ such that, for $u \leqslant \varphi_t(c_t)$, (a) $M_{tu} = M_{t0}$ near c_t, and (b) M_{tu} restricted to $D_M(c_t)$ is an isotopy of $D_M(c_t)$ to $D_M'(c_t)$, say, with $\operatorname{int}(D_M'(c_t)) \cap N_j = \varnothing$ for each j such that $t \in B_j$. We call such a c_t a *primary* circle of C_t^i. Since $D_M'(c_t) \cap N_i = c_t$, then by the characteristic property of the 3-manifold M^3, exactly one of the two disks into which N_i is cut by c_t, say $D_N(c_t)$, is such that the 2-sphere $D_M'(c_t) \cup D_N(c_t)$ bounds a 3-ball $B(c_t)$ in M^3. Note that $B(c_t) \cap N_j = \varnothing$ for each $j \neq i$ such that $t \in B_j$, since $\partial B(c_t) \cap N_j = \varnothing$.

The isotopy M_{tu} for $\varphi_t(c_t) \leqslant u \leqslant \varphi_t(c_t) + \varepsilon$ is now constructed to eliminate c_t by isotoping $D_M'(c_t)$ across $B(c_t)$ to $D_N(c_t)$, and a little beyond. If there are any other circles of C_t^i in $\operatorname{int}(D_N(c_t))$ remaining at time $u = \varphi_t(c_t)$, this isotopy also eliminates them (see the figure); we call such circles *secondary* circles of C_t^i.



A key property of this isotopy eliminating the primary circle $c_t \in C_t^i$ during $[\varphi_t(c_t), \varphi_t(c_t) + \varepsilon]$ is that it does not move M_t near any circles of C_t^j for $j \neq i$, since $B(c_t) \cap N_j = \varnothing$.

If the interval $[\varphi_t(c_t), \varphi_t(c_t) + \varepsilon]$ overlaps another interval $[\varphi_t(c_t'), \varphi_t(c_t') + \varepsilon]$ for a primary circle $c_t' \in C_t^j$, then by (5) the disks $D_M'(c_t)$ and $D_M'(c_t')$ are disjoint, and by (6), $i \neq j$, so the disks $D_N(c_t)$ and $D_N(c_t')$ are disjoint. It follows that $B(c_t)$ and $B(c_t')$ are disjoint. The two isotopies eliminating c_t and c_t' during these overlapping intervals thus have disjoint supports and can be performed independently. Hence for fixed t a well-defined isotopy M_{tu}, $0 \leqslant u \leqslant 1$, is obtained, eliminating each primary circle $c_t \in C_t$ during the u-interval $[\varphi_t(c_t), \varphi_t(c_t) + \varepsilon]$. From the construction it is clear that $M_{t1} \cap N_i = \varnothing$ for $t \in B_i$.

Observe that for circles of C_t^i, the distinction between primary and secondary is independent of $t \in B_i$. For, by (6) the ordering of the circles of C_t^i is independent of $t \in B_i$; and, as noted earlier, an isotopy which eliminates a primary circle of C_t^j for $j \neq i$ does not affect any circle of C_t^i.

The isotopy M_{tu} of M_t may not depend continuously on t, for the reason that as t leaves a ball B_i, the circles of C_t^i are deleted from C_t and hence an isotopy eliminating a primary circle $c_t \in C_t^i$ during $[\varphi_t(c_t), \varphi_t(c_t) + \varepsilon]$ is suddenly not performed. This can be remedied by the following tapering process. For each i, let $B_i' \subset \operatorname{int}(B_i)$ be a concentric ball such that $\{\operatorname{int}(B_i')\}$ still covers D^k. Let Z_i be a tapering cylinder in $D^k \times [0, 1]$ with base $B_i \times \{0\}$ and top $B_i' \times \{1\}$, having a radius in $D^k \times \{u\}$ which decreases constantly as u goes from 0 to 1. The refined prescription for M_{tu} then is: for an isotopy eliminating a primary circle $c_t \in C_t^i$ during $[\varphi_t(c_t), \varphi_t(c_t) + \varepsilon]$, use only the portion of this isotopy supported by the part of $[\varphi_t(c_t), \varphi_t(c_t) + \varepsilon]$ inside Z_i. In other words, outside Z_i we simply forget about the slice N_i and the way M_{tu} intersects N_i. This creates no problems since isotopies eliminating primary circles of C_t^i leave circles of C_t^j unchanged for $j \neq i$.

To make M_{tu} depend continuously on t it remains only to choose the isotopy eliminating a primary $c_t \in C_t^i$ during $[\varphi_t(c_t), \varphi_t(c_t) + \varepsilon]$ to vary continuously with t. This can be done by choosing for one $t \in B_i$ a diffeomorphism of a neighborhood of $B(c_t)$ onto a standard model, then extending this diffeomorphism to all $t \in B_i$ by isotopy extension, and then using these diffeomorphisms to pull back a standard disjunction isotopy in the model. We can do this for each i separately, and for each primary $c_t \in C_t^i$ in the order given by φ_t, since intervals $[\varphi_t(c_t), \varphi_t(c_t) + \varepsilon]$ for primary c_t's overlap only when their $B(c_t)$'s are disjoint.

The family of isotopies M_{tu} provides by isotopy extension a family of isotopies f_{tu} of the given $f_t \in \mathfrak{S}$ such that $f_{t1} \in \mathfrak{S}'$. For t near ∂D^k we can choose the slices N_i

to be disjoint from M_t (since f_t is then in $\mathcal{E}'$), so $M_{tu} = M_t$ and (we may assume) $f_{tu} = f_t$ for $t \in \partial D^k$. Thus we have shown $\pi_k(\mathcal{E}, \mathcal{E}') = 0$ for any k. $\square$

Proof of the Corollary. We shall use the following two assertions:

(a) The space of embeddings $S^2 \to S^2 \times \mathbf{R}$ deformation retracts onto the subspace of diffeomorphisms $S^2 \to S^2 \times \{0\}$.

(b) The space of diffeomorphisms of $S^2 \times I$ deformation retracts onto the subspace of diffeomorphisms taking slices $S^2 \times \{x\}$ to slices $S^2 \times \{y\}$.

We leave it as an exercise for the reader to verify that (a) and (b) follow from the Smale Conjecture. (In fact, they are each equivalent to the Smale Conjecture.)

To prove the Corollary, let $f_t: S^1 \times S^2 \to S^1 \times S^2$, $t \in D^k$, represent an element of $\pi_k(\mathrm{Diff}(S^1 \times S^2), \mathrm{Diff}_s(S^1 \times S^2))$, where the subscript s denotes diffeomorphisms taking slices $\{x\} \times S^2$ to slices $\{y\} \times S^2$. If $* \in S^1$ is a basepoint, then by the Theorem, we may assume $f_t | \{*\} \times S^2 \in \mathcal{E}'$ for all $t \in D^k$. The projection of $f_t(\{*\} \times S^2)$ onto S^1 is then an arc varying continuously with t, so we may choose $x_t \in S^1$ varying continuously with t and disjoint from this arc. Thus $f_t(\{*\} \times S^2) \cap \{x_t\} \times S^2 = \varnothing$ for all t. After composing f_t with a rotation in the S^1 factor of $S^1 \times S^2$, we may assume this x_t is a constant point $x_0 \in S^1$, $x_0 \neq *$. Next we apply (a) to isotope f_t so that $f_t(\{*\} \times S^2) = \{*\} \times S^2$. Then we apply (b) to make $f_t \in \mathrm{Diff}_s(S^1 \times S^2)$ for all $t \in D^k$. This shows that $\mathrm{Diff}_s(S^1 \times S^2) \to \mathrm{Diff}(S^1 \times S^2)$ is a homotopy equivalence. Since $\mathrm{Diff}(S^1) \simeq O(2)$ and $\mathrm{Diff}(S^2) \simeq O(3)$ [**5**], the Corollary follows. $\square$

Remarks. For the nontrivial S^2-bundle over S^1, this argument shows the diffeomorphism group has the homotopy type of the subgroup of diffeomorphisms taking fibers to fibers by elements of $O(3) \subset \mathrm{Diff}(S^2)$. In the case of a manifold of the form $M = M_1 \# M_2$, where M_1 and M_2 are irreducible (and not S^3), a similar argument shows that $\mathrm{Diff}(M)$ deformation retracts onto the subgroup of diffeomorphisms which preserve a fixed copy of the nontrivial $S^2 \subset M$.

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